Math. Commun. **18**(2013), 97–107

# Study on existence of solutions for some nonlinear functional-integral equations with applications

Deepmala[1,*] and Hemant Kumar Pathak[1]

[1] *School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur (C.G.), 492 010, India*

Received April 24, 2012; accepted December 11, 2012

**Abstract.** In this paper, using the technique of measure of noncompactness in Banach algebra, we prove an existence theorem for a nonlinear integral equation which contains as particular cases a lot of integral and functional-integral equations considered in nonlinear analysis and its applications. Our claim is also illustrated with the applications to some nonlinear functional-integral equation for proving the existence results.

**AMS subject classifications**: 47H10

**Key words**: Banach algebra, fixed point, functional-integral equation, measure of noncompactness

## 1. Introduction

It is well-known that the differential and integral equations that arise in many physical problems are mostly nonlinear and fixed point theory provides a powerful tool for obtaining the solutions of such equations which otherwise are difficult to solve by other ordinary methods. In this paper, we consider solvability of a certain functional-integral equation which contains as particular cases a lot of integral and functional-integral equations, which are applicable in several real world problems of engineering, mechanics, physics, economics and so on (see [6, 7]). The authors consider the following functional-integral equation:

$$\begin{aligned} x(t) = &\left( u(t, x(t)) + f\left( t, \int_0^t p(t, s, x(s)) ds, x(\alpha(t)) \right) \right) \\ &\times g\left( t, \int_0^a q(t, s, x(s)) ds, x(\beta(t)) \right), \quad \text{for } t \in [0, a]. \end{aligned} \tag{1}$$

The main tool used in our result is a fixed point theorem which satisfies the Darbo condition with respect to a measure of noncompactness in the Banach algebra of continuous functions in the interval $[0, a]$. In Section 2, we introduce some preliminaries and use them to obtain our main results in Section 3. In the last section, we provide some examples that verify the application of this kind of nonlinear functional-integral equations.

*Corresponding author. *Email addresses:* `dmrai23@gmail.com` (Deepmala), `hkpathak05@gmail.com` (H. K. Pathak)

 

## 2. Preliminaries

In this section, we recall basic results which we will need further on. Assume that $E$ is a real Banach space with the norm $\|.\|$ and zero element $\theta$. Denote by $B(x,r)$ the closed ball centered at $x$ with radius $r$ and the symbol $B_r$ stands for the ball $B(\theta, r)$.
For $X$ being a nonempty subset of $E$, by $\overline{X}$ and Conv$X$ we denote the closure and the convex closure of $X$, respectively. We denote the standard algebraic operations on sets by the symbols $\lambda\ X$ and $X+Y$. Finally, the family of all nonempty and bounded subsets of $E$ is denoted by $m_E$ and its subfamily consisting of all relatively compact sets is denoted by $n_E$. We accept the following definition of the measure of noncompactness [1].

**Definition 1.** *Let $X \in m_E$ and*

$$\mu(X) = \inf\left\{\delta > 0 : X = \bigcup_{i=1}^{m} X_i \text{ with } diam X_i \leq \delta,\ i = 1, 2, ..., m\right\},$$

*where we denote*

$$diam X = \sup\{\|x-y\| : x, y \in X\}.$$

*Clearly, $0 \leq \mu(X) < \infty$. $\mu(X)$ is called the Kuratowski measure of noncompactness.*

**Theorem 1.** *Let $X, Y \in m_E$ and $\lambda \in \mathbb{R}$. Then*

*(i)* $\mu(X) = 0$ *if and only if* $X \in n_E$;

*(ii)* $X \subseteq Y$ *implies* $\mu(X) \leq \mu(Y)$;

*(iii)* $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$;

*(iv)* $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$;

*(v)* $\mu(\lambda X) = |\lambda|\mu(X)$, *where* $\lambda X = \{\lambda x : x \in X\}$;

*(vi)* $\mu(X+Y) \leq \mu(X) + \mu(Y)$, *where* $X + Y = \{x + y : x \in X, y \in Y\}$;

*(vii)* $|\mu(X) - \mu(Y)| \leq 2d_h(X,Y)$, *where $d_h(X,Y)$ denotes the Hausdorff distance of $X$ and $Y$, i.e.*

$$d_h(X,Y) = \max\left\{\sup_{y\in Y} d(y,X), \sup_{x\in X} d(x,Y)\right\}.$$

*Here $d(a, A)$ denotes the distance of an element $a$ to a set $A$.*

Further on, every function $\mu : m_E \to [0, \infty)$, satisfying conditions $(i) - (vi)$ of Theorem 1, will be called a regular measure of noncompactness in the Banach space [1].
Now let us assume that $\Omega$ is a nonempty subset of a Banach space $E$ and $S : \Omega \to E$ is a continuous operator transforming bounded subsets of $\Omega$ into bounded ones. Moreover, let $\mu$ be a regular measure of noncompactness in $E$.

**Definition 2** (See [1]). *We say that $S$ satisfies the Darbo condition with a constant $k$ with respect to measure $\mu$ provided*

$$\mu(SX) \le k\mu(X),$$

*for each $X \in m_E$ such that $X \subset \Omega$.*
*If $k < 1$, then $S$ is called a contraction with respect to $\mu$.*

In the sequel, we will work in the space $C[a,b]$ consisting of all real functions defined and continuous on the interval $[a,b]$. The space $C[a,b]$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [a,b]\}.$$

Obviously, the space $C[a,b]$ has also the structure of Banach algebra.
In our considerations, we will use a regular measure of noncompactness defined in [2] (cf also [1]).

In order to recall the definitions of that measure let us fix a set $X \in m_{C[a,b]}$. For $x \in X$ and for a given $\epsilon > 0$ denote by $\omega(x,\epsilon)$ the modulus of continuity of $x$, i.e.,

$$\omega(x,\epsilon) = \sup\{|x(t) - x(s)| : t, s \in [a,b], |t-s| \le \epsilon\}.$$

Further, put

$$\omega(X,\epsilon) = \sup\{\omega(x,\epsilon) : x \in X\},$$
$$\omega_0(X) = \lim_{\epsilon \to 0} w(X,\epsilon).$$

It may be shown [2] that $\omega_0(X)$ is a regular measure of noncompactness in $C[a,b]$.
For our purpose we will need the following theorem [2].

**Theorem 2.** *Assume that $\Omega$ is a nonempty, bounded, convex and closed subset of $C[a,b]$ and the operators $P$ and $T$ transform continuously the set $\Omega$ into $C[a,b]$ in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, assume that the operator $S = P.T$ transform $\Omega$ into itself. If the operators $P$ and $T$ satisfy on the set $\Omega$ the Darbo condition with the constant $k_1$ and $k_2$, respectively, then the operator $S$ satisfies the Darbo condition on $\Omega$ with the constant*

$$\|P(\Omega)\|\, k_2 + \|T(\Omega)\|\, k_1.$$

*Particularly, if $\|P(\Omega)\|\, k_2 + \|T(\Omega)\|\, k_1 < 1$, then $S$ is a contraction with respect to the measure $\omega_0$ and it has at least one fixed point in the set $\Omega$.*

## 3. Main results

In this section, we will study solvability of the nonlinear functional-integral equation(1) for $x \in C[a,b]$.
We formulate the assumptions under which equation (1) will be investigated. Namely, we assume the following hypothesis.

$(H_1)$ The functions $u : [0, a] \times \mathbb{R} \to \mathbb{R}$, $f, g : [0, a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $l, m \geq 0$ such that

$$|u(t,0)| \leq l,$$
$$|f(t,0,x)| \leq m,$$
$$|g(t,0,x)| \leq m.$$

$(H_2)$ There exist the continuous functions $a_1, a_2, a_3, a_4, a_5 : [0, a] \to [0, a]$ such that

$$|u(t,x_1) - u(t,x_2)| \leq a_1(t)|x_1 - x_2|,$$
$$|f(t,y_1,x_1) - f(t,y_2,x_2)| \leq a_2(t)|y_1 - y_2| + a_3(t)|x_1 - x_2|,$$
$$|g(t,y_1,x_1) - g(t,y_2,x_2)| \leq a_4(t)|y_1 - y_2| + a_5(t)|x_1 - x_2|,$$

for all $t \in [0, a]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$(H_3)$ The functions $p = p(t, s, x)$ and $q = q(t, s, x)$ act continuously from the set $[0, a] \times [0, a] \times \mathbb{R}$ into $\mathbb{R}$ and the functions $\alpha(t)$ and $\beta(t)$ transform continuously the interval $[0, a]$ into itself.

$(H_4)$ There exists a non negative constant $k$ such that

$$\max\{a_1(t), a_2(t), a_3(t), a_4(t), a_5(t)\} \leq k, \text{ for } t \in [0, a].$$

$(H_5)$ (sub-linearity condition) There exist constants $\alpha$ and $\beta$ such that

$$|p(t,s,x)| \leq c_1 + c_2|x|,$$
$$|q(t,s,x)| \leq c_1 + c_2|x|.$$

$(H_6)$ $4\gamma\eta < 1$ and $ac_2 \geq 1$, for $\gamma = k + kac_2$ and $\eta = kac_1 + l + m$.

The following result is obtained by using the above hypothesis.

**Theorem 3.** *Under assumptions $(H_1) - (H_6)$ equation (1) has at least one solution in the Banach algebra $C = C[0, a]$.*

**Proof.** Let us consider the operators $F$ and $G$ defined on the Banach algebra $C$ by the formula

$$(Fx)(t) = u(t, x(t)) + f\left(t, \int_0^t p(t, s, x(s))ds, x(\alpha(t))\right),$$
$$(Gx)(t) = g\left(t, \int_0^a q(t, s, x(s))ds, x(\beta(t))\right),$$

for $t \in [0, a]$.
From assumptions $(H_1)$ and $(H_3)$, it follows that $F$ and $G$ transform the algebra $C$ into itself.
Further, let us define the operator $T$ on the algebra $C$ by putting

$$Tx = (Fx)(Gx).$$

Obviously, $T$ transforms $C$ into itself.
Now, let us fix $x \in C$. Then, using our assumptions for $t \in [0,a]$, we get

$$\begin{aligned}
|(Tx)(t)| &= |(Fx)(t)|.|(Gx)(t)| \\
&= \left| u(t,x(t)) + f\left(t, \int_0^t p(t,s,x(s))ds, x(\alpha(t))\right) \right| . \\
&\quad \left| g\left(t, \int_0^a q(t,s,x(s))ds, x(\beta(t))\right) \right| \\
&\le \Bigg( |u(t,x(t)) - u(t,0)| + |u(t,0)| \\
&\quad + \left| f\left(t, \int_0^t p(t,s,x(s))ds, x(\alpha(t))\right) - f(t,0,x(\alpha(t))) \right| \\
&\quad + |f(t,0,x(\alpha(t)))| \Bigg) \\
&\quad \Bigg( \left| g\left(t, \int_0^a q(t,s,x(s))ds, x(\beta(t))\right) - g(t,0,x(\beta(t))) \right| \\
&\qquad + |g(t,0,x(\beta(t)))| \Bigg) \\
&\le \left( a_1(t)|x(t)| + l + a_2(t) \left| \int_0^t p(t,s,x)ds \right| + m \right) \\
&\quad \left( a_4(t) \left| \int_0^a q(t,s,x)ds \right| + m \right) \\
&\le (k|x(t)| + l + ka(c_1 + c_2|x(t)|) + m)(ka(c_1 + c_2|x(t)|) + m) \\
&\le ((k + kac_2)\|x\| + kac_1 + l + m)^2.
\end{aligned}$$

Let $\gamma = k + kac_2$ and $\eta = kac_1 + l + m$, then from the above estimate, it follows easily that

$$\|Fx\| \le \gamma \|x\| + \eta, \tag{2}$$
$$\|Gx\| \le \gamma \|x\| + \eta, \tag{3}$$
$$\|Tx\| \le (\gamma \|x\| + \eta)^2. \tag{4}$$

for $x \in C[0,a]$.
From (4), we deduce that the operator $T$ maps the ball $B_r \subset C[0,a]$ into itself for $r_1 \le r \le r_2$, where

$$r_1 = \frac{(1 - 2\gamma\eta) - \sqrt{1 - 4\gamma\eta}}{2\gamma^2}.$$

$$r_2 = \frac{(1 - 2\gamma\eta) + \sqrt{1 - 4\gamma\eta}}{2\gamma^2}.$$

Also, from estimate (2) and (3), we obtain

$$\|FB_r\| \le \gamma\ r + \eta, \tag{5}$$
$$\|GB_r\| \le \gamma\ r + \eta. \tag{6}$$

Next, we show that the operator $F$ is continuous on the ball $B_r$. To do this, fix $\epsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x-y\| \leq \epsilon$. Then for $t \in [0,a]$, we get

$$
\begin{aligned}
|(Fx)(t)-(Fy)(t)| &= \bigg| u(t,x(t)) + f\left(t, \int_0^t p(t,s,x(s))ds, x(\alpha(t))\right) \\
&\quad -u(t,y(t)) - f\left(t, \int_0^t p(t,s,y(s))ds, y(\alpha(t))\right)\bigg| \\
&\leq a_1(t)|x(t)-y(t)| + \bigg| f\left(t, \int_0^t p(t,s,x(s))ds, x(\alpha(t))\right) \\
&\quad -f\left(t,\int_0^t p(t,s,x(s))ds, y(\alpha(t))\right) + f\left(t,\int_0^t p(t,s,x(s))ds, y(\alpha(t))\right) \\
&\quad -f\left(t, \int_0^t p(t,s,y(s))ds, y(\alpha(t))\right)\bigg| \\
&\leq a_1(t)|x(t)-y(t)| + a_3(t)|x(\alpha(t))-y(\alpha(t))| \\
&\quad +a_2(t)\left|\int_0^t p(t,s,x(s))ds - \int_0^t p(t,s,x(s))ds\right| \\
&\leq 2k\,\|x-y\| + k\ a\ \omega(p,\epsilon),
\end{aligned}
$$

where $\omega(p,\epsilon) = \sup\{|p(t,s,x)-p(t,s,y)| : t,s \in [0,a]; x,y \in [-r,r]; \|x-y\| \leq \epsilon\}$. Since we know that $p = p(t,s,x)$ is uniformly continuous on the bounded subset $[0,a]\times[0,a]\times[-r,r]$, we infer that $\omega(p,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, the operator $F$ is continuous on $B_r$. Similarly, one can easily show that $G$ is continuous on $B_r$ and consequently we deduce that $T$ is continuous on $B_r$.

Now, we show that the operators $F$ and $G$ satisfy the Darbo condition with respect to the measure $\omega_0$ as defined in Section 2, in the ball $B_r$. Take a nonempty subset $X$ of $B_r$ and $x \in X$, then for a fixed $\epsilon > 0$ and $t_1, t_2 \in [0,a]$ such that without loss of generality we may assume that $t_1 \leq t_2$ and $t_2 - t_1 \leq \epsilon$, we obtain

$$
\begin{aligned}
|(Fx)(t_2)-(Fx)(t_1)| &= \bigg| u(t_2,x(t_2)) + f\left(t_2, \int_0^{t_2} p(t_2,s,x(s))ds, x(\alpha(t_2))\right) \\
&\quad -u(t_1,x(t_1)) + f\left(t_1, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_1))\right)\bigg| \\
&\leq |u(t_2,x(t_2))-u(t_2,x(t_1))| + |u(t_2,x(t_1))-u(t_1,x(t_1))| \\
&\quad +\bigg| f\left(t_2, \int_0^{t_2} p(t_2,s,x(s))ds, x(\alpha(t_2))\right) \\
&\quad -f\left(t_2, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_2))\right)\bigg| \\
&\quad +\bigg| f\left(t_2, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_2))\right) \\
&\quad -f\left(t_1, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_1))\right)\bigg|
\end{aligned}
$$

$$\begin{aligned}
&\leq a_1(t)|x(t_2)-x(t_1)| + |u(t_2,x(t_1)) - u(t_1,x(t_1))| \\
&\quad + a_2(t)\left|\int_0^{t_2} p(t_2,s,x(s))ds - \int_0^{t_1} p(t_1,s,x(s))ds\right| \\
&\quad + \left| f\left(t_2, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_2))\right)\right. \\
&\quad \left. - f\left(t_1, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_2))\right)\right| \\
&\quad + \left| f\left(t_1, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_2))\right)\right. \\
&\quad \left. - f\left(t_1, \int_0^{t_1} p(t_1,s,x(s))ds, x(\alpha(t_1))\right)\right| \qquad (7)
\end{aligned}$$

For simplicity, we define the following quantities

$$\begin{aligned}
\omega_u(\epsilon,.) &= \sup\{|u(t,x)-u(t',x)| : t,t' \in [0,a]; |t-t'| \leq \epsilon; x \in [-r,r]\}, \\
\omega_p(\epsilon,.,.) &= \sup\{|p(t,s,x)-p(t',s,x)| : t,t' \in [0,a]; |t-t'| \leq \epsilon; x \in [-r,r]\}, \\
\omega_f(\epsilon,.,.) &= \sup\{|f(t,y,x)-f(t',y,x)| : t,t' \in [0,a]; |t-t'| \leq \epsilon; \\
&\quad x \in [-r,r], y \in [-k'a,k'a]\}, \\
k' &= \sup\{|p(t,s,x)| : t,s \in [0,a]; x \in [-r,r]\}.
\end{aligned}$$

Then using relation (7) we obtain the following

$$\begin{aligned}
|(Fx)(t_2)-(Fx)(t_1)| &\leq 2k|x(\alpha(t_2))-x(\alpha(t_1))| + \omega_u(\epsilon,.) \\
&\quad + k[\omega_p(\epsilon,.,.).a + k'\epsilon] + \omega_f(\epsilon,.,.).
\end{aligned}$$

This yields the following estimate

$$\omega(Fx,\epsilon) \leq 2k\omega(x,\omega(\alpha,\epsilon)) + \omega_u(\epsilon,.) + k[\omega_p(\epsilon,.,.).a + k'\epsilon] + \omega_f(\epsilon,.,.).$$

In view of our assumption we infer that the functions $u = u(t,x)$ and $f = f(t,y,x)$ are uniformly continuous on $[0,a]\times\mathbb{R}$ and $[0,a]\times\mathbb{R}\times\mathbb{R}$.
Hence, we deduce that $\omega_u(\epsilon,.) \to 0, \omega_p(\epsilon,.,.) \to 0, \omega_f(\epsilon,.,.) \to 0$ as $\epsilon \to 0$. Thus,

$$\omega_0(FX) \leq 2k\omega_0(X). \qquad (8)$$

Similarly, we can show that

$$\omega_0(GX) \leq 2k\omega_0(X). \qquad (9)$$

Finally, from (5), (6), (8) and (9) and Theorem 2, we infer that the operator $T$ satisfies the Darbo condition on $B_r$ with respect to the measure $\omega_0$ with constant $(\gamma r+\eta)\ 2k + (\gamma r+\eta)\ 2k$. Also, we have

$$\begin{aligned}
(\gamma r+\eta)\ 2k + (\gamma r+\eta)\ 2k &= 4k(\gamma r+\eta) \\
&= 4k(\gamma r_1+\eta) \\
&= 4k\left(\gamma\left(\frac{(1-2\gamma\eta)-\sqrt{1-4\gamma\eta}}{2\gamma^2}\right)+\eta\right) \\
&= 4k\left(\frac{1-\sqrt{1-4\gamma\eta}}{2\gamma}\right) < 1.
\end{aligned}$$

Hence, the operator $T$ is a contraction on $B_r$ with respect to $\omega_0$. Thus, by applying Theorem 2 we get that $T$ has at least one fixed point in $B_r$. Consequently, the nonlinear functional-integral equation (1) has at least one solution in $B_r$. This completes the proof. □

## 4. Applications

In this section, we give some examples of classical integral and functional equations considered in the applied problems of nonlinear analysis which are particular cases of equation (1).

- If $g(t, y, x) = 1$, then equation (1) is in the following form which was studied in [12].

$$x(t) = u(t, x(t)) + f\left(t, \int_0^t p(t, s, x(s))ds, x(\alpha(t))\right). \tag{10}$$

- For $u(t, x) = 0$, we obtain the following nonlinear functional-integral equation studied in [4, 11].

$$x(t) = f\left(t, \int_0^t p(t, s, x(s))ds, x(\alpha(t))\right) \times g\left(t, \int_0^a q(t, s, x(s))ds, x(\beta(t))\right). \tag{11}$$

- $f(t, y, x) = y$ and $g(t, y, x) = 1$, then we get the following functional-integral equation studied in [3].

$$x(t) = u(t, x(t)) + \int_0^t p(t, s, x(s))ds. \tag{12}$$

- If $u(t, x) = 0$, $g(t, y, x) = 1$ and $f(t, y, x) = u(t, x)y$, then equation (1) has the following form as in the paper [13].

$$x(t) = u(t, x(t)) \int_0^t p(t, s, x(s))ds. \tag{13}$$

- If $u(t, x) = 0$, $g(t, y, x) = 1$ and $f(t, y, x) = a(t) + y$, then we get the following well known nonlinear Volterra integral equation

$$x(t) = a(t) + \int_0^t p(t, s, x(s))ds. \tag{14}$$

- If $u(t, x) = 0$, $f(t, y, x) = 1$ and $g(t, y, x) = b(t) + y$, then we obtain Urysohn integral equation

$$x(t) = b(t) + \int_0^a q(t, s, x(s))ds. \tag{15}$$

- If $u(t,x)=0$, $f(t,y,x)=a(t)+y$ and $g(t,y,x)=y$, then equation (1) has the form examined in the paper [8].

$$x(t)=a(t)\int_0^a q(t,s,x(s))ds+\left(\int_0^t p(t,s,x(s))ds\right)\left(\int_0^a q(t,s,x(s))ds\right). \tag{16}$$

- Moreover, if $u(t,x)=0$, $f(t,y,x)=1$, $g(t,y,x)=1+xy$ and $q(t,s,y)=\frac{t}{t+s}\varphi(s)x$, $\beta(t)=t$, then equation (1) has the form

$$x(t)=1+x(t)\int_0^a \frac{t}{t+s}\varphi(s)x(s)ds. \tag{17}$$

  The above equation is the famous quadratic integral equation of Chandrasekhar type [5] which is applied in the theories of radiative transfer, neutron transport and kinetic energy of gases (see [5, 9, 10]).

On the other hand, equation (1) also covers the well known functional equation of the first order having the form

$$x(t)=f_1(t,x(\alpha(t))),$$

for this it is sufficient to put $g(t,y,x)=1$, $u(t,x)=0$ and $f(t,y,x)=f_1(t,x)$.

Now, we present an example of a functional-integral equation and consequently, see the existence of its solutions by using Theorem 3.

**Example 1.** *Consider the following nonlinear functional integral equation:*

$$x(t)=\left(\frac{1}{5}\sin\left(\frac{t}{4}\right)+\frac{1}{4}\int_0^t ts\cos(x(s))ds\right)\cdot\left(\frac{1}{3}\int_0^1 t\sin\left(\frac{sx(s)}{1+x(s)}\right)ds\right), \tag{18}$$

*for* $t\in[0,1]$.
*Observe that equation (18) is a special case of equation (1). Let us take* $u:[0,1]\times\mathbb{R}\to\mathbb{R}$, $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $p,q:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ *and comparing (18) with (1), we get*

$$u(t,x(t))=\frac{1}{5}\sin(\frac{t}{4}),f(t,y_1,x)=\frac{1}{4}y_1,g(t,y_2,x)=\frac{t}{3}y_2,$$

$$p(t,s,x)=ts\cos(x(s)),\ \ q(t,s,x)=\sin\left(\frac{sx(s)}{1+x(s)}\right).$$

*It is easy to prove that these functions are continuous and satisfy the hypothesis* $(H_2)$ *with* $a_1=a_3=a_5=0, a_2=\frac{1}{4}, a_4=\frac{1}{3}$. *In this case,* $k=\max\{0,\frac{1}{3},\frac{1}{4}\}=\frac{1}{3}$. *Moreover,*

$$|u(t,0)|\leq\frac{1}{5},\ \ |f(t,0,x)|=0,\ \ |g(t,0,x)|=0,\ \ |p(t,s,x)|\leq 0+1|x|, |q(t,s,x)|\leq 0+1|x|.$$

*It is observed that* $c_1 = 0, c_2 = 1$ *and* $l = \frac{1}{5}, m = 0,\ a = 1$ *and* $ac_2 = 1 \geq 1$ . *Also,*

$$4\gamma\eta = 4(k + kac_2)(kac_1 + l + m) = 4\left(\frac{2}{3}\right).\left(\frac{1}{5}\right) = \frac{8}{15} < 1.$$

*Hence, all the hypotheses from* $(H_1)-(H_6)$ *are satisfied. Applying the result obtained in Theorem 3, we deduce that equation (18) has at least one solution in Banach algebra* $C[0,1]$.

**Example 2.** *Consider the following functional integral equation:*

$$x(t) = \left(\frac{1}{7}\cos\left(\frac{e^{-t}}{1+t}\right)+\frac{1}{2}\int_0^t\left(\frac{t}{1+t+s}\right)\sin(x(s))ds\right).\left(\frac{1}{3}\int_0^1\frac{x(s)}{2+x(s)}ds\right), \quad (19)$$

*where* $t \in [0,1]$.
*Observe that this equation is a special case of equation (1). In this example one can easily verify that the assumptions of our existence Theorem 3 are satisfied, i.e. equation (19) has at least one solution in Banach algebra* $C[0,1]$.

## Acknowledgment

The authors would like to thank the referees for their helpful suggestions. The research of the second author (corresponding author) is supported by the Department of Science and Technology, Government of India under INSPIRE Fellowship Program No. DST/INSPIRE Fellowship/2010/[182].

## References

[1] J. BANAS, K. GOEBEL, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.
[2] J. BANAS, M. LECKO, *Fixed points of the product of operators in Banach algebra*, Panamer. Math. J. **12**(2002), 101–109.
[3] J. BANAS, B. RZEPKA, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. **284**(2003), 165–173.
[4] J. BANAS, K. SADARANGANI, *Solutions of some functionalintegral equations in Banach algebra*, Math. Comput. Modelling **38**(2003), 245–250.
[5] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford Univ. Press, London, 1950.
[6] C. CORDUNEANU, *Integral Equations and Applications*, Cambridge University Press, New York, 1990.
[7] K. DEIMLING, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
[8] B. C. DHAGE, *On $\alpha$-condensing mappings in Banach algebras*, The Math. Student **63**(1994), 146–152.
[9] S. HU, M. KHAVANIN, W. ZHUANG, *Integral equations arising in the kinetic theory of gases*, Appl. Anal. **34**(1989), 261–266.
[10] C. T. KELLY, *Approximation of solutions of some quadratic integral equations in transport theory*, J. Integral Eq. **4**(1982), 221–237.
[11] K. MALEKNEJAD, R. MOLLAPOURASL, K. NOURI, *Study on existence of solutions for some nonlinear functionalintegral equations*, Nonlinear Anal. **69**(2008), 2582–2588.

[12] K. MALEKNEJAD, K. NOURI, R. MOLLAPOURASL, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simulat. **14**(2009), 2559–2564.

[13] K. MALEKNEJAD, K. NOURI, R. MOLLAPOURASL, *Investigation on the existence of solutions for some nonlinear functional-integral equations*, Nonlinear Anal. **71**(2009), 1575–1578.