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Π^0_1-ordinal analysis beyond first-order arithmetic

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Abstract. In this paper we give an overview of an essential part of a Π^0_1 ordinal analysis of Peano Arithmetic (PA) as presented by Beklemishev ([2]). This analysis is mainly performed within the polymodal provability logic GLP_ω.
We reflect on ways of extending this analysis beyond PA. A main difficulty in this is to find proper generalizations of the so-called Reduction Property. The Reduction Property relates reflection principles to reflection rules.
In this paper we prove a result that simplifies the reflection rules. Moreover, we see that for an ordinal analysis the full Reduction Property is not needed. This latter observation is also seen to open up ways for applications of ordinal analysis relative to some strong base theory.

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1. Introduction

Primitive Recursive Arithmetic (PRA) is a rather weak formal theory about natural numbers which is among various philosophers, logicians and mathematicians held to be a good candidate for the concept of finitism (see e.g., [18]). The concept of finitism tries to capture those mathematical truths and that part of mathematical reasoning which is true beyond doubt and which does not use strong assumptions on infinite mathematical entities.

In his seminal paper from 1936 ([12]) Gentzen showed that PRA together with some clearly non-finitist notion of transfinite induction for easy formulas along a rather small ordinal could prove the consistency of Peano Arithmetic (PA).

This result can be seen as a partial realization of Hilbert's programme where finitist theories are to prove the consistency of strong mathematical theories. Of course, since Gödel's incompleteness results we know that this program is not viable but Gentzen's consistency proof seems to clearly isolate the non-finitist part needed for such a consistency proof.

Since Gentzen's consistency proof, the scientific community has tried to calibrate the proof-strength of various theories other than PA. The amount of transfinite induction needed in these consistency proofs is referred to as the proof-theoretic

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ordinal of a theory. There are various ways of defining and computing these ordinals and for most natural theories these methods all yield the same ordinals.

Among these methods, the more novel one was introduced by Beklemishev [1, 2] and it is based on modal provability logics. The corresponding ordinals are referred to as the Π^0_1 ordinals. In this paper we shall sketch the method for computing these Π^0_1 ordinals. So far, the method has only been applied successfully to theories as PA and its kin. In this paper we reflect on ways of extending this analysis beyond PA.

A main difficulty in this is to find proper generalizations of the so-called Reduction Property. The Reduction Property relates reflection principles to reflection rules. In this paper we prove a result that simplifies reflection rules. Moreover, we see that the full Reduction Property is not needed for an ordinal analysis. This latter observation is also seen to open up ways for applications of ordinal analysis relative to some strong base theory.

Before we can start looking into the ordinal analysis, we must first introduce some basic knowledge concerning arithmetic and provability logics.

2. Prerequisites

All results in this section are given without proofs. For further background the reader is referred to standard textbooks like [6] or [13].

2.1. Arithmetic

By the language of arithmetic we understand in this paper the language based on symbols $\{0, S, +, \times, \exp, \leq, =\}$, where exp denotes the function $x \mapsto 2^x$.

Formulas of arithmetic are stratified in complexity classes as usual. Thus, Δ^0_0 formulas are first-order formulas where all quantifiers refer to numbers and are bounded by some term t as in $\forall\, x{\leq}t$, where of course $x \notin t$.

We define $\Sigma^0_0 := \Pi^0_0 := \Delta^0_0$. If $\varphi(\vec{x}, \vec{y}) \in \Sigma^0_n$, then $\forall \vec{x}\ \varphi(\vec{x}, \vec{y}) \in \Pi^0_{n+1}$ and likewise, if $\varphi(\vec{x}, \vec{y}) \in \Pi^0_n$, then $\exists \vec{x}\ \varphi(\vec{x}, \vec{y}) \in \Sigma^0_{n+1}$.

Similarly we define the hierarchies Π^n_m, where now the number of nth-order quantifiers is counted although in this paper we shall at most need second order quantifiers.

By EA we denote the arithmetic theory of *Elementary Arithmetic*. This theory is formulated in the language of arithmetic. Apart from the defining axioms for the symbols in the language, EA has an induction axiom I_φ for each Δ^0_0 formula $\varphi(\vec{x}, y)$ (that may contain exp):

$$I_\varphi(\vec{x}): \quad \varphi(\vec{x}, 0) \wedge \forall\, y\ (\varphi(\vec{x}, y) \to \varphi(\vec{x}, y+1)) \ \to\ \forall y \varphi(\vec{x}, y).$$

By EA^+ we denote EA plus the axiom that states that super-exponentiation – the function that maps x to the x times iteration of exp– is a total function.

By $\mathrm{I}\Sigma_n$ we denote the theory that is as EA except that it now has induction axioms I_φ for all formulas $\varphi(\vec{x}) \in \Sigma^0_n$. The theory of PA is the union of all $\mathrm{I}\Sigma_n$ in that it has induction axioms for all arithmetic formulas.

2.2. Transfinite induction

Greek letters will often denote ordinals and as usual we denote by ε_0 the supremum of $\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. Apart from considering induction along natural numbers we shall consider induction along transfinite orderings too. If $\langle \Gamma, \prec \rangle$ is a natural arithmetical representation in EA of some ordinal, by $\mathsf{TI}[X, \Gamma]$ we denote the collection of transfinite induction axioms for all formulas in X:

$$\forall y \, \big(\forall \, y' {\prec} y \; \varphi(\vec{x}, y') \to \varphi(\vec{x}, y)\big) \;\to\; \forall y \; \varphi(\vec{x}, y) \qquad \text{with } \varphi(\vec{x}, y) \in X.$$

2.3. Formalized metamathematics

Throughout this paper we shall use representations in arithmetic of various metamathematical notions. In particular, we fix some Gödel numbering to represent formulas and other syntactical objects in arithmetic.

Moreover, we assume that we can represent r.e. theories in a suitable way so that we can speak of "the formula φ is provable in the theory T" whose formalization we shall denote by $\Box_T \varphi$. Dually, we shall use the notion of "the formula φ is consistent with the theory T" which is denoted by $\mathsf{Con}_T(\varphi)$ or $\Diamond_T \varphi$.

If we write $\Box_T \varphi(\dot{x})$, by that we denote a formula whose free variable is x, and so that provably for every x, the formula $\Box_T \varphi(\dot{x})$ is equivalent to $\Box_T \varphi(\overline{x})$. Here $\overline{x}$ denotes the numeral of x, that is,

$$\overline{x} = \overbrace{S \ldots S}^{x \text{ times}} 0.$$

3. Provability logics

The logics GLP_Λ provide provability logics for a series of provability predicates/modalities $[\alpha]$ of increasing strength.

Definition 1. *Let Λ be an ordinal. By GLP_Λ we denote the poly-modal propositional logic that has for each $\alpha < \Lambda$ a modality $[\alpha]$ (that syntactically binds as the negation symbol). The axioms of GLP_Λ are all propositional logical tautologies in this signature together with instantiations of the following schemes:*

$$\begin{aligned}
[\alpha](A \to B) &\to ([\alpha]A \to [\alpha]B) && \forall \alpha < \Lambda; \\
[\alpha]([\alpha]A \to A) &\to [\alpha]A && \forall \alpha < \Lambda; \\
[\alpha]A &\to [\beta]A && \forall \alpha \leq \beta < \Lambda; \\
\langle \alpha \rangle A &\to [\beta]\langle \alpha \rangle A && \forall \alpha < \beta < \Lambda.
\end{aligned}$$

As always, we have that $\langle \alpha \rangle A := \neg[\alpha]\neg A$. The rules are Modus Ponens and a Necessitation rule for each modality α below Λ, that is, $\frac{A}{[\alpha]A}$.

By GLP we shall denote class-size logic which is the "union" of GLP_Λ over all $\Lambda \in \mathsf{On}$. The closed fragment of GLP_Λ is the set of its theorems that do not contain any propositional variables and it is denoted by GLP^0_Λ. It turns out that GLP^0_Λ is already a very rich structure that is strong enough to perform major parts of our ordinal analysis. Some privileged inhabitants of GLP^0_Λ are the so-called *worms*. They are just iterated consistency statements in GLP^0_Λ.

Definition 2 (S^Λ)**.** *$\top \in S^\Lambda$, and if both $A \in S^\Lambda$ and $\beta < \Lambda$, then $\langle\beta\rangle A \in S^\Lambda$.*

We can define an order $<_0$ on S^Λ by $A <_0 B \ :\Leftrightarrow\ \mathsf{GLP}^0_\Lambda \vdash B \to \langle 0\rangle A$. It is known ([4, 5]) that this ordering makes S^Λ into a well-order.

3.1. The Reduction Property

Japaridze ([15]) showed GLP_ω to be arithmetically sound and complete if we interpret $[n]$ as "provable by n applications of the ω-rule". Ignatiev then showed in [14] that this completeness result actually holds for a wide range of arithmetical readings of $[n]$. In particular, we still have completeness when reading $[n]$ as a natural formalization of "provable in EA together with all true Π^0_n sentences".

For the remainder of the section, let $[n]$ refer to this latter reading. The advantage of this reading is that certain worms can be easily linked to reflection principles and fragments of arithmetic:

Lemma 1. $\mathrm{EA} + \langle n+2\rangle\top \ \equiv\ \mathrm{EA} + \mathsf{RFN}_{\Sigma_{n+2}}(\mathrm{EA}) \ \equiv\ \mathrm{I}\Sigma_{n+1}$.

Proof. We shall refrain from distinguishing a modal formula from its arithmetical interpretation if the context allows us to. Thus, in this statement, $\langle n+2\rangle\top$ clearly refers to the formalized statement that EA together with all true Π^0_{n+2}-formulas is consistent.

By $\mathsf{RFN}_{\Sigma_{n+1}}(\mathrm{EA})$ we denote the set of axioms $\{[0]_{\mathrm{EA}}\sigma(\dot{x}) \to \sigma(x) \mid \sigma \in \Sigma_{n+1}\}$.

The $\mathrm{EA} + \langle n+2\rangle\top \equiv \mathrm{EA} + \mathsf{RFN}_{\Sigma_{n+2}}(\mathrm{EA})$ equivalence is actually rather easy and can be found in [3]. The remaining equivalence $\mathrm{EA} + \mathsf{RFN}_{\Sigma_{n+2}}(\mathrm{EA}) \equiv \mathrm{I}\Sigma_{n+1}$ is a classical result by Leivant [16]. □

We can write $\mathsf{RFN}_{\Sigma_n}(EA)$ also as $\pi(x) \to \Diamond_{\mathrm{EA}}\pi(\dot{x})$ for $\pi(x) \in \Pi^0_n$. This in turn can be studied as a rule rather than an implication: $\frac{\pi(x)}{\Diamond_{\mathrm{EA}}\pi(\dot{x})}$. In this rule we can vary both the complexity class to which $\pi(x)$ belongs and the notion of provability used (here just $\Diamond_{\mathrm{EA}}$ which is $\langle 0\rangle_{\mathrm{EA}}$) giving rise to a scala of different rules. In [2, 3], these rules are introduced and studied.

Definition 3. *The Reflection Rule Π^0_m–$\mathsf{RR}^n(\mathcal{U})$ is defined as $\frac{\pi(x)}{\langle n\rangle_{\mathcal{U}}\pi(\dot{x})}$, where $\pi \in \Pi^0_m$.*

The following theorem is called the Reduction Property. Its proof can be found in either one of [2, 3].

Theorem 1. *The theory* $\mathrm{EA} + \mathsf{RFN}_{\Sigma_{n+1}}$ *is Π^0_{n+1}-conservative over* $\mathrm{EA} + \Pi^0_{n+1}$–$\mathsf{RR}^n(\mathrm{EA})$.

The Reduction Property can be stated and proved under more general conditions but for the current purpose this presentation suffices. At first glance it might seem a mere technicality but it implies various classical results like Parson's result that $\mathrm{I}\Sigma_1$ is Π^0_2-conservative over PRA. Moreover, as we shall see, it is one of the main ingredients in our ordinal analysis.

3.2. Simplifying the Reflection Rule

In this subsection we shall see that we can simplify considerably the family of reflection rules. We prefer to work in a general setting here. Thus, let $[n]_{\mathcal{U}}$ be any series of provability predicates over a theory $\mathcal{U}$ that is sound for GLP. Moreover, we have for each $n \in \omega$ that the formalized deduction theorem holds:

$$\mathcal{U} \vdash [n]_{\mathcal{U}+\varphi}\psi \iff \mathcal{U} \vdash [n]_{\mathcal{U}}(\varphi \to \psi).$$

The Reflection Rule as studied in the GLP project has currently two parameters n and m:

$$\Pi^0_m{-}\mathsf{RR}^n(U+\varphi) \ := \ \frac{\psi}{\langle n\rangle(\varphi \wedge \psi)} \quad \text{for } \varphi \in \Pi_m.$$

By virtue of easy lemmas below we shall see that we can drop the parameter m as over $\mathcal{U}$, for $m > n$ all the versions turn out to be equivalent, and for $m \leq n$ the rule is just equivalent to the axiom $\langle n\rangle\varphi$. Thus, we propose to just speak of the $\mathsf{RR}^n(U+\varphi)$:

$$\mathsf{RR}^n(U+\varphi) \ := \ \frac{\psi}{\langle n\rangle(\psi \wedge \varphi)}$$

without any restriction on the complexity of ψ. In the remainder of this subsection, we shall assume that $\langle n\rangle\varphi$ is of complexity Π^0_{n+1}. However, if this were not the case, the arguments go through exactly the same by replacing each occurrence of Π^0_{n+1} by $\widetilde{\Pi^0_{n+1}}$, where $\widetilde{\Pi^0_{n+1}}$ represents some natural complexity class to which $\langle n\rangle\varphi$ belongs.

Definition 4. *Let $Q^0_n(\varphi) = \langle n\rangle_{\mathcal{U}}\varphi$ and $Q^{k+1}_n(\varphi) = \langle n\rangle_{\mathcal{U}}(\varphi \wedge Q^k_n(\varphi))$.*

Lemma 2. *Let $l, m, n \in \omega$ and $l > n < m$. We have that*

$$\mathcal{U} + \Pi_l - \mathsf{RR}^n(\mathcal{U}+\varphi) \ \equiv \ \mathcal{U} + \Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi) \ \equiv \ \mathcal{U} + \{Q^k_n(\varphi) \mid k \in \omega\}.$$

Proof. As the complexity of $Q^k_n(\varphi)$ is Π_{n+1} for any k and φ, it is easy to see by an induction on k that for any $k, m, n \in \omega$ we have

$$\Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi) \ \vdash Q^k_n(\varphi)$$

so that $\mathcal{U} + \Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi) \ \supseteq \ \mathcal{U} + \{Q^k_n(\varphi) \mid k \in \omega\}$.

For the reverse inclusion we do induction on the number of applications of the rule $\Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi)$. So, suppose that for some $\chi \in \Pi_m$ we have that

$$\mathcal{U} + \Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi) \vdash \chi.$$

By the IH we have $\mathcal{U} \vdash Q^k_n(\varphi) \to \chi$ for some natural number k. But then by necessitation we have $\mathcal{U} \vdash [n](Q^k_n(\varphi) \to \chi)$, whence $\mathcal{U} + Q^{k+1}_n(\varphi) \vdash \langle n\rangle_{\mathcal{U}}(\varphi \wedge \chi)$ as was to be shown. □

Lemma 3. *Let $m, n \in \omega$ with $n \geq m$. We have that*

$$\mathcal{U} + \Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi) \ \equiv \ \mathcal{U} + \langle n\rangle\varphi.$$

Proof. Clearly, by one application of the $\Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi)$ rule we obtain $\frac{\top}{\langle n\rangle\varphi}$. Thus

$$\mathcal{U}+\langle n\rangle\varphi \subseteq \mathcal{U}+\Pi_m - \mathsf{RR}^n(\mathcal{U}+\varphi).$$

To prove the converse implication we show that $\mathcal{U}+\langle n\rangle\varphi$ is closed under the rule. Thus, reason in $\mathcal{U}+\langle n\rangle\varphi$ and suppose we have proved ψ with $\psi \in \Pi_m$. As $\psi \in \Sigma_{n+1}$ we have that $\psi \to [n]\psi$. We combine this with $\langle n\rangle\varphi$ to obtain the required $\langle n\rangle(\psi \wedge \varphi)$. □

We note that a similar argument applies to GLP_Λ once we have fixed suitable formulas $Q^k_\alpha(\varphi)$ there and have specified complexity classes for formulas of the form $\langle\alpha\rangle\psi$.

3.3. The Reduction Property revisited

More generality, we can define for GLP formulas –not just worms– an ordering over GLP:

$$\varphi <_\alpha \psi \ \Leftrightarrow\ \mathsf{GLP} \vdash \psi \to \langle\alpha\rangle\varphi.$$

With respect to these orderings, consistency statements behave very well and admit some sort of fundamental sequence. For any formula φ we defined $Q^k_\alpha(\varphi)$ for $k \in \omega$ by $Q^0_\alpha(\varphi) := \langle\alpha\rangle\varphi$ and $Q^{k+1}_\alpha(\varphi) := \langle\alpha\rangle(\varphi \wedge Q^k_\alpha(\varphi))$. With these formulas at hand we can state part of the fundamental sequence result to the effect that the formulas $\{Q^k_n(\varphi)\}_{k\in\omega}$ substitute a fundamental sequence of $\langle n+1\rangle\varphi$.

Lemma 4. *For each $k \in \omega$ we have that $\mathsf{GLP} \vdash \langle\alpha+1\rangle\varphi \to Q^k_\alpha(\varphi)$, whence also $\mathsf{GLP} \vdash \langle\alpha+1\rangle\varphi \to \langle\alpha\rangle Q^k_\alpha(\varphi)$.*

A proof of this lemma is not hard and it can be found, e.g., in [3]. The other half of the fundamental sequence result is by virtue of the above just recasting the Reduction Property in terms of GLP.

Theorem 2. $\mathrm{EA}+\langle n+1\rangle\varphi$ *is Π_{n+1}-conservative over* $\mathrm{EA}+\{Q^k_n(\varphi) \mid k \in \omega\}$.

Proof. By Lemma 4 we see that $\mathrm{EA}+\{Q^k_n(\varphi) \mid k \in \omega\} \subseteq \mathrm{EA}+\langle n+1\rangle\varphi$. The Π_{n+1}-conservativity follows directly from the Reduction Property –Theorem 1– and Lemma 2 above. □

The main ingredient of the proof of the Reduction Property is a cut-elimination argument. Thus, as was noted in previous papers, the theorem above –Theorem 2– is formalizable as soon as the superexponential function is provably total and in particular in EA^+. From this fact we get a powerful result concerning provable equi-consistency (see e.g. [3]):

Theorem 3. *For $m \leq n$ we have that* $\mathrm{EA}^+ \vdash \langle m\rangle\langle n+1\rangle\varphi \ \leftrightarrow\ \forall k\ \langle m\rangle Q^k_n(\varphi)$.

Proof. We reason in EA^+ and prove the equivalence by contraposition. Lemma 4 is actually already provable in EA so that we see

$$\exists k\ [m]_{Q^k_n(\varphi)}\bot \to [m]_{\langle n+1\rangle\varphi}\bot.$$

For the other direction we invoke the Reduction Property as stated in Theorem 2.

So, still reasoning in EA^+, we suppose that $[m]_{\langle n+1\rangle\varphi}\bot$. Let π be the conjunction of Π^0_m sentences that are used in the $\mathrm{EA} + \langle n+1\rangle\varphi$ proof of $\bot$. Thus, we get that $[0]_{\langle n+1\rangle\varphi}\neg\pi$. As $\neg\pi \in \Pi^0_{n+1}$ by the formalized reduction property we get that $[0]_{Q^k_n(\varphi)}\neg\pi$ for some (possibly non-standard) number k. The latter implies $[m]_{Q^k_n(\varphi)}\bot$ and we are done. □

4. A Π^0_1-ordinal analysis for PA

The following theorem with the proof can be found in full detail in [3]. We present here the main part of the proof but refer to certain claims made here too [3].

Theorem 4. $\mathrm{EA}^+ + \mathsf{TI}[\Pi^0_1, \varepsilon_0] \vdash \mathsf{Con}(\mathrm{PA})$

Proof. It is well-known that the equivalence between reflection, induction and consistency as stated in Lemma 1 can actually be formalized in EA^+. Thus, we reason in EA^+ and observe that we have $\mathrm{PA} \subseteq \mathrm{EA} + \{\langle 1\rangle\top, \langle 2\rangle\top, \langle 3\rangle\top, \langle 4\rangle\top, \ldots\}$. Consequently, $\mathsf{Con}(\mathrm{EA} + \{\langle 1\rangle\top, \langle 2\rangle\top, \langle 3\rangle\top, \langle 4\rangle\top, \ldots\}) \to \mathsf{Con}(\mathrm{PA})$ and we shall complete our proof by showing $\mathsf{Con}(\mathrm{EA} + \{\langle 1\rangle\top, \langle 2\rangle\top, \langle 3\rangle\top, \langle 4\rangle\top, \ldots\})$. For this, it suffices to show

$$\forall n\ \langle 0\rangle\langle n\rangle\top. \tag{1}$$

We shall prove this by transfinite induction. It is known that $\langle S^\omega, <_0\rangle$ is provably in EA isomorphic to $\langle \varepsilon_0, <\rangle$. Thus it suffices to perform a transfinite induction over the structure $\langle S^\omega, <_0\rangle$. Clearly

$$\forall\, A{\in}S^\omega\ \langle 0\rangle A \tag{2}$$

implies (1), so we shall prove (2) by transfinite induction over $\langle S^\omega, <_0\rangle$. We set out to prove $\forall\, A{\in}S^\omega\ (\forall\, A'{<_0}A\ \langle 0\rangle A' \to \langle 0\rangle A)$ from which (2) follows, and distinguish three cases:

1. $A = \top$ in which case we have $\langle 0\rangle\top$ as EA^+ proves the consistency of EA.

2. A is of the form $\langle 0\rangle B$ for some worm B.

   It is well-known that $\mathrm{EA}^+ \vdash \mathsf{RFN}_{\Sigma^0_1}(\mathrm{EA})$. So in particular, as $[0]B$ is a Σ^0_1-sentence, we get $[0][0]B \to [0]B$. Thus also $\langle 0\rangle B \to \langle 0\rangle\langle 0\rangle B$. However, as $B <_0 A$ we have by the induction hypothesis that $\langle 0\rangle B$ and we are done.

3. A is of the form $\langle n+1\rangle B$ for some worm B and natural number n.

   So, we need to prove $\langle 0\rangle\langle n+1\rangle B$. By Theorem 3 we get that

   $$\langle 0\rangle\langle n+1\rangle B \leftrightarrow \forall k\ \langle 0\rangle Q^k_n(B).$$

   However, as for each $k \in \omega$ we have by Lemma 4 that $Q^k_n(B) <_0 \langle n+1\rangle B$, we are done by the induction hypothesis.

□

On the basis of Theorem 4 one could decide to call ε_0 the proof-theoretical ordinal of PA. Like many other ordinal analyses, the current analysis is susceptible to plugging in pathological ordinal notation systems so as to get way weaker or stronger proof-theoretical ordinals for PA. However, we feel confident to judge ourselves which notation system is natural enough to use and which not.

We shall now briefly say why this particular ordinal is called the Π^0_1 ordinal of PA. If we define Turing progressions EA^α of EA by transfinite induction in the standard way as $\mathrm{EA}^0 := \mathrm{EA}$, and $\mathrm{EA}^\alpha := \cup_{\beta<\alpha}(\mathrm{EA}_\beta + \mathsf{Con}(\mathrm{EA}_\beta))$, we can define a Π^0_1 proof theoretical ordinal based on these EA^α. For a target theory T we define $|T|_{\Pi^0_1}$ –the Π^0_1 proof theoretical ordinal of T– to be the smallest α for which EA^α comprises all the Π^0_1 consequences of T.

For natural theories T and natural ordinal notation systems, this ordinal will coincide with the ordinal obtained by an analysis presented in Theorem 4. Moreover, for $T = \mathrm{PA}$ and various sub-systems T of PA, it is known that $|T|_{\Pi^0_1}$ coincides with all the other known ordinal analyses like $|T|_{\Pi^1_1}$ or $|T|_{\Pi^0_2}$.

We mention these other proof-theoretical ordinals here without further detail and just provide some context. In this same spirit it is worth mentioning that $|T|_{\Pi^0_1}$ is more fine-grained than any of the others. For example, $|\mathrm{PA} + \mathsf{Con}(\mathrm{PA})|_{\Pi^1_1} = |\mathrm{PA} + \mathsf{Con}(\mathrm{PA})|_{\Pi^0_2} = |\mathrm{PA}|_{\Pi^0_2} = \varepsilon_0$, whereas $|\mathrm{PA} + \mathsf{Con}(\mathrm{PA})|_{\Pi^0_1} = \varepsilon_0 \cdot 2$.

5. Ingredients for going beyond PA

The paradigm for Π^0_1 is nice in that it provides a more fine-grained analysis than all other ordinal analyses around. In a sense, it provides the finest analysis possible as different true theories will at least differ on Π^0_1 sentences. A critique to the paradigm is that the analysis has so far only been performed for rather weak mathematical theories: PA and its kin.

If we wish to address stronger theories than PA, there are two paths one can take. In the next subsection we discuss one such path where the base theory is strengthened. In the remaining subsection we speak about the approach where we strengthen GLP_ω to GLP_Λ with $\Lambda > \omega$.

5.1. Relative Π^0_1 ordinal analysis

We can choose to stay within GLP_ω and strengthen our base theory $\mathcal{X}$. So, if we wish to analyze some target theory $\mathcal{U}$ with the Π^0_1 paradigm relative to $\mathcal{X}$, the question translates to how often one should iterate the Turing progression based on $\mathcal{X}$ to comprise all the Π^0_1 consequences of $\mathcal{U}$. In the next section we shall analyze this in further detail.

5.2. Beyond GLP_ω

Another choice to strengthen the applicability of the paradigm is to use modal provability logics that go beyond GLP_ω. Currently most efforts of taking the paradigm further are along these lines. There are two main aspects involved here. The first is to extend the modal theory of GLP beyond GLP_ω and the other is to find suitable (hyper)arithmetical interpretations of the modalities $[\alpha]$ involved.

5.2.1. The modal theory

By now, the modal theory of GLP_Λ is rather well studied and understood. A first and seminal step in this direction was taken by Beklemishev in [4]. In particular, the paper focused on the closed fragment GLP^0 of GLP and studied the worms therein. It was shown that the orderings $<_0$ are well behaved also in the class-size GLP^0 and define a well order provided the irreflexivity of $<_0$.

The irreflexivity of $<_0$ has been shown both in [5] and [10]. In particular, [10] provides a class-size universal model for GLP^0. The ordering $<_0$ and natural and important generalizations are now well studied and understood as presented in [4, 8, 9, 7].

Although there are various important and interesting questions open in the modal theory of the logics GLP_Λ, it seems that all modal theory is in place to move the Π^0_1 ordinal analysis beyond PA.

5.2.2. Hyperarithmetic interpretations and the Reduction Property

Currently the aim the GLP project is to provide an ordinal analysis of predicative analysis whose classical proof-theoretical ordinal is the Feferman-Schütte ordinal Γ_0. Various natural candidates of provability notions have been seen to be sound and complete for GLP_{Γ_0}. However, for none of this interpretations a natural generalization of the Reduction Property has been established, so far.

In the final section of this paper we shall briefly mention some of these generalized provability notions. In the next section we shall see how the need of a full Reduction Property can be circumvented.

6. Reduction Property, equi-consistency and relative ordinal analysis

In this section we shall see how we can minimize the ingredients needed for a consistency proof as presented in Theorem 4. In particular, we shall not need the full Reduction Property but rather some of its weak versions in terms of equi-consistency.

We shall see that the following steps suffice. Below, let $\mathcal{U}$ denote the target theory which we wish to perform ordinal analysis of.

1. We fix some base theory $\mathcal{X}$ over which most of our arguments will be performed;

2. We find some notions of consistency over $\mathcal{X}$ of increasing strength

$$\{\langle 0\rangle_{\mathcal{X}}\varphi, \langle 1\rangle_{\mathcal{X}}\varphi, \langle 2\rangle_{\mathcal{X}}\varphi, \langle 3\rangle_{\mathcal{X}}\varphi, \ldots\}$$

so that the following properties are obtained (we shall drop subscripts $\mathcal{X}$)

(a) The notion $\langle n\rangle_{\mathcal{T}}$ grows monotone both in n and in $\mathcal{T}$ and for all natural numbers n, theories $\mathcal{T}$, and formulas φ, ψ we have that provably in some weak theory but certainly in $\mathcal{X}$

$$\langle n\rangle_{\mathcal{T}+\varphi}\psi \ \leftrightarrow\ \langle n\rangle_{\mathcal{T}}(\psi \wedge \varphi);$$

(b) The logic GLP is sound for the corresponding dual provability operators $[n]_{\mathcal{X}}$;

(c) We have that (provably in some weak theory but certainly in $\mathcal{X}$)

$$\mathcal{U} \subseteq \mathcal{X} + \{\langle 0\rangle_{\mathcal{X}}\top, \langle 1\rangle_{\mathcal{X}}\top, \langle 2\rangle_{\mathcal{X}}\top, \langle 3\rangle_{\mathcal{X}}\top, \ldots\};$$

(d) The theory $\mathcal{X} + \langle n+1\rangle\top$ is equi-consistent with the theory $\mathcal{X} + \{Q^k_n(\top) \mid k \in \omega\}$, where $Q^0_n(\varphi) = \langle n\rangle\varphi$ and $Q^{k+1}_n(\varphi) = \langle n\rangle(\varphi \wedge Q^k_n(\varphi))$. This equi-consistency should be provable in some weak extension $\mathcal{X}^+$ of $\mathcal{X}$.

We shall now see that these ingredients suffice to perform a consistency proof of $\mathcal{U}$ relative to $\mathcal{X}$ formalized in $\mathcal{X}^+$.

Theorem 5. *Suppose we have fixed $\mathcal{X}$ and consistency notions as above. Then*

$$\mathcal{X}^+ + \mathsf{TI}(\widetilde{\Pi^0_1}, \varepsilon_0) \vdash \mathsf{Con}(\mathcal{U}),$$

where $\widetilde{\Pi^0_1}$ is some complexity class that corresponds to the consistency notion $\langle 0\rangle_{\mathcal{X}}$.

Proof. The proof is similar to that of Theorem 4. We reason in $\mathcal{X}^+$. By 2c above, we have that $\mathcal{U} \subseteq \mathcal{X} + \{\langle 0\rangle_{\mathcal{X}}\top, \langle 1\rangle_{\mathcal{X}}\top, \langle 2\rangle_{\mathcal{X}}\top, \langle 3\rangle_{\mathcal{X}}\top, \ldots\}$, whence also

$$\langle 0\rangle_{\{\langle 0\rangle_{\mathcal{X}}\top, \langle 1\rangle_{\mathcal{X}}\top, \langle 2\rangle_{\mathcal{X}}\top, \langle 3\rangle_{\mathcal{X}}\top, \ldots\}}\top \to \langle 0\rangle_{\mathrm{ZFC}}\top.$$

Clearly, by 2a we have that

$$\langle 0\rangle_{\{\langle 0\rangle_{\mathcal{X}}\top, \langle 1\rangle_{\mathcal{X}}\top, \langle 2\rangle_{\mathcal{X}}\top, \langle 3\rangle_{\mathcal{X}}\top, \ldots\}}\top \quad\leftrightarrow\quad \forall n\ \langle 0\rangle\langle n\rangle\top.$$

We now reason inside some weak extension $\mathcal{X}^+$ of $\mathcal{X}$ and conclude by using transfinite induction and showing that $\forall n\ \langle 0\rangle_{\mathcal{X}}\langle n\rangle_{\mathcal{X}}\top$. Clearly it suffices to show that for all worms A in GLP_ω we have that $\langle 0\rangle_{\mathcal{X}}A$. Thus, we set out to prove

$$\forall A\ [\forall B{<_0}A\ \langle 0\rangle_{\mathcal{X}}B \to \langle 0\rangle_{\mathcal{X}}A]. \tag{3}$$

We choose $\mathcal{X}^+$ strong enough so that it at least contains $\mathsf{RFN}_{\widetilde{\Sigma^0_1}}(\mathcal{X})$ in order to have

1. $\mathcal{X}^+ \vdash \langle 0\rangle_{\mathcal{X}}\top$ and,
2. $\mathcal{X}^+ \vdash \langle 0\rangle_{\mathcal{X}}\varphi \ \to\ \langle 0\rangle_{\mathcal{X}}\langle 0\rangle_{\mathcal{X}}\varphi$.

These two observations account for a proof of (3) for the empty worm and worms of the form $\langle 0\rangle A'$. For worms of the form $\langle n+1\rangle A'$ we see by 2d that

$$\langle 0\rangle_{\mathcal{X}}\langle n+1\rangle_{\mathcal{X}}A' \ \leftrightarrow\ \forall k\langle 0\rangle_{\mathcal{X}}\langle n\rangle_{\mathcal{X}}Q^k_n(A').$$

But, as by 2b, GLP is sound for our modalities, we get that all the $Q^k_n(A')$ are $<_0$-below $\langle n+1\rangle\top$ and we have the right-hand side from the induction hypothesis. □

For the sake of presentation we have chosen $\mathcal{X}$ and $\mathcal{U}$ such that in some sense $\frac{\mathrm{EA}}{\mathrm{PA}} = \frac{\mathcal{X}}{\mathcal{U}}$ in which case it would be justified to say that the Π^0_1-proof theoretic ordinal of $\mathcal{U}$ *relative to* $\mathcal{X}$ is ε_0.

It is clear that Theorem 5 above can be extended to larger orderings once we have extended our notion of the fundamental sequence as in Definition 4 also for modalities with limit ordinals. This is unproblematic in principle but may slightly depend on the choice of fundamental sequences of the ordinals inside the modalities. The important observation is that the use of the full Reduction Property can be avoided.

7. Going beyond PA: recent developments

In this final section we just wish to briefly report on ongoing work to find arithmetical interpretations for GLP_Λ with $\Lambda > \omega$.

7.1. Truth-predicates and reflection

Lev Beklemishev and Evgeniy Dashkov, both at Moscow State University, have been studying interpretations of $\mathsf{GLP}_{\omega\cdot 2}$ where an additional truth-predicate for arithmetical formulas is added to the language of arithmetic. Within this framework they can express reflection for arithmetical formulas and slightly beyond.

The work is still unpublished but they have presented some results where they go up to $[\omega + \omega]$ while preserving the full Reduction Property at the price of giving up the nice modal logic GLP. They rather switch to a positive fragment of GLP to account for the fact that certain reflection principles (at limit stages) are not finitely axiomatizable.

7.2. Omega rule interpretations

The author and David Fernández Duque study ([11]) an interpretation for GLP_Λ for recursive ordinals Λ where each $[\alpha]$ modality is interpreted in second order arithmetic as "provable in T with a proof-tree whose nested omega rules have order type at most α". Moreover, Andrés Cordón Franco, David Fernández Duque, and Félix Lara Martín from the University of Sevilla, also in collaboration with the author, are studying an ordinal analysis of predicative analysis based on this interpretation. However, not much is known to what extent the Reduction Property holds for this interpretation.

7.3. Levy's reflection results

In discussions with the author Joan Bagaria from ICREA and the University of Barcelona suggested the following set-theoretical reading of our modalities $[n]$ for $n \in \omega$. Let $\mathcal{X}$ be the theory $\mathrm{ZFC} - \{\mathsf{Repl} + \mathsf{Inf}\}$. It is established in a paper by Levy ([17]) that

$$\mathrm{ZFC} \equiv \mathcal{X} + \mathsf{RFN}(\mathcal{X}).$$

Here, RFN refers to the following notion of reflection: For each (externally quantified) natural number n, by $\mathsf{RFN}_{\Sigma_n}(\mathcal{X})$ we denote the following principle

$$\forall\,\varphi{\in}\Sigma_n\,\forall a\,\exists\,\alpha{\in}\mathsf{On}\ [V_\alpha \models \varphi(a)\ \Leftrightarrow\ \models_n \varphi(a)].$$

Here, $\models_n$ refers to partial truth predicates that are known to exist for ZFC and subtheories. At first sight it seems that replacement is needed to define the entities V_α. However, in the absence of replacement one can work with the Scott-rank instead and define $V_\alpha := \{x \mid \mathsf{rank}(x) \leq \alpha\}$, where $\mathsf{rank}(x) \leq \alpha$ *is* definable in $\mathcal{X}$ making use of the transitive closure. We now define classes that collect the ordinals α for which the partial universes V_α are Σ_n elementary substructures of V:

$$C^{(n)} := \{\alpha \mid V_\alpha \prec_{\Sigma_n} V\}.$$

It is a theorem by Levy that the classes $C^{(n)}$ are Π_n definable in $\mathcal{X}$. Next, we define

$$\langle n\rangle_{\mathcal{T}}\varphi\ :\Leftrightarrow\ \exists\,\alpha{\in}C^{(n)}\ [V_\alpha \models \mathcal{T} \wedge V_\alpha \models \varphi].$$

It seems that all 2 (a)–(c) are satisfied for this notion of provability. In particular, we have that $\langle n\rangle\varphi \to [m]\langle n\rangle\varphi$ since $\langle n\rangle_{\mathcal{T}}$ is definable in a Σ_{n+1}-fashion. As we cannot obtain that

$$\mathrm{ZFC} \nvdash \mathsf{TI}(\Pi^0_1, \varepsilon_0),$$

we must conclude that (d) does not hold and that the two theories are not equiconsistent.

7.4. On a relative proof-theoretical ordinal of ZFC

We conclude by a simple observation on a proof theoretical ordinal of ZFC. It is generally believed that an ordinal analysis for ZFC is currently way out of reach. With the methods presented here one might hope that at least an ordinal analysis relative to some strong base theory of ZFC might be possible.

However, if such a relative analysis were to be given, it is most likely to be formalizable within ZFC itself. As ZFC proves transfinite induction over any well-ordering, this implies that the order type –however small– involved in such a relative ordinal analysis of ZFC must be represented inside ZFC in such a way that ZFC does not prove it is indeed a well-order.

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