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The signless Laplacian spectral radii of modified graphs

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Abstract. In this paper, various modifications of a connected graph G are regarded as perturbations of its signless Laplacian matrix. Several results concerning the resulting changes to the signless Laplacian spectral radius of G are obtained by solving intermediate eigenvalue problems of the second type.
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1. Introduction

Let G be a simple graph with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$. The signless Laplacian eigenvalues of G are the eigenvalues of $Q(G)$; they are real numbers (since $Q(G)$ is symmetric). As usual, $\theta_1(G) \geq \theta_2(G) \geq \cdots \geq \theta_n(G)$ are the signless Laplacian eigenvalues of G in non-increasing order. The largest signless Laplacian eigenvalue of G, i.e. $\theta_1(G)$, is also called the signless Laplacian spectral radius of G. For a connected graph G, $Q(G)$ is non-negative (i.e., all entries are non-negative) and irreducible, and by the Perron-Frobenius theorem for non-negative matrices, $\theta_1(G)$ has multiplicity one and there exists a unique positive unit eigenvector $\boldsymbol{x}$ corresponding to $\theta_1(G)$. We shall refer to such an eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ as the Perron vector of $Q(G)$, where the positive real number x_i corresponds to the vertex v_i for $i = 1, 2, \ldots, n$. Then the following description is well-known:

$$\theta_1(G) = (\boldsymbol{x}, Q(G)\boldsymbol{x}) = \sum_{v_i v_j \in E(G)} (x_i + x_j)^2 \tag{1}$$

The study of graph perturbations is concerned primarily with changes in eigenvalues which result from modifications of a graph. Maas [5], Rowlinson [3, 4] and Zhou [7] obtained several results concerning the resulting changes to the spectral radius of G (the largest eigenvalue of $A(G)$) by solving intermediate eigenvalue prob-

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lems of the second type. In this paper, by applying the same techniques to the signless Laplacian matrix of G, we obtain several results on the resulting changes to the signless Laplacian spectral radius of G.

2. Intermediate eigenvalue problems of second type

We present the results required from [6] (also see [2]) in terms of an n-dimensional Euclidean space $\mathbb{V}$ in which the inner product of vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ is denoted by $(\boldsymbol{y},\boldsymbol{z}) = \boldsymbol{y}^T\boldsymbol{z}$. Let $\widetilde{Q}$ be a symmetric linear transformation of $\mathbb{V}$, and let $\widetilde{H}$ be a positive transformation of $\mathbb{V}$. For any symmetric transformation T of $\mathbb{V}$, let $\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq \lambda_n(T)$ denote the eigenvalues of T. We also use $\lambda_{\min}(T)$ and $\lambda_{\max}(T)$ to denote the smallest and largest eigenvalues of T, respectively. The general problem is to find lower bounds for the eigenvalues $\lambda_i(\widetilde{Q}+\widetilde{H})$ which can be readily calculated from $\widetilde{H}$ and appropriate invariants of $\widetilde{Q}$.

A second inner product may be defined on $\mathbb{V}$ by $[\boldsymbol{y},\boldsymbol{z}] = (\widetilde{H}\boldsymbol{y},\boldsymbol{z})$. Choose any basis $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ for $\mathbb{V}$ and, using the new inner product, let P_r be the orthogonal projection onto the subspace of $\mathbb{V}$ spanned by $\boldsymbol{v}_1,\dots,\boldsymbol{v}_r$ $(r = 1,...,n)$. Thus $P_n = \boldsymbol{I}$, and if we define $P_0 = \boldsymbol{0}$, we have $[P_{r-1}\boldsymbol{y},\boldsymbol{y}] \leq [P_r\boldsymbol{y},\boldsymbol{y}]$, whence $((\widetilde{Q} + \widetilde{H}P_{r-1})\boldsymbol{y},\boldsymbol{y}) \leq ((\widetilde{Q} + \widetilde{H}P_r)\boldsymbol{y},\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathbb{V}$ $(r = 1,\dots,n)$. Moreover, for each $r \in \{0,1,\dots,n\}$, $\widetilde{H}P_r$ is a symmetric transformation of the original inner product space $\mathbb{V}$. For any symmetric transformation T of $\mathbb{V}$, $\lambda_i(T)$ is the minimum of $\max\{(T\boldsymbol{y},\boldsymbol{y}) : \|\boldsymbol{y}\| = 1, \boldsymbol{y} \in \mathbb{U}\}$ taken over all i-dimensional subspaces $\mathbb{U}$ of $\mathbb{V}$. It follows that $\lambda_i(\widetilde{Q}) \leq \lambda_i(\widetilde{Q} + \widetilde{H}P_1) \leq \lambda_i(\widetilde{Q} + \widetilde{H}P_2) \leq \cdots \leq \lambda_i(\widetilde{Q} + \widetilde{H}P_{n-1}) \leq \lambda_i(\widetilde{Q} + \widetilde{H}P_n)$ for $i = 1,2,\dots,n$. The problem of determining the eigenvalues of $\widetilde{Q} + \widetilde{H}P_r$ for some r is called an intermediate eigenvalue problem of the second type.

Let $\widetilde{Q}\boldsymbol{u}_i = \widetilde{\lambda}_i\boldsymbol{u}_i$ $(i = 1,\dots,n)$, where $\widetilde{\lambda_1},\widetilde{\lambda_2},\dots,\widetilde{\lambda_n}$ are the eigenvalues of $\widetilde{Q}$ and $\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_n$ are orthonomal. Choose $\boldsymbol{v}_i = \widetilde{H}^{-1}\boldsymbol{u}_i$ $(i = 1,\dots,n)$. Then $P_r\boldsymbol{u}_j = \sum\limits_{i=1}^{r}\gamma_{ij}\boldsymbol{v}_i$, where (γ_{ij}) is the inverse of the $r \times r$ Gram matrix T_r whose (i,j)-entry is $[\boldsymbol{v}_i,\boldsymbol{v}_j]$ $(i,j = 1,\dots,r)$. Moreover, the matrix of $\widetilde{Q}+\widetilde{H}P_r$ with respect to the basis $\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\}$ is

$$\begin{bmatrix} \widetilde{\lambda_1} & & & \\ & \widetilde{\lambda_2} & & \\ & & \ddots & \\ & & & \widetilde{\lambda_n} \end{bmatrix} + \begin{bmatrix} T_r^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \tag{2}$$

In what follows, we take $\mathbb{V} = \mathbb{R}^n$, $(\boldsymbol{y},\boldsymbol{z}) = \boldsymbol{y}^T\boldsymbol{z}$ and identify a linear transformation on $\mathbb{R}^n$ and its matrix with respect to the transformation of $\mathbb{R}^n$. Let $\boldsymbol{J}_n$ be the $n \times n$ matrix with all entries equal to 1, and $\boldsymbol{I}_n$ the $n \times n$ identity matrix. If Q and $Q + H$ are the signless Laplacian matrices of a graph G and its modified graph G', respectively, then we take

$$\widetilde{Q} = -Q - (\lambda_{\max}(H) + \delta)\boldsymbol{I}, \text{ and } \widetilde{H} = (\lambda_{\max}(H) + \delta)\boldsymbol{I} - H, \text{ where } \delta > 0.$$

Thus $\widetilde{H}$ is positive and $\widetilde{Q}+\widetilde{H}=-Q-H$. Hence, we have

$$\theta_1(G')=\lambda_{\max}(Q+H)=-\lambda_{\min}(\widetilde{Q}+\widetilde{H})\le-\lambda_{\min}(\widetilde{Q}+\widetilde{H}P_r).$$

For given r, we can choose δ to optimize the upper bound for $\theta_1(G')$.

3. The main results

In this section, we consider three types of modifications of a connected graph G: the first one is obtained from G by relocating an edge, the second one is obtained from G by adding edges between the vertices of an independent set such that it induces a clique, and the third one is obtained from G by adding a new vertex with a prescribed set of neighbors. Using the technique mentioned in Section 2, we obtain upper bounds for the signless Laplacian spectral radii of the three types of modified graphs just mentioned, respectively.

Let G be a connected graph. Here we apply the results of Section 2 with $r=1$ and $\boldsymbol{u}_1$ the Perron vector $\boldsymbol{x}$ of $Q(G)$. When the edge v_iv_j of G is replaced by v_kv_l, there are essentially two cases to consider:

(a) v_i,v_j,v_k,v_l are distinct,

(b) $v_i=v_l$ and v_i,v_j,v_k are distinct.

In case (a), without loss of generality, we take $v_i=v_1$, $v_j=v_2$, $v_k=v_3$, $v_l=v_4$, so that

$$H=\begin{bmatrix} H' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \text{where } H'=\begin{bmatrix} -1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}.$$

Thus $\lambda_{\max}(H)=2$. Let $\widetilde{Q}=-Q-(2+\delta)\boldsymbol{I}_n$ and $\widetilde{H}=(2+\delta)\boldsymbol{I}_n-H$. Therefore we have

$$\widetilde{H}^{-1}=\begin{bmatrix} \widetilde{H}' & \mathbf{0}\\ \mathbf{0} & \frac{1}{2+\delta}\boldsymbol{I}_{n-4}\end{bmatrix},\ \text{where } \widetilde{H}'=\begin{bmatrix} \frac{3+\delta}{(3+\delta)^2-1} & \frac{-1}{(3+\delta)^2-1} & 0 & 0\\ \frac{-1}{(3+\delta)^2-1} & \frac{3+\delta}{(3+\delta)^2-1} & 0 & 0\\ 0 & 0 & \frac{1+\delta}{(1+\delta)^2-1} & \frac{1}{(1+\delta)^2-1}\\ 0 & 0 & \frac{1}{(1+\delta)^2-1} & \frac{1+\delta}{(1+\delta)^2-1}\end{bmatrix}.$$

Hence we find that

$$\begin{aligned}\gamma_{11}&=\frac{1}{[\widetilde{H}^{-1}\boldsymbol{u}_1,\widetilde{H}^{-1}\boldsymbol{u}_1]}\\ &=\frac{1}{(\boldsymbol{u}_1,\widetilde{H}^{-1}\boldsymbol{u}_1)}=\frac{\delta(\delta+2)(\delta+4)}{-\delta(x_1+x_2)^2+(\delta+4)(x_3+x_4)^2+\delta(\delta+4)}.\end{aligned}\tag{3}$$

If the signless Laplacian eigenvalues of G are $\theta_1\ge\theta_2\ge\cdots\ge\theta_n$, then the eigenvalues of $\widetilde{Q}$ are $\widetilde{\theta}_i=-\theta_i-2-\delta$ for $i=1,2,\ldots,n$. Hence the eigenvalues of $\widetilde{Q}+\widetilde{H}P_1$ are $-\theta_1-2-\delta+\gamma_{11}$ and $-\theta_i-2-\delta$ $(i=2,3,\ldots,n)$. Thus $-\lambda_{\min}(\widetilde{Q}+\widetilde{H}P_1)$

is $\theta_1+2+\delta-\gamma_{11}$ if $\gamma_{11}\le\theta_1-\theta_2$, and $\theta_2+2+\delta$ if $\gamma_{11}\ge\theta_1-\theta_2$. As a function of $\delta>0$, γ_{11} has range $(0,\infty)$ and so we may choose $\delta>0$ such that $\gamma_{11}=\theta_1-\theta_2$. Then $\theta_1(G')\le\theta_1(G)+2+\delta-\gamma_{11}$, and to ensure that $\theta_1(G')<\theta_1(G)$, we require $\gamma_{11}>\delta+2$. Now from (3) we have

$$\gamma_{11}^{-1}\delta(\delta+4)(-\gamma_{11}+\delta+2)=\alpha-\delta\beta,$$

where $\alpha=4(x_3+x_4)^2$ and $\beta=(x_1+x_2)^2-(x_3+x_4)^2$.

If $\beta>0$ and $\delta>\alpha\beta^{-1}>0$, then $\gamma_{11}>\delta+2$. If $\beta\le 0$, then by the Rayleigh-Ritz Theorem, we have

$$\theta_1(G')\ge(\boldsymbol{x},Q(G')\boldsymbol{x})=\sum_{v_iv_j\in E(G)}(x_i+x_j)^2-\beta\ge\sum_{v_iv_j\in E(G)}(x_i+x_j)^2=\theta_1(G).$$

These results can be summarized as follows.

Theorem 1. *Let G be a connected graph of order n with distinct vertices v_i, v_j, v_k, v_l such that $v_iv_j\in E(G)$ and $v_kv_l\notin E(G)$. Suppose that G' is obtained from G by replacing edge v_iv_j with v_kv_l. Let $\theta_1\ge\theta_2\ge\cdots\ge\theta_n$ be the signless Laplacian eigenvalues of G and let $\boldsymbol{x}=(x_1,x_2,\dots,x_n)^T$ be the Perron vector of $Q(G)$.*

(i) If $(x_i+x_j)^2\le(x_k+x_l)^2$, then $\theta_1(G')\ge\theta_1(G)$;

(ii) if $(x_i+x_j)^2>(x_k+x_l)^2$ and $\theta_1-\theta_2>\frac{2[(x_i+x_j)^2+(x_k+x_l)^2]}{(x_i+x_j)^2-(x_k+x_l)^2}$, then $\theta_1(G')<\theta_1(G)$.

To deal with case (b), without loss of generality, we assume that $v_i=v_l=v_1$, $v_j=v_2$ and $v_k=v_3$. Thus

$$H=\begin{bmatrix}H' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \text{where } H'=\begin{bmatrix}0 & -1 & 1\\ -1 & -1 & 0\\ 1 & 0 & 1\end{bmatrix}.$$

Note that $\lambda_{\max}(H)=\sqrt{3}$. Let $\widetilde{Q}=-Q-(\sqrt{3}+\delta)\boldsymbol{I}_n$ and $\widetilde{H}=(\sqrt{3}+\delta)\boldsymbol{I}_n-H$. Similarly, we find that

$$\gamma_{11}=\frac{\delta(\delta+\sqrt{3})(\delta+2\sqrt{3})}{\alpha-(\delta+\sqrt{3})\beta+\delta(\delta+2\sqrt{3})},$$

where $\alpha=(x_1+x_2)^2+(x_1+x_3)^2+(x_2-x_3)^2$ and $\beta=(x_1+x_2)^2-(x_1+x_3)^2$. Hence the eigenvalues of $\widetilde{Q}+\widetilde{H}P_1$ are $-\theta_1-\sqrt{3}-\delta+\gamma_{11}$ and $-\theta_i-\sqrt{3}-\delta$ $(i=2,3,\dots,n)$. Again we may choose $\delta>0$ such that $\gamma_{11}=\theta_1-\theta_2$, and arguing as before we find that $\gamma_{11}>\delta+\sqrt{3}$ when $\beta>0$ (i.e., $x_2>x_3$) and $\theta_1-\theta_2>\alpha\beta^{-1}+\sqrt{3}$.

The corresponding theorem in this case is therefore as follows.

Theorem 2. *Let G be a connected graph of order n with distinct vertices v_i, v_j, v_k such that $v_iv_j\in E(G)$ and $v_iv_k\notin E(G)$. Suppose that G' is obtained from G by replacing edge v_iv_j with v_iv_k. Let $\theta_1\ge\theta_2\ge\cdots\ge\theta_n$ be the signless Laplacian eigenvalues of G and $\boldsymbol{x}=(x_1,x_2,\dots,x_n)^T$ be the Perron vector of $Q(G)$.*

(i) If $x_j\le x_k$, then $\theta_1(G')\ge\theta_1(G)$;

(ii) *if* $x_j > x_k$ *and* $\theta_1 - \theta_2 > \frac{(\sqrt{3}+1)(x_i+x_j)^2-(\sqrt{3}-1)(x_i+x_k)^2+(x_j-x_k)^2}{(x_i+x_j)^2-(x_i+x_k)^2}$, *then* $\theta_1(G') < \theta_1(G)$.

Before we illustrate Theorems 1 and 2 with an example, the following lemma is needed.

Lemma 1 ([1]). *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B a principal sub-matrix of A. Let B have eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ $(m \leq n)$. Then the inequalities $\lambda_{n-m+i} \leq \rho_i \leq \lambda_i$ hold for $i = 1, 2, \ldots, m$.*

Example 1. *Let G be the wheel $K_1 \nabla C_n$ $(n > 10)$ with vertex v_0 adjacent to each vertex of the cycle $v_1 v_2 \cdots v_n v_1$. Then the signless Laplacian spectral radius of $K_1 \nabla C_n$ is $\theta_1(K_1 \nabla C_n) = \frac{n+5+\sqrt{n^2-6n+25}}{2}$. Lemma 1 implies that $\theta_2(K_1 \nabla C_n) \leq \lambda_{\max}(Q_{v_0}(K_1 \nabla C_n)) = 5$, where $Q_{v_0}(K_1 \nabla C_n)$ is the sub-matrix of $Q(K_1 \nabla C_n)$ obtained by deleting the row and column corresponding to the vertex v_0. Thus $\theta_1 - \theta_2 > \theta_1 - 5$. The Perron vector of $Q(K_1 \nabla C_n)$ is $(\alpha, \underbrace{\beta, \ldots, \beta}_{n})$, where $\alpha, \beta > 0$, $\theta_1 \alpha = n\alpha + n\beta$, $\theta_1 \beta = 5\beta + \alpha$ and $\alpha^2 + n\beta^2 = 1$. Let G' be the graph obtained from $K_1 \nabla C_n$ by replacing edge $v_0 v_2$ with $v_1 v_3$. To apply Theorem 1, with $i = 0$, $j = 2$, $k = 1$ and $l = 3$, we require $\theta_1 - \theta_2 > 2\frac{(\alpha+\beta)^2+4\beta^2}{(\alpha+\beta)^2-4\beta^2}$. It suffices to establish that $\theta_1 - 5 > 2\frac{\theta_1^2+4(\theta_1-n)^2}{\theta_1^2-4(\theta_1-n)^2}$, which holds for $n > 10$ since $n+1 < \theta_1 < n+2$ and $(\theta_1+2)(2n-\theta_1)(3\theta_1-2n) - 4\theta_1^2 > 0$ for $n > 10$. Then $\theta_1(G') < \theta_1(K_1 \nabla C_n)$ by Theorem 1. Now, let G'' be the graph obtained from $K_1 \nabla C_n$ by replacing edge $v_1 v_0$ with $v_1 v_3$. To apply Theorem 2 with $i = 1$, $j = 0$ and $k = 3$, we require $\theta_1 - \theta_2 > \frac{(\sqrt{3}+2)\alpha^2+3(2-\sqrt{3})\beta^2+2\sqrt{3}\alpha\beta}{\alpha^2+2\alpha\beta-3\beta^2}$, and this condition holds for $n > 10$. Then $\theta_1(G'') < \theta_1(K_1 \nabla C_n)$ by Theorem 2.*

Let G be a connected graph, and let $S \subset V(G)$ be any independent set with $|S| \geq 2$. Let G^S be the graph obtained from G by adding edges between the vertices in S such that it induces a clique. For any independent set $S \subset V(G)$ with $|S| \geq 2$, the following theorem gives an upper bound for $\theta_1(G^S)$.

Theorem 3. *Let G be a connected graph of order n with signless Laplacian eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $Q(G)$, and let $S \subset V(G)$ be an independent set with $s = |S| \geq 2$. Then*

$$\theta_1(G^S) \leq \theta_1 + 2(s-1) + \delta - \gamma_{11},$$

where

$$\gamma_{11} = \frac{\delta(s+\delta)[2(s-1)+\delta]}{\delta(s-2)\sum_{i=1}^{s} x_i^2 + [2(s-1)+\delta]\left(\sum_{i=1}^{s} x_i\right)^2 + \delta(s+\delta)} = \theta_1 - \theta_2.$$

Proof. Without loss of generality, suppose that $S = \{v_1, \ldots, v_s\}$. Let Q and $Q + H$ be the signless Laplacian matrices of G and G^S, respectively. Note that the largest eigenvalue of H is $2(s-1)$. Let $\widetilde{Q} = -Q - [2(s-1)+\delta]\boldsymbol{I}_n$ and $\widetilde{H} = [2(s-1)+\delta)]\boldsymbol{I}_n - H$. Hence the eigenvalues of $\widetilde{Q}$ are $\widetilde{\theta_i} = -\theta_i - 2(s-1) - \delta$ $(i = 1, \dots, n)$ and

$$\widetilde{H}^{-1} = \begin{bmatrix} \frac{1}{s+\delta} I_n + \frac{1}{\delta(s+\delta)} \boldsymbol{J}_s & \mathbf{0} \\ \mathbf{0} & \frac{1}{2(s-1)+\delta} \boldsymbol{I}_{n-s} \end{bmatrix}.$$

We use the results in Section 2 with $r = 1$ and $\boldsymbol{u}_1 = \boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$. The matrix of the transformation $\widetilde{Q} + \widetilde{H}P_1$ with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ is $diag(\widetilde{\theta_1} + \gamma, \widetilde{\theta_2}, \dots, \widetilde{\theta_n})$, where

$$\begin{aligned}\gamma_{11} = \frac{1}{[\widetilde{H}^{-1}\boldsymbol{u}_1, \widetilde{H}^{-1}\boldsymbol{u}_1]} &= \frac{1}{(\boldsymbol{u}_1, \widetilde{H}^{-1}\boldsymbol{u}_1)} \\ &= \frac{\delta(s+\delta)[2(s-1)+\delta]}{\delta(s-2)\sum\limits_{i=1}^{s} x_i^2 + [2(s-1)+\delta]\left(\sum\limits_{i=1}^{s} x_i\right)^2 + \delta(s+\delta)}.\end{aligned}$$

Then $\lambda_{\min}(\widetilde{Q} + \widetilde{H}P_1) = \min\{\widetilde{\theta_1} + \gamma, \widetilde{\theta_2}\}$ and so

$$\theta_1(G^S) = -\lambda_{\min}(\widetilde{Q}+\widetilde{H}) \le -\lambda_{\min}(\widetilde{Q}+\widetilde{H}P_1) = \max\{\theta_1+2(s-1)+\delta-\gamma,\ \theta_2+2(s-1)+\delta\}.$$

As a function of δ $(\delta > 0)$, γ_{11} has range $(0, \infty)$ and so we may choose $\delta > 0$ such that $\gamma_{11} = \theta_1 - \theta_2$. This completes the proof. $\square$

Example 2. *Let G be a complete bipartite graph $K_{m,m}$ with $m \ge 2$. Note that $\theta_1(K_{m,m}) = 2m$ and $\theta_2(K_{m,m}) = m$. Let S be any independent set of G with $s = 2$. Then $\theta_1(G^S) \le m + 2 + \delta$, where $\delta > 0$ satisfies $\delta^2 - (m-2)\delta - 2 = 0$. If $m = 5$, then $\theta_1(G^S) < 10.5616$, while $\theta_1(G^S) = 10.5367$ by direct computations.*

Let G be a connected graph of order n, and let S be a non-empty subset of $V(G)$. Let G_S be the graph obtained from G by adding a new vertex whose neighbors are the vertices in S. Using the same argument as Theorem 3 in [7], the following upper bounds on $\theta_1(G_S)$ for any $S \subseteq V(G)$ can be obtained.

Theorem 4. *Let G be a connected graph of order n with signless Laplacian eigenvalues $\theta_1 \ge \theta_2 \ge \cdots \ge \theta_n$. Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of $Q(G)$, and let S be a non-empty subset of $V(G)$ with $|S| = s$. Let $a = \sum\limits_{i=1}^{s} x_i$ and $b = \sum\limits_{i=1}^{s} x_i^2$. If*

$$\frac{s+b}{s+1}\theta_1 < \frac{a^2+s^2}{s} + 2\frac{s+b}{s+1}\theta_2 < \frac{s+b}{s+1}\theta_1 + \sqrt{4a^2 + \left(\frac{s^2-a^2}{s} - \frac{s+b}{s+1}\theta_1\right)^2}, \quad (4)$$

then $\theta_1(G_S) \le \theta_1 + \epsilon$, where $\epsilon > 0$ and $\epsilon = 1 - (\theta_1 - \theta_2)$, $\epsilon = 1 - b - (\theta_1 - \theta_2)$, or ϵ satisfies the equation

$$\epsilon^3 + (2\theta_1 - \theta_2 - 1)\epsilon^2 + [(\theta_1 - b)(\theta_1 - \theta_2) - \theta_1 - a^2 - s(1-b)]\epsilon - (\theta_1 - \theta_2)[\theta_1 b + a^2 + s(1-b)] = 0;$$

otherwise, $\theta_1(G_S) \le \theta_2 + s + 1$.

Example 3. *Let* $G = K_1 \nabla C_6$ *with vertex* v_0 *adjacent to each vertex of the cycle* $v_1v_2v_3v_4v_5v_6v_1$ *and* $S = \{v_0\}$. *Note that* $\theta_1(G) = 8$ *and* $\theta_2(G) = 4$. *It is easy to check that (4) holds. Then by Theorem 4, we have* $\theta_1(G_S) \le 8+\epsilon$, *where* $\epsilon^3+11\epsilon^2+\frac{103}{5}\epsilon-\frac{116}{5}=0$. *Clearly,* $\varepsilon < 0.75$, *and we have* $\theta_1(G_S) \le 8.75$, *while* $\theta_1(G_S) = 8.7355$ *by direct computation.*

Example 4. *Let* G *be a 4-cycle* $C_4 = v_1v_2v_3v_4v_1$ *and* $S = \{v_1, v_3\}$. *Then* $\theta_1 = 4$ *and* $\theta_2 = 2$. *Theorem 4 implies that* $\theta_1(G_S) \le 5$, *while by direct computation, we have* $\theta_1(G_S) = 5$.

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