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An invariance principle for the law of the iterated logarithm for vector-valued additive functionals of Markov chains

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Abstract. In this note, we prove the Strassen's strong invariance principle for vector-valued additive functionals of a Markov chain via the martingale argument and the theory of fractional coboundaries.
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1. Introduction

Let $(X_n)_{n\geq 0}$ denote a stationary ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a measurable space $(\mathcal{X}, \mathcal{B})$. Let $Q(x, dy)$ be its transition kernel and π the stationary initial distribution. Fix an integer $d \geq 1$ and for $p \geq 1$ let $L^p(\pi)$ denote the space of (equivalence classes of) $\mathcal{B}$-measurable functions $g : \mathcal{X} \to \mathbb{R}^d$ such that $\|g\|_p^p := \int_{\mathcal{X}} |g(x)|^p \pi(dx) < \infty$, and let $L_0^p(\pi)$ denote the set of $g \in L^p(\pi)$ for which $\int_{\mathcal{X}} g d\pi = 0$. Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Now fix an $\mathbb{R}^d$-valued function $g \in L_0^2(\pi)$. For $n \geq 0$, define

$$S_{n+1} = S_{n+1}(g) := \sum_{i=0}^{n} g(X_i) \ \text{ and } \ S_0 = 0. \tag{1}$$

For the question of central limit type results for S_n, there have been numerous studies from many angles and under different assumptions; see Maxwell and Woodroofe [6], Derriennic and Lin [4] and references therein.

This short note is a natural continuation of Maxwell and Woodroofe [6]. Our goal is to consider the problem that S_n satisfies the law of the iterated logarithm (LIL) under some proper conditions. Since the appearance of Strassen's paper [13], almost sure invariance principles for the law of iterated logarithm have been obtained for a large class of independent and dependent sequence $(Y_n)_{n\geq 1}$; see Hall and Heyde [5], and Philipp and Stout [10]. Here, the Skorokhod representation plays an important

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role. However, we encounter the essential difficulties, when considering the vector-valued martingale, since Monrad and Philipp [9] proved that it is impossible to embed a general $\mathbb{R}^d$-valued martingale in an $\mathbb{R}^d$-valued Gaussian process.

In this paper, we mainly take along the lines of Maxwell and Woodroofe [6], and use Berger's strong approximation [1]. Moreover, we identify the lim sup in the functional LIL just the square root of the trace of the diffusion matrix corresponding to the functional central limit theorems (CLT). Recently, Zhao and Woodroofe [14] published a LIL for stationary processes that is stronger than our results in the case of $d = 1$, please refer to Miao and Yang [8]. However, the present note has the advantage to make clear that the reasonings used to get the functional CLT in its "almost sure" sense lead easily to the functional form of the multidimensional LIL (Zhao and Woodroofe [14] considered only the classical scalar form of this law but their reasoning could apply to a more general situation).

2. Main results

For introducing our main results, we need some notations. Let $C([0,1],\mathbb{R}^d)$ be the Banach space of continuous maps from $[0,1]$ to $\mathbb{R}^d$, endowed with the supremum norm $|||\cdot|||$, using the Euclidean norm in $\mathbb{R}^d$. Denote by K the set of absolutely continuous maps $f \in C([0,1], \mathbb{R}^d)$, such that

$$f(0) = 0, \qquad \int_0^1 |\dot{f}(t)|^2 dt \le 1,$$

where $\dot{f}$ denotes the derivative of f determined almost everywhere with respect to Lebesgue measure. Obviously, K is relatively compact and closed. Define

$$\xi_n(t) = (2n \log\log n)^{-1/2}[S_k + (nt-k)g(X_k)]$$

for $t \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 0, 1, 2, \cdots, n-1$. In order to avoid difficulties in specification, we adopt the convention that $\log\log x = 1$, if $0 < x \le e^e$. Then, ξ_n is a random element with values in $C([0,1], \mathbb{R}^d)$. In addition, given a function $h \in L^1(\pi)$, we define an operator

$$Qh(x) = \int h(y)Q(x,dy), \quad \pi\text{-a.e. } x \in \mathcal{X}.$$

Obviously, Q is a contraction on $L^p(\pi)$ for $p \ge 1$.

Theorem 1. *Let $g \in L_0^2(\pi)$ and assume that there exists an $\alpha \in (0, 1/2)$ such that*

$$\|\sum_{i=0}^{n-1} Q^i g\|_2 = O(n^\alpha). \tag{2}$$

Then, the sequence of functions $(\xi_n(\cdot),\, n \ge 1)$ is almost surely relatively compact in the space $C([0,1], \mathbb{R}^d)$, and the set of its limit points as $n \to \infty$, coincides with $\sqrt{\mathrm{tr}(\mathfrak{D})}K$, where $\mathrm{tr}(\cdot)$ denotes the trace operator of a matrix and $\mathfrak{D} = \mathbb{E}(M_1 M_1^t) = \int HH^t d\pi_1$ is the diffusion matrix corresponding to the functional central limit theorem, where M_1, H and π_1 are defined in Section 3.

Theorem 2. *Let $g \in L_0^2(\pi)$ and assume that there exists an $\alpha \in (0, 1/2)$ for which (2) is satisfied. Then*

$$\limsup |S_n|/\sqrt{2n \log\log n} = \sqrt{\mathrm{tr}(\mathfrak{D})}, \qquad \mathbb{P}\text{-}a.s. \tag{3}$$

It is worthwhile to give some comments on Theorem 1 and Theorem 2.

Remark 1. *Maxwell and Woodroofe [6] showed the functional CLT for S_n when $d = 1$, under condition (2). However, the CLT is proved there under the following weaker condition*

$$\sum_{n=1}^{\infty} n^{-3/2} \| \sum_{i=0}^{n-1} Q^i g\|_2 < \infty. \tag{4}$$

Recently, for $d = 1$, Peligrad and Utez [11] proved the functional CLT under condition (4) by developing some new maximal inequality for stationary sequences. It is well known that the LIL is closely related to the CLT in some sense, hence the functional LIL should be true under condition (4). But, under our present framework, the functional LIL cannot be proved under condition (4) since the fractional coboundary theory of Derriennic and Lin [3] is no longer valid. Cuny [2] also treated the LIL for S_n when $d = 1$ under some conditions. For an update account of condition (4) and some other forms, please refer to Merlevède, Peligrad and Peligrad [7] and references therein.

3. Proof of main results

For $\varepsilon > 0$, let h_ε be the solution of the equation

$$(1 + \varepsilon)h = Qh + g,$$

where Q is defined as in Section 2. In fact,

$$h_\varepsilon = \sum_{n=1}^{\infty} (1 + \varepsilon)^{-n} Q^{n-1} g. \tag{5}$$

Note that $h_\varepsilon \in L^p(\pi)$, if $g \in L^p(\pi)$. Let π_1 be the joint distribution of X_0 and X_1, so that $\pi_1(dx_0, dx_1) = Q(x_0, dx_1)\pi(dx_0)$; denote the L^2-norm on $L^2(\pi_1)$ by $\| \cdot \|_1$; and let

$$H_\varepsilon(x_0, x_1) = h_\varepsilon(x_1) - Qh_\varepsilon(x_0)$$

for x_0, $x_1 \in \mathcal{X}$. For any $\varepsilon > 0$, let

$$M_n(\varepsilon) = \sum_{i=0}^{n-1} H_\varepsilon(X_i, X_{i+1}) \quad \text{and} \quad R_n(\varepsilon) = Qh_\varepsilon(X_0) - Qh_\varepsilon(X_n),$$

hence, by simple computations,

$$S_n(g) = M_n(\varepsilon) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon). \tag{6}$$

For convenience, we summarize the results of Maxwell and Woodroofe [6] as the following theorem.

Theorem 3 (Theorem MW)**.** *Assume that* $g \in L_0^2(\pi)$ *and that there exists an* $\alpha \in (0, 1/2)$ *for which (2) is satisfied. Then we have*

1. *The limit* $H = \lim_{\varepsilon \to 0^+} H_\varepsilon$ *exists in* $L^2(\pi_1)$*. Moreover, if one defines*

$$M_n = \sum_{i=0}^{n-1} m_i,$$

where $m_i = H(X_i, X_{i+1})$*, then* $(m_n)_{n \geq 0}$ *is a stationary and ergodic* $\mathbb{P}$*-square integrable martingale difference sequence, with respect to the filtration* $\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n)\}_{n \geq 0}$*;*

2. $\|h_\varepsilon\|_2 = O(\varepsilon^{-\alpha})$*, and if* $R_n = S_n - M_n = M_n(\varepsilon) - M_n + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon)$*, then*

$$\mathbb{E}(|R_n|^2) = O(n^{2\alpha}).$$

3.1. Proof of Theorem 1

For $0 \leq t \leq 1$, define

$$\zeta_n(t) = (2n \log\log n)^{-1/2} M_{[nt]},$$
$$\eta_n(t) = (2n \log\log n)^{-1/2} B(nt),$$

where M_n is as defined in Section 2 and $B(\cdot)$ is an $\mathbb{R}^d$-valued Brownian motion with mean 0 and diffusion matrix $\mathfrak{D}$. Theorem 1 of Strassen [13] shows that $(\eta_n(\cdot))_{n\geq 1}$ is relatively compact and the set of its limit points coincides with $\sqrt{\mathrm{tr}(\mathfrak{D})}K$.

Notice that by part (1) of Theorem MW, $(M_n)_{n \geq 1}$ is a square integrable martingale with strictly stationary increments. Moreover,

$$\mathbb{E}(< u, m_0 >^2) < \infty \quad \text{and} \quad \mathbb{E}(< u, m_0 >) = 0, \text{ for all } u \in \mathbb{R}^d, \tag{7}$$

where, $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^d$. Therefore, Corollary 4.1 of Berger [1] implies that,

Without changing its distribution, one can redefine the sequence $(M_n)_{n \geq 1}$ *on a new probability space* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ *on which there exists an* $\mathbb{R}^d$*-valued Brownian motion* $(B(t))_{t \geq 0}$ *with mean 0 and diffusion matrix* $\mathfrak{D}$ *such that*

$$|M_{[t]} - B(t)| = o((t \log\log t)^{1/2}), \quad \hat{\mathbb{P}}\text{-}a.s. \quad (as\ t \to \infty). \tag{8}$$

where, $\mathfrak{D} = \lim_{n \to \infty} n^{-1} Cov(M_n)$*.*

Remark 2. *Birkhoff-Khinchin's ergodic theorem and a simple calculation show that* $\mathfrak{D} = \mathbb{E}(M_1 M_1^t) = \int H H^t d\pi_1$ *is the diffusion matrix corresponding to the functional central limit theorem; please see Rassoul-Agha and Seppäläinen [12].*

That is to say,

$$\sup_{0 \leq t \leq 1} |M_{[nt]} - B(nt)| = o((2n \log\log n)^{1/2}), \quad \mathbb{P}\text{-}a.s.$$

Hence,

$$\|\|\zeta_n - \eta_n\|\| = (2n \log\log n)^{-1/2} \sup_{0 \le t \le 1} |M_{[nt]} - B(nt)| \\ = o(1), \quad \mathbb{P}\text{-}a.s.$$

Define

$$\tilde{\zeta}_n(t) = (2n \log\log n)^{-1/2}[M_k + (nt-k)m_k]$$

for $t \in [\frac{k}{n}, \frac{k+1}{n}),\ k = 0, 1, 2, \cdots, n-1$. Then $\tilde{\zeta}_n \in C([0,1], \mathbb{R}^d)$ and

$$\sup_{t \in [0,1)} |\zeta_n(t) - \tilde{\zeta}_n(t)| = (2n \log\log n)^{-1/2} \max_{0 \le k \le n-1} |m_k|.$$

Next, we give the order estimation of $\max_{0 \le k \le n-1} |m_k|$. Since the fact that

$$\frac{1}{n} \sum_{1}^{n} |m_k|^2 \longrightarrow \mathrm{tr}(\mathfrak{D}), \ \ \mathbb{P}\text{-}a.s.$$

we have $\max_{0 \le k \le n-1} |m_k| = o(n^{1/2})$ a.s. Hence,

$$\|\|\zeta_n - \tilde{\zeta}_n\|\| = \sup_{t \in [0,1)} |\zeta_n(t) - \tilde{\zeta}_n(t)| = o(1), \qquad \mathbb{P}\text{-}a.s.$$

The above discussions immediately yield the following claim:

$(\tilde{\zeta}_n(\cdot), n \ge 1)$ *is almost surely relatively compact and the set of its limit points coincides with* $\sqrt{\mathrm{tr}(\mathfrak{D})}K$.

We now turn to deal with the negligible term R_n in the sense of functional LIL. Firstly, let us recall the concept of the Dunford-Schwarz (DS) operator; see Derriennic and Lin [3]. We call T a DS operator on L^1 of a probability space, if T is a contraction of L^1 such that $\|Tf\|_\infty \le \|f\|_\infty$ for every $f \in L^\infty$. If θ is a measure preserving transformation in a probability space (Ω, Σ, μ), then the operator $Tf = f \circ \theta$ is a DS operator on $L^1(\mu)$. More generally, any Markov transition operator P with an invariant probability measure yields a positive DS operator.

Lemma 1 (Lemma DL, [3])**.**

1. *Let T be a contraction in a Banach space X, and let $0 < \beta < 1$. If* $\sup_n \|\frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k y\| < \infty$, *then $y \in (I-T)^\alpha X$ for every $0 < \alpha < \beta$.*

2. *Let T be a DS operator in $L^1(\mu)$ of a probability space, and fix $1 < p < \infty$, with $q = p/(p-1)$. Let $0 < \alpha < 1$, and $f \in (I-T)^\alpha L^p$. If $\alpha > 1 - \frac{1}{p} = \frac{1}{q}$, then* $\frac{1}{n^{1/p}} \sum_{k=0}^{n-1} T^k f \to 0$ *a.e.*

Note that Lemma DL was originally proved for $d = 1$; however, we can make the shift operating coordinate-wise, hence the original proof of Derriennic and Lin is enough for our case. To apply the above lemma, i.e., the theory of fractional coboundaries named by Derriennic and Lin [3], we need to construct a DS operator. On $\mathcal{X} \times \mathcal{X}$, define

$$f(x_0, x_1) = g(x_0) - H(x_0, x_1),$$

then

$$\begin{aligned} R_n =& S_n - M_n \\ =& \sum_{i=0}^{n-1} [g(X_i) - H(X_i, X_{i+1})] \\ =& \sum_{i=0}^{n-1} f(X_i, X_{i+1}). \end{aligned} \tag{9}$$

Let θ be the shift map on the path space $\mathcal{X}^{\mathbb{N}}$ for the Markov chain which is a contraction on $L^2(\mathbb{P})$. The DS operator is the shift θ. For a sequence $x = (x_i)_{i\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$, define $F(x) = f(x_0, x_1)$, then we have

$$F \in L^2(\mathbb{P}) \quad \text{and} \quad R_n = \sum_{k=0}^{n-1} F \circ \theta^k.$$

From part (2) of Theorem MW, there exists a constant $1/2 < \beta < 1 - \alpha$, such that

$$\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} F \circ \theta^k \right\| < \infty, \tag{10}$$

Since part (1) of Lemma DL and $0 < \alpha < 1/2$, we have $F \in (I - \theta)^\eta L^2(\mathbb{P})$, for some $\eta \in (1/2, 1 - \alpha)$. By part (2) of Lemma DL, we have

$$\frac{1}{n^{1/2}} R_n \to 0, \qquad \mathbb{P}\text{-}a.s.$$

Furthermore, applying an elementary property of real convergent sequences, we immediately get

$$\max_{0\le k\le n} |R_k| = o((2n \log\log n)^{1/2}), \qquad \mathbb{P}\text{-}a.s.$$

Consequently,

$$(2n \log\log n)^{-1/2} \sup_{0\le t\le 1} |R_{[nt]}| \to 0, \qquad \mathbb{P}\text{-}a.s. \tag{11}$$

From the above discussions, we complete the proof of Theorem 1.

3.2. Proof of Theorem 2

Here, we take along the lines of the proof of Theorem 4.8 in Hall and Heyde [5]. Let $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. For any $\mathbb{R}^d$-valued function f, denote $f = (f_1, f_2, \cdots, f_d)^t$. By the definition of K, for any $f \in \sqrt{tr(\mathfrak{D})}K$, we have

$$|f(t)|^2 = \sum_{i=1}^{d}(\int_0^t \dot{f}_i(s)ds)^2 \leq \sum_{i=1}^{d}(\int_0^t \dot{f}_i(s)^2ds)\int_0^t 1ds \leq tr(\mathfrak{D})t \tag{12}$$

where the first inequality is due to the Cauchy-Schwarz's inequality. So, $|f(t)| \leq \sqrt{tr(\mathfrak{D})t}$. It follows that $\sup_{t\in[0,1]}|f(t)| \leq \sqrt{tr(\mathfrak{D})}$. Hence, by Theorem 1

$$\limsup_n \sup_{t\in[0,1]} |\xi_n(t)| \leq \sqrt{tr(\mathfrak{D})}, \qquad \mathbb{P}-a.s. \tag{13}$$

and setting $t = 1$,

$$\limsup_n |S_n|/\sqrt{2n\log\log n} \leq \sqrt{\mathrm{tr}(\mathfrak{D})}, \qquad \mathbb{P}-a.s. \tag{14}$$

On the other hand, we put $f(t) = t\sqrt{\frac{tr(\mathfrak{D})}{d}}\sum_{i=1}^d e_i,\ t \in [0,1]$. Then, $f \in \sqrt{tr(\mathfrak{D})}K$ and so for $\mathbb{P}-a.s.\,\omega$, there exists a sequence $n_k = n_k(\omega)$ such that

$$\xi_{n_k}(\cdot)(\omega) \stackrel{|||\cdot|||}{\longrightarrow} f(\cdot). \tag{15}$$

Particularly, $f(1) = \sqrt{\frac{tr(\mathfrak{D})}{d}}\sum_{i=1}^d e_i$, $|\xi_{n_k}(1)(\omega)| \longrightarrow |f(1)|$. That is to say,

$$|S_{n_k}|/\sqrt{2n_k\log\log n_k} \longrightarrow \sqrt{\mathrm{tr}(\mathfrak{D})}, \qquad \mathbb{P}-a.s. \tag{16}$$

This completes the proof of Theorem 2.

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