Classification of quadrics in a double isotropic space

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Abstract. *This paper gives the classification of second order surfaces in a double isotropic space $I_3^{(2)}$. Although quadrics in $I_3^{(2)}$ have been investigated earlier [e.g. 8 or 9], this paper offers a new method based on linear algebra. The definition of invariants of a quadric with respect to the group of motions in $I_3^{(2)}$ makes it possible to determine the type of a quadric without reducing its equation to a canonical form. For that purpose isometric properties of conics in the isotropic plane and affine properties of quadrics are used.*

Key words: *quadrics, plane isotropic geometry, geometry of the double isotropic space*

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1. Double isotropic space

Let $P_3(R)$ be a three-dimensional real projective space, quadruples $(x_0 : x_1 : x_2 : x_3) \neq (0:0:0:0)$ the projective coordinates, ω a plane in P_3, and $A_3 = P_3 \backslash \omega$ the derived affine space. An affine space A_3 is called a double isotropic space $I_3^{(2)}$ if in A_3 a metric is induced by an *absolute* $\{\omega, f, F\}$, consisting of the line f in the plane of infinity ω and the point $F \in f$. The geometry of $I_3^{(2)}$ could be seen in e.g. Brauner [5]. In the affine model, where

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}, \quad z = \frac{x_3}{x_0}, \quad (x_0 \neq 0) \tag{1}$$

the absolute figure is determined by $\varpi \equiv x_0 = 0, f \equiv x_0 = x_1 = 0$, and $F\,(0:0:0:1)$.

All regular projective transformations that keep the absolute figure fixed form a 9-parametric group

$$G_9 \begin{cases} \overline{x_0} = x_0 \\ \overline{x_1} = c_1x_0 + c_2x_1 \\ \overline{x_2} = c_3x_0 + ax_1 + c_4x_2 \\ \overline{x_3} = c_5x_0 + bx_1 + cx_2 + c_6x_3 \end{cases}, c_1, \dots, c_6, a, b, c \in R, \text{ and } c_2c_4c_6 \neq 0. \tag{2}$$

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We call it the *group of similarities* of the double isotropic space $I_3^{(2)}$.

The unimodular transformations in G_9 form a 6-parametric *motion group* of $I_3^{(2)}$, i.e.,

$$G_6 \begin{cases} \overline{x_0} = x_0 \\ \overline{x_1} = c_1x_0 + x_1 \\ \overline{x_2} = c_3x_0 + ax_1 + x_2 \\ \overline{x_3} = c_5x_0 + bx_1 + cx_2 + x_3 \end{cases} \tag{3}$$

In the plane of infinity ω, G_6 induces a 3-parametric group

$$G_3 \begin{cases} \overline{x_1} = x_1 \\ \overline{x_2} = ax_1 + x_2 \\ \overline{x_3} = bx_1 + cx_2 + x_3 \end{cases} \tag{4}$$

G_3 is the *motion group* of an isotropic plane, see [7].

2. Quadric equation

A quadric equation is a second-degree equation in three variables that can be written in the form

$$\begin{aligned} Q(x,y,z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz \\ + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0, \end{aligned} \tag{5}$$

where $a_{11}, \ldots, a_{00} \in R$ and at least one of the coefficients $a_{11}, \ldots, a_{23} \neq 0$ [1].

Using the matrix notation,

$$Q(x,y,z) \equiv [1\ x\ y\ z] \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = 0. \tag{6}$$

We will need

$$\Delta = \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}, \quad \Delta_2 = |a_{33}| = a_{33}, \tag{7}$$

the 3×3- minors $D_{i,j}$ $(i,j = 1,2,3)$ of Δ, the 2×2- minors Δ_{ij} $(i,j = 1,2,3)$ of Δ_0, as well as

$$\begin{aligned} &\Gamma_0 = \gamma_{01} + \gamma_{02} + \gamma_{03} = \begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{02} \\ a_{02} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{03} \\ a_{03} & a_{33} \end{vmatrix}, \\ &\Gamma_1 = \Delta_{11} + \Delta_{22} + \Delta_{33}, \quad (\Delta_{11} = \Delta_1), \quad \Gamma = \Gamma_0 + \Gamma_1, \qquad \text{and} \\ &\alpha_0 = a_{11} + a_{22} + a_{33}, \quad \alpha_1 = \alpha_0 + a_{00}. \end{aligned} \tag{8}$$

As it is known, see [4], the sign of the determinant Δ as well as that of Δ_0 is invariant with respect to the affine (linear, regular) transformations. The goal is to determine the invariants of quadrics with respect to the group G_6 of motions in $I_3^{(2)}$. For that purpose let us apply on the quadric equation (5) the mapping from G_6 given by

$$\begin{cases} x = \overline{x} \\ y = a\overline{x} + \overline{y} \\ z = b\overline{x} + c\overline{y} + \overline{z} \end{cases}, \quad a, b, c \in R. \tag{9}$$

We obtain

$$Q(x, y, z) \equiv [1\ \overline{x}\ \overline{y}\ \overline{z}] \begin{bmatrix} \overline{a}_{00} & \overline{a}_{01} & \overline{a}_{02} & \overline{a}_{03} \\ \overline{a}_{01} & \overline{a}_{11} & \overline{a}_{12} & \overline{a}_{13} \\ \overline{a}_{02} & \overline{a}_{12} & \overline{a}_{22} & \overline{a}_{23} \\ \overline{a}_{03} & \overline{a}_{13} & \overline{a}_{23} & \overline{a}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} = 0, \tag{10}$$

where,

$$\begin{aligned} \overline{a}_{00} &= a_{00}, \\ \overline{a}_{01} &= a_{01} + aa_{02} + ba_{03}, \\ \overline{a}_{02} &= a_{02} + ca_{03}, \\ \overline{a}_{03} &= a_{03}, \\ \overline{a}_{11} &= a_{11} + 2aa_{12} + 2ba_{13} + 2aba_{23} + a^2a_{22} + b^2a_{33}, \\ \overline{a}_{12} &= a_{12} + aa_{22} + ba_{23} + ca_{13} + aca_{23} + bca_{33}, \\ \overline{a}_{13} &= a_{13} + aa_{23} + ba_{33}, \\ \overline{a}_{22} &= a_{22} + 2ca_{23} + c^2a_{33}, \\ \overline{a}_{23} &= a_{23} + ca_{33}, \\ \overline{a}_{33} &= a_{33}. \end{aligned} \tag{11}$$

These yield: $\overline{\Delta} = \Delta$, $\overline{\Delta_0} = \Delta_0$, $\overline{\Delta_1} = \Delta_1$, $\overline{\Delta_2} = \Delta_2$.
For example,

$$\overline{\Delta_1} = \begin{vmatrix} \overline{a}_{22} & \overline{a}_{23} \\ \overline{a}_{23} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{22} + 2ca_{23} + c^2a_{33} & a_{23} + ca_{33} \\ a_{23} + ca_{33} & a_{33} \end{vmatrix} = \cdots = a_{22}a_{33} - a_{23}^2 = \Delta_1.$$

3. Diagonalization of the quadratic form

The quadratic form within equation (5) is a homogenous polynomial of the second degree:

$$\begin{aligned} K(x, y, z) &\equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz \\ &= [x\ y\ z] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \end{aligned} \tag{12}$$

The question is whether and when it is possible to obtain the *canonical form* (see [6]) of the quadratic form (12), i. e., $a_{12} = a_{13} = a_{23} = 0$, using transformations of the group G_6. From (11) we derive that

$$\overline{a}_{12} = \overline{a}_{13} = \overline{a}_{23} = 0 \Rightarrow c = -\frac{a_{23}}{a_{33}}, b = \frac{a_{12}a_{23} - a_{13}a_{22}}{a_{22}a_{33} - a_{23}^2}, a = \frac{a_{13}a_{23} - a_{12}a_{33}}{a_{22}a_{33} - a_{23}^2}, \quad (13)$$

that is,

$$c = -\frac{a_{23}}{\Delta_2}, \quad b = \frac{\Delta_{13}}{\Delta_1}, \quad a = \frac{\Delta_{12}}{\Delta_1} \quad (14)$$

In such a way we obtain

$$K(x,y,z) = [\overline{x} \ \ \overline{y} \ \ \overline{z}] \begin{bmatrix} \overline{a}_{11} & 0 & 0 \\ 0 & \overline{a}_{22} & 0 \\ 0 & 0 & \overline{a}_{33} \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} \quad (15)$$

where,

$$\overline{a}_{11} = \frac{\Delta_0}{\Delta_1}, \quad \overline{a}_{22} = \frac{\Delta_1}{\Delta_2}, \quad \overline{a}_{33} = \Delta_2. \quad (16)$$

4. Isotropic values of the matrix

Using the analogies with the results given in Beban-Brkic [2] obtained for the isotropic plane, we have the following:

Definition 1. *Let* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ *be any real symmetric matrix, and let us denote* $\Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$, $\Delta_2 = |a_{33}| = a_{33}$. *Then the values* $I_1 = \frac{\Delta_0}{\Delta_1}$, $I_2 = \frac{\Delta_1}{\Delta_2}$, $I_3 = \Delta_2$, $(\Delta_1, \Delta_2 \neq 0)$, *are called* isotropic values *of the matrix* A.

Definition 2. *We say that the real symmetric* 3×3*-matrix* A allows the isotropic diagonalization *if there exists a matrix* $G = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ *such that* $G^T AG$ *is a diagonal matrix, i.e.,* $G^T AG = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ *where* I_1, I_2, I_3 *are the isotropic values of the matrix* A*. We say that matrix* G isotropically diagonalizes A.

It can be seen from what has been shown earlier that the following propositions hold:

Proposition 1. *The isotropic values* I_1, I_2, I_3 *as well as their products* $I_1 I_2 I_3 = \Delta_0$, $I_2 I_3 = \Delta_1$ *are invariant with respect to the group of motions of the double isotropic space* $I_3^{(2)}$.

Proposition 2. *Let* A *be a matrix from* Definition 1*. Then there is a matrix* $G = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ *with* $a = \frac{\Delta_{12}}{\Delta_1}$, $b = \frac{\Delta_{13}}{\Delta_1}$, $c = -\frac{a_{23}}{\Delta_2}$, *which under the condition* $\Delta_1, \Delta_2 \neq 0$ *isotropically diagonalizes* A.

Proposition 3. *It is always possible to reduce the quadratic form (12) by an isotropic motion to the canonical form (15) except for*

a) $\Delta_2 = 0,\ \Delta_1 \neq 0$;

b) $\Delta_2 = 0,\ \Delta_1 = 0,\ \Delta_0 \neq 0$;

c) $\Delta_2 = \Delta_1 = \Delta_0 = 0$.

5. The absolute plane ω

The quadric equation (5) written in homogenous coordinates $(x_0 : x_1 : x_2 : x_3)$ has the form

$$\begin{aligned} a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + 2a_{01}x_1x_0 \\ + 2a_{02}x_2x_0 + 2a_{03}x_3x_0 + a_{00}x_0^2 = 0. \end{aligned} \tag{17}$$

With $x_0 = 0$ we obtain the cross-section of the quadric with the plane of infinity ω, i.e.,

$$k_\omega = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0, \tag{18}$$

where, $(x_1 : x_2 : x_3)$ are plane projective coordinates. The affine coordinates ξ, η in ω are given by

$$\xi = \frac{x_2}{x_1}, \quad \eta = \frac{x_3}{x_1}, \tag{19}$$

and the absolute figure is determined by

$$F(0:0:1), \quad f \equiv x_1 = 0. \tag{20}$$

Equation (18) obtains the form

$$\begin{aligned} k_\omega &\equiv a_{22}\xi^2 + a_{33}\eta^2 + 2a_{12}\xi + 2a_{13}\eta + 2a_{23}\xi\eta + a_{11} \\ &= [1\ \xi\ \eta] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix} = 0. \end{aligned} \tag{21}$$

As it has been shown in [2], the basic conic invariants with respect to the group of motions of the isotropic plane ω, in terms of the quadrics invariants, are Δ_0, Δ_1, and Δ_2, given in relation (7).

6. Isotropic classification of quadrics

It is known that in the isotropic plane there are 20 different types of conics, see [3], [7]. We will use the classification given in [3], made with respect to the conics isometric invariants in the isotropic plane, as well as the results related to k_ω obtained in the same paper:

<table>
<tr><td rowspan="10">$\Delta_2 \neq 0$</td><td rowspan="3">$\Delta_1 > 0$</td><td rowspan="2">$\Delta_0 \neq 0$</td><td>$\Delta_0\Delta_2 > 0$</td><td colspan="4">imaginary ellipse</td><td rowspan="10">1^{st} family</td></tr>
<tr><td>$\Delta_0\Delta_2 < 0$</td><td colspan="4">real ellipse</td></tr>
<tr><td>$\Delta_0 = 0$</td><td colspan="5">imaginary pair of straight lines</td></tr>
<tr><td rowspan="3">$\Delta_1 < 0$</td><td rowspan="2">$\Delta_0 \neq 0$</td><td>$\Delta_2\Delta_0 > 0$</td><td colspan="4">2^{nd} type hyperbola</td></tr>
<tr><td>$\Delta_2\Delta_0 < 0$</td><td colspan="4">1^{st} type hyperbola</td></tr>
<tr><td>$\Delta_0 = 0$</td><td colspan="5">pair of intersecting straight lines</td></tr>
<tr><td rowspan="4">$\Delta_1 = 0$</td><td>$\Delta_0 \neq 0$</td><td colspan="5">parabola</td></tr>
<tr><td rowspan="3">$\Delta_0 = 0$</td><td>$\Delta_{22} < 0$</td><td colspan="4">pair of parallel lines</td></tr>
<tr><td>$\Delta_{22} > 0$</td><td colspan="4">imaginary pair of parallel lines</td></tr>
<tr><td>$\Delta_{22} = 0$</td><td colspan="4">two coinciding lines</td></tr>
<tr><td rowspan="10">$\Delta_2 = 0$</td><td>$\Delta_1 < 0$</td><td>$\Delta_0 \neq 0$</td><td colspan="5">special hyperbola</td><td rowspan="2">2^{nd} family</td></tr>
<tr><td>$\Delta_1 < 0$</td><td>$\Delta_0 = 0$</td><td colspan="5">pair of lines, one being an isotropic line</td></tr>
<tr><td rowspan="8">$\Delta_1 = 0$</td><td>$\Delta_0 \neq 0$</td><td colspan="5">parabolic circle</td><td rowspan="4">3^{rd} family</td></tr>
<tr><td rowspan="7">$\Delta_0 = 0$</td><td>$\Delta_{33} < 0$</td><td colspan="4">pair of isotropic lines</td></tr>
<tr><td>$\Delta_{33} > 0$</td><td colspan="4">imaginary pair of isotropic lines</td></tr>
<tr><td>$\Delta_{33} = 0$</td><td>$a_{22} \neq 0$</td><td colspan="3">two coinciding isotropic lines</td></tr>
<tr><td>$\Delta_{22} < 0$</td><td colspan="4">straight line + f</td><td rowspan="4">4^{th} family</td></tr>
<tr><td rowspan="3">$\Delta_{22} = 0$</td><td>$\Delta_{33} < 0$</td><td rowspan="3">$a_{22} = 0$</td><td colspan="2">isotropic line + f</td></tr>
<tr><td rowspan="2">$\Delta_{33} = 0$</td><td>$a_{11} \neq 0$</td><td>double f</td></tr>
<tr><td>$a_{11} = 0$</td><td>all points in plane</td></tr>
</table>

So, for quadrics classification we will use the conics division in four families depending on their relation towards the absolute figure given by relation (20). We will distinguish the following:

k_ω belongs to the 1^{st} family $\equiv$ it does not have an isotropic direction $\Rightarrow$ 1^{st} *family of quadrics*; consisting of two subfamilies defined by

$$\begin{cases} k_\omega \cap f = \emptyset\,; & \alpha \ subfamily \\ k_\omega \cap f \neq \emptyset\,; & \beta \ subfamily \end{cases} \quad ;$$

k_ω belongs to the 2^{nd} family $\equiv$ it has one isotropic direction $\Rightarrow$ 2^{nd} *family of quadrics*;

k_ω belongs to the 3^{rd} family $\equiv$ it has a double isotropic direction $\Rightarrow$ 3^{rd} *family of quadrics*;

k_ω belongs to the 4^{th} family $\equiv$ it contains an absolute line f $\Rightarrow$ 4^{th} *family of quadrics.*

On the other hand, according to Brauner [5], in $I_3^{(2)}$ we distinguish six classes of straight lines. Those are:

- Nonisotropic lines $\equiv$ lines l with the property $l \cap f = \emptyset$;
- Isotropic lines $\equiv$ lines l with the property $l \cap f \neq \emptyset$, $l \cap f \neq F$;
- Double isotropic lines $\equiv$ lines l with the property $l \cap f = F$;
- Nonisotropic lines in the plane of infinity $\omega \equiv$ lines for which $l \cap f = \emptyset$;
- Isotropic lines in the plane of infinity $\omega \equiv$ lines for which $l \cap f = F$, $l \neq f$;
- Absolute line f.

Hence, quadrics in $I_3^{(2)}$ will also be classified according to the direction of the longitudinal axes, i.e., according to the straight line class the axes belong to. We will distinguish:

- Nonisotropic surfaces;
- Isotropic surfaces;
- Double isotropic surfaces.

6.1. k_ω belongs to the 1^{st} family

If k_ω belongs to the 1^{st} family, then according to the conics isometric invariants, there are two possibilities:

(i) $\Delta_2 \neq 0,\ \Delta_1 \neq 0$;

(ii) $\Delta_2 \neq 0,\ \Delta_1 = 0$.

Ad (i) Under such conditions it is always possible to reduce the conic equation (21) by isotropic motions into the form

$$k_\omega \equiv \frac{\Delta_1}{\Delta_2}\xi^2 + \Delta_2\eta^2 + \frac{\Delta_0}{\Delta_1} = 0,\ \text{i.e.,} \tag{22}$$

$$k_\omega \equiv I_2\xi^2 + I_3\eta^2 + I_1 = 0. \tag{23}$$

In the homogenous coordinates,

$$k_\omega \equiv I_2x_2^2 + I_3x_3^2 + I_1x_1^2 = 0, \tag{24}$$

and finally, in the affine space coordinates,

$$k_\omega \equiv I_2y^2 + I_3z^2 + I_1x^2 = 0. \tag{25}$$

All quadrics having (25) as the cross-section with the plane of infinity ω could be written in the form, see [8]:

$$I_1x^2 + I_2y^2 + I_3z^2 + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0,\ \text{i.e.,}$$
$$\frac{\Delta_0}{\Delta_1}x^2 + \frac{\Delta_1}{\Delta_2}y^2 + \Delta_2z^2 + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0. \tag{26}$$

The possibilities are the following:

Ad 1) k_ω is an imaginary ellipse. Choosing

$$\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 \neq 0, \Delta_0\Delta_2 > 0, \tag{27}$$

it follows that (21) represents an equation of an imaginary ellipse. By means of three translations of the coordinate system, in the direction of x, y, and z-axes, the quadric equation (26) obtains the form

$$\frac{\Delta_0}{\Delta_1}x^2 + \frac{\Delta_1}{\Delta_2}y^2 + \Delta_2z^2 + t = 0, \tag{28}$$

where $t = \dfrac{\Delta}{\Delta_0}$ is invariant. Surface (28) defined by:

1.1. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, and $\Delta > 0$, will be called an ***imaginary ellipsoid of the α subfamily***;

1.2. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, and $\Delta < 0$, will be called an ***ellipsoid of the α subfamily***;

1.3. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, and $\Delta = 0$, will be called an ***imaginary cone of the α subfamily***.

Ad 2) k_ω is a real ellipse. With

$$\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 \neq 0, \Delta_0\Delta_2 < 0, \tag{29}$$

an equation of an ellipse is obtained. The quadric equation (26) can be transformed into

$$-\left|\frac{\Delta_0}{\Delta_1}\right| x^2 + \left|\frac{\Delta_1}{\Delta_2}\right| y^2 + |\Delta_2|\, z^2 + t = 0, \quad t = \frac{\Delta}{\Delta_0} \tag{30}$$

wherefrom three types of surfaces can be distinguished:

2.1. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$ and $\Delta > 0$, will be called a ***nonisotropic one-sheet hyperboloid of the α subfamily***;

2.2. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$ and $\Delta < 0$, will be called a ***nonisotropic two-sheet hyperboloid of the α subfamily***;

2.3. $\Delta_2 \neq 0$, $\Delta_1 > 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$ and $\Delta = 0$, will be called a ***nonisotropic cone of the α subfamily***.

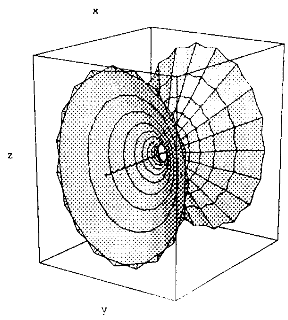

Figure 1. *Nonisotropic one-sheet hyperboloid of the α subfamily*

Ad 3) k_ω is an imaginary pair of straight lines. Choosing

$$\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 = 0, \tag{31}$$

(21) represents an imaginary pair of straight lines. It is easy to verify that the quadric quation (26) under such conditions can be transformed into

$$\frac{\Delta_1}{\Delta_2} y^2 + \Delta_2 z^2 + 2a_{01}x + a_{00} = 0, \tag{32}$$

wherefrom we derive two possibilities:

3.1. $a_{01} \neq 0$; After a translation of the coordinate system in the direction of the x-axis, equation (32) becomes of the form

$$\frac{\Delta_1}{\Delta_2} y^2 + \Delta_2 z^2 + 2a_{01}x = 0, \tag{33}$$

where the remaining coefficient $a_{01} = \sqrt{-\frac{\Delta}{\Delta_1}}$ is invariant as well. A surface defined by:

3.1.1. $\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 = 0$, and $\Delta < 0$, will be called a ***nonisotropic elliptical paraboloid of the*** α ***subfamily.***

3.2.

$$a_{01} = 0 \Rightarrow \frac{\Delta_1}{\Delta_2} y^2 + \Delta_2 z^2 + a_{00} = 0, \tag{34}$$

wherefrom it follows that $\Delta = 0$, and $a_{00} = \frac{D_{11}}{\Delta_1}$, where D_{11} is invariant with respect to the group of motions of $I_3^{(2)}$, and is therefore a_{00}. A surface defined by:

3.2.1. $\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 = 0$, and $\Delta = 0$, $D_{11}\Delta_2 > 0$, will be called a ***nonisotropic imaginary cylinder of the*** α ***subfamily***;

3.2.2. $\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 = 0$, and $\Delta = 0$, $(\Gamma < 0 \vee D_{11}\alpha_1 < 0)$, will be called a ***nonisotropic elliptical cylinder of the*** α ***subfamily***;

3.2.3. $\Delta_2 \neq 0, \Delta_1 > 0, \Delta_0 = 0$, and $\Delta = 0$, $D_{11}\Delta_2 = 0$, will be called a ***pair of imaginary planes with a nonisotropic cross-section of the*** α ***subfamily***.

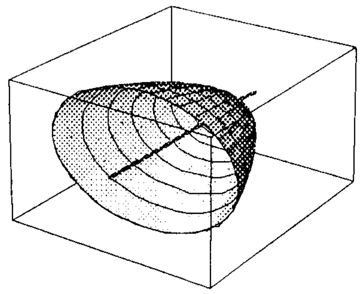

Figure 2. *Nonisotropic elliptical paraboloid of the α subfamily*

Here follows the proof for D_{11} being invariant under isotropic motions: From (11) and (34) we derive that

$$\overline{D}_{11} = \begin{vmatrix} \overline{a}_{00} & \overline{a}_{02} & \overline{a}_{03} \\ \overline{a}_{02} & \overline{a}_{22} & \overline{a}_{23} \\ \overline{a}_{03} & \overline{a}_{23} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{00} & a_{02}+ca_{03} & a_{03} \\ a_{02}+ca_{03} & a_{22}+2ca_{23}+c^2a_{33} & a_{23}+ca_{33} \\ a_{03} & a_{23}+ca_{33} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{22}+c^2a_{33} & ca_{33} \\ 0 & ca_{33} & a_{33} \end{vmatrix} = a_{00}a_{22}a_{33} = D_{11}.$$

Ad 4) $\mathbf{k}_\omega$ is a $\mathbf{1}^{st}$ type hyperbola. Choosing

$$\Delta_2 \neq 0, \Delta_1 < 0, \Delta_0 \neq 0, \Delta_0\Delta_2 < 0, \tag{35}$$

(21) represents an equation of the 1^{st} type hyperbola. The quadric equation given in (26) can be transformed into

$$-\left|\frac{\Delta_0}{\Delta_1}\right| x^2 + \left|\frac{\Delta_1}{\Delta_2}\right| y^2 - |\Delta_2|\, z^2 + t = 0, \quad t = \frac{\Delta}{\Delta_0}, \tag{36}$$

wherefrom three surfaces can be distinguished:

4.1. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta > 0$, will be called an ***isotropic one-sheet hyperboloid of the β subfamily***;

4.2. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta < 0$, will be called an ***isotropic two-sheet hyperboloid of the β subfamily***;

4.2.1. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta = 0$, will be called an ***isotropic cone of the β subfamily***.

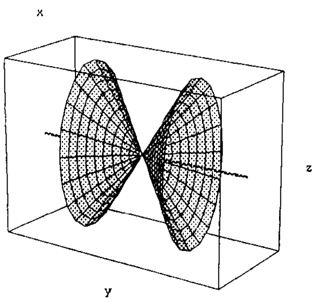

Figure 3. *Isotropic cone of the β subfamily*

Ad 5) $\mathbf{k}_\omega$ is a $\mathbf{2}^{nd}$ type hyperbola. With

$$\Delta_2 \neq 0, \Delta_1 < 0, \Delta_0 \neq 0, \Delta_0\Delta_2 > 0, \tag{37}$$

(21) is an equation of the 2^{nd} type hyperbola. The quadric equation (26) can be written in the form:

$$-\left|\frac{\Delta_0}{\Delta_1}\right| x^2 - \left|\frac{\Delta_1}{\Delta_2}\right| y^2 + |\Delta_2|\, z^2 + t = 0, \quad t = \frac{\Delta}{\Delta_0}. \tag{38}$$

The possibilities are:

5.1. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta > 0$, will be called a ***double isotropic one-sheet hyperboloid of the β subfamily***;

5.2. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta < 0$, will be called a ***double isotropic two-sheet hyperboloid of the β subfamily***;

5.3. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, $\Delta_0\Delta_2 > 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta = 0$, will be called a ***double isotropic cone of the β subfamily.***

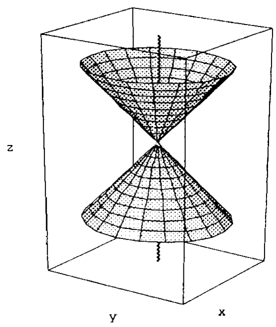

Figure 4. *Double isotropic cone of the β subfamily*

Ad 6) $\mathbf{k}_\omega$ is a pair of intersecting straight lines. With

$$\Delta_2 \neq 0, \Delta_1 < 0, \Delta_0 = 0, \tag{39}$$

(21) represents a pair of intersecting straight lines. It is easy to verify that the quadrics equation (26) under such conditions can be transformed into

$$\left|\frac{\Delta_1}{\Delta_2}\right| y^2 - |\Delta_2|\, z^2 + 2a_{01}x + a_{00} = 0, \tag{40}$$

wherefrom we derive two possibilities:

6.1.

$$a_{01} \neq 0 \Rightarrow \left|\frac{\Delta_1}{\Delta_2}\right| y^2 - |\Delta_2|\, z^2 + 2a_{01}x = 0, \quad a_{01} = \sqrt{-\frac{\Delta}{\Delta_1}}. \tag{41}$$

A surface defined by:

6.1.1. $\Delta_2 \neq 0, \Delta_1 < 0, \Delta_0 = 0$, and $\Delta > 0$, will be called a ***nonisotropic hyperbolical paraboloid of the β subfamily.***

6.2.

$$a_{01} = 0 \Rightarrow \left|\frac{\Delta_1}{\Delta_2}\right| y^2 - |\Delta_2|\, z^2 + a_{00} = 0, \tag{42}$$

wherefrom it follows that $\Delta = 0$, and $a_{00} = \dfrac{D_{11}}{\Delta_1}$. A surface defined by:

6.2.1. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 = 0$, and $\Delta = 0$, $D_{11} \neq 0$, will be called a ***nonisotropic hyperbolical cylinder of the β subfamily***;

6.2.2. $\Delta_2 \neq 0$, $\Delta_1 < 0$, $\Delta_0 = 0$, and $\Delta = 0$, $D_{11} = 0$, will be called a ***pair of planes with a nonisotropic cross-section of the β subfamily***.

Ad (ii) Under the conditions $\Delta_2 \neq 0$, and $\Delta_1 = 0$, it is always possible to reduce the conic equation (21) by isotropic motions into the form

$$k_\omega \equiv \Delta_2 \eta^2 + 2a_{12}\xi = 0, \text{ or} \tag{43}$$

$$k_\omega \equiv \Delta_2 \eta^2 + a_{11} = 0, \tag{44}$$

depending on Δ_0 being different from or equal to zero. The remaining coefficients $a_{12} = \sqrt{-\dfrac{\Delta_0}{\Delta_2}}$, and $a_{11} = \dfrac{\Delta_{22}}{\Delta_2}$. Following the earlier described procedure, all quadrics in $I_3^{(2)}$ related to k_ω given by equation (43) or (44) in the affine space coordinates, are of the form

$$\Delta_2 z^2 + 2\sqrt{-\frac{\Delta_0}{\Delta_2}}xy + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0, \quad \text{or} \tag{45}$$

$$\frac{\Delta_{22}}{\Delta_2}x^2 + \Delta_2 z^2 + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0. \tag{46}$$

Ad 7) $\mathbf{k}_\omega$ is a parabola. With

$$\Delta_2 \neq 0, \Delta_1 = 0, \Delta_0 \neq 0, \tag{47}$$

(43) represents an equation of the parabola. The quadric equation (45), using translations of the coordinate system in the direction of the x, y, and $z-$axes, is reduced to

$$\Delta_2 z^2 + 2\sqrt{-\frac{\Delta_0}{\Delta_2}}xy + 2a_{00} = 0, \quad a_{00} = \frac{\Delta}{\Delta_0} \tag{48}$$

wherefrom three surfaces can be distinguished:

7.1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta > 0$ will be called a ***nonisotropic one sheet hyperboloid of the β subfamily***;

7.2. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta < 0$ will be called a ***nonisotropic two-sheet hyperboloid of the β subfamily***;

7.3. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_0\Delta_2 < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, and $\Delta = 0$ will be called a ***nonisotropic real cone of the β subfamily.***

Ad 8) $\mathbf{k}_\omega$ is a pair of parallel lines. From

$$\Delta_2 \neq 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} < 0, \tag{49}$$

it follows that (44) represents an equation of a pair of parallel lines. The equation of the associated quadrics can be reduced to

$$\left|\frac{\Delta_{22}}{\Delta_2}\right| x^2 - |\Delta_2|\, z^2 + 2a_{02}y + a_{00} = 0, \tag{50}$$

wherefrom we derive two possibilities:

8.1.
$$a_{02} \neq 0 \Rightarrow \left|\frac{\Delta_{22}}{\Delta_2}\right| x^2 - |\Delta_2|\, z^2 + 2a_{02}y = 0, \quad a_{02} = \sqrt{-\frac{\Delta}{\Delta_{22}}} \tag{51}$$

Surface (51) defined by:

8.1.1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta > 0$, will be called an ***isotropic hyperbolical paraboloid of the β subfamily.***

8.2.
$$a_{02} = 0 \Rightarrow \left|\frac{\Delta_{22}}{\Delta_2}\right| x^2 - |\Delta_2|\, z^2 + a_{00}y = 0. \tag{52}$$

It is easy to be seen that $\Delta = 0$, and that $a_{00} = \dfrac{D_{22}}{\Delta_{22}}$, with D_{22} being invariant under isotropic motions *(see case 3.2.)*. A surface defined by:

8.2.1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta = 0$, $D_{22} \neq 0$, will be called an ***isotropic hyperbolical cylinder of the β subfamily;***

8.2.2. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta = 0$, $D_{22} = 0$, will be called a ***pair of planes with an isotropic cross-section of the*** β ***subfamily***.

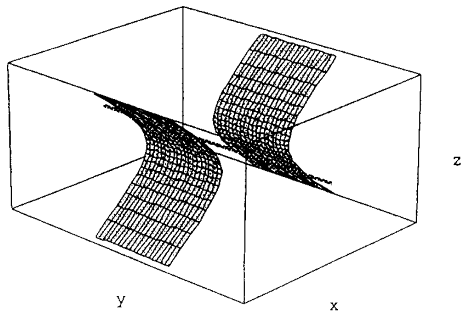

Figure 5. *Isotropic hyperbolical cylinder of the* β *subfamily*

Ad 9) k_ω is an imaginary pair of parallel lines

From

$$\Delta_2 \neq 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} > 0, \tag{53}$$

it follows that (44) represents an equation of a conic from the title. The associated quadric equation can be reduced to

$$\frac{\Delta_{22}}{\Delta_2}x^2 + \Delta_2 z^2 + 2a_{02}y + a_{00} = 0, \tag{54}$$

wherefrom similarly to the previous case we derive two possibilities:

9.1.

$$a_{02} \neq 0 \Rightarrow \frac{\Delta_{22}}{\Delta_2}x^2 + \Delta_2 z^2 + 2a_{02}y = 0, \quad a_{02} = \sqrt{-\frac{\Delta}{\Delta_{22}}}. \tag{55}$$

Surface (55) defined by:

9.1.1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} > 0$, and $\Delta < 0$, will be called an ***isotropic elliptical paraboloid of the*** β ***subfamily.***

9.2.

$$a_{02} = 0 \Rightarrow \frac{\Delta_{22}}{\Delta_2}x^2 + \Delta_2 z^2 + a_{00} = 0, \tag{56}$$

with $\Delta = 0$, and $a_{00} = \frac{D_{22}}{\Delta_{22}}$. Surface (56) defined by:

9.2.1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} > 0$, and $\Delta = 0$, $(\Gamma < 0 \vee D_{22}\alpha_1 < 0)$, will be called an ***isotropic elliptical cylinder of the*** β ***subfamily***;

9.2.2. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} > 0$, and $\Delta = 0$, $D_{22}\Delta_2 > 0$, will be called an ***isotropic imaginary cylinder of the β subfamily***;

9.2.3. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} > 0$, and $\Delta = 0$, $D_{22}\Delta_2 = 0$, will be called an ***imaginary pair of planes with an isotropic cross-section of the β subfamily.***

Ad 10) $\mathbf{k}_\omega$ consists of two coinciding lines. Choosing

$$\Delta_2 \neq 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} = 0, \tag{57}$$

(44) represents an equation of a conic from the title. The equation of the joined quadrics can be reduced to

$$\Delta_2 z^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0, \quad a_{01} = \sqrt{-\frac{D_{22}}{\Delta_2}}, \quad a_{02} = \sqrt{-\frac{D_{11}}{\Delta_2}}. \tag{58}$$

On the other hand, on the assumption that in this case the surface equation disjoints in two linear factors, compared with (58), $a_{01} = a_{02} = 0$ is derived. Therefore, the surface equation is of the form

$$\Delta_2 z^2 + a_{00} = 0, \quad a_{00} = \frac{\gamma_{03}}{\Delta_2}, \tag{59}$$

where the sign of γ_{03} is invariant. Three surfaces can be distinguished:

10.1 $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, and $\Delta = 0$, $D_{11} = D_{22} = 0$, $\gamma_{03} < 0$, will be called a ***pair of nonisotropic parallel planes of the β subfamily***;

10.2. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, and $\Delta = 0$, $D_{11} = D_{22} = 0$, $\gamma_{03} > 0$, will be called an ***imaginary pair of nonisotropic parallel planes of the β subfamily***;

10.3. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, and $\Delta = 0$, $D_{11} = D_{22} = 0$, $\gamma_{03} = 0$, will be called ***two coinciding nonisotropic planes of the β subfamily.***

Here follows the proof that the sign of γ_{03} is invariant under isotropic motions. From (8) and (59) it follows that

$$\begin{aligned}\Gamma_0 = \gamma_{01} + \gamma_{02} + \gamma_{03} &= \begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{02} \\ a_{02} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{03} \\ a_{03} & a_{33} \end{vmatrix} \\ &= \begin{vmatrix} a_{00} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{00} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{00} & 0 \\ 0 & a_{33} \end{vmatrix} \\ &= \gamma_{03}.\end{aligned}$$

On the other hand, according to (11), we obtain

$$\begin{aligned}\overline{\Gamma}_0 = \overline{\gamma}_{01} + \overline{\gamma}_{02} + \overline{\gamma}_{03} &= \begin{vmatrix} \overline{a}_{00} & \overline{a}_{01} \\ \overline{a}_{01} & \overline{a}_{11} \end{vmatrix} + \begin{vmatrix} \overline{a}_{00} & \overline{a}_{02} \\ \overline{a}_{02} & \overline{a}_{22} \end{vmatrix} + \begin{vmatrix} \overline{a}_{00} & \overline{a}_{03} \\ \overline{a}_{03} & \overline{a}_{33} \end{vmatrix} \\ &= \begin{vmatrix} a_{00} & 0 \\ 0 & b^2 a_{33} \end{vmatrix} + \begin{vmatrix} a_{00} & 0 \\ 0 & c^2 a_{33} \end{vmatrix} + \begin{vmatrix} a_{00} & 0 \\ 0 & a_{33} \end{vmatrix} \\ &= \cdots = \Gamma(b^2 + c^2 + 1) = \gamma_{03}(b^2 + c^2 + 1).\end{aligned}$$

6.2. k_ω belongs to the 2^{nd} family

Since k_ω belongs to the 2^{nd} family, with respect to the conics isometric invariants, the following conditions are fulfilled:

$$\Delta_2 = 0, \Delta_1 < 0. \tag{60}$$

Under such conditions, using isotropic motions it is always possible to reduce the conic equation (20) into

$$k_\omega \equiv 2a_{23}\xi\eta + a_{11} = 0, \tag{61}$$

where $a_{11} = \dfrac{\Delta_0}{\Delta_1}$ and $a_{23} = \sqrt{-\Delta_1}$. In the affine space coordinates:

$$k_\omega = a_{11}x^2 + 2a_{23}yz = 0. \tag{62}$$

All quadrics having (62) as the cross-section with the plane of infinity ω could be written in the form

$$a_{11}x^2 + 2a_{23}yz + 2a_{01}x + 2a_{02}y + a_{03}z + a_{00} = 0. \tag{63}$$

The possibilities are the following:

Ad 11) k$_\omega$ is a special hyperbola. With

$$\Delta_2 = 0, \Delta_1 < 0, \Delta_0 \neq 0, \tag{64}$$

(61) is an equation of a special hyperbola. Using translations in the direction of the x, y, and $z-$axes, the quadric equation given in (63) can be reduced to

$$\frac{\Delta_0}{\Delta_1}x^2 + 2\sqrt{-\Delta_1}yz + a_{00} = 0, \quad a_{00} = \frac{\Delta}{\Delta_0}. \tag{65}$$

The surface defined by:

11.1. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, and $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta > 0$, will be called an ***one-sheet hyperboloid of the second family*** (isotropic and nonisotropic);

11.2. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, and $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta < 0$, will be called an ***two-sheet hyperboloid of the second family*** (isotropic and nonisotropic);

11.3. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 \neq 0$, and $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta = 0$, will be called a ***real cone of the second family*** (isotropic and nonisotropic).

Ad 12) k$_\omega$ is a pair of lines, one being an isotropic line. Choosing

$$\Delta_2 = 0, \Delta_1 < 0, \Delta_0 = 0, \tag{66}$$

it follows that (61) represents an equation of a conic from the title. The quadric equation (63) is reduced to

$$2\sqrt{-\Delta_1}yz + 2a_{01}x + a_{00} = 0, \tag{67}$$

wherefrom we derive two possibilities:

12.1.

$$a_{01} \neq 0 \Rightarrow 2\sqrt{-\Delta_1}yz + 2a_{01}x = 0, \quad a_{00} = \sqrt{-\frac{\Delta}{\Delta_1}}. \tag{68}$$

Surface (68) defined by:

12.1.1. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 = 0$, and $\Delta > 0$, will be called a ***nonisotropic hyperbolical paraboloid of the second family***.

12.2.

$$a_{01} \neq 0 \Rightarrow 2\sqrt{-\Delta_1}yz + 2a_{00} = 0, \quad a_{00} = \frac{D_{11}}{\Delta_1}. \tag{69}$$

The surface defined by:

12.2.1. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 = 0$, and $\Delta = 0$, $D_{11} \neq 0$, will be called a ***nonisotropic hyperbolical cylinder of the second family***;

12.2.2. $\Delta_2 = 0$, $\Delta_1 < 0$, $\Delta_0 = 0$, and $\Delta = 0$, $D_{11} = 0$, will be called a ***pair of planes with a nonisotropic cross-section of the second family***.

6.3. k_ω belongs to the 3^{rd} family

If k_ω belongs to the 3^{rd} family, the following conditions are fulfilled:

$$\Delta_2 = 0, \Delta_1 = 0. \tag{70}$$

In addition, on the assumption that $a_{22} \neq 0$, using isotropic motions, it is always possible to reduce the conic equation (21) into the form

$$k_\omega \equiv a_{22}\xi^2 + 2a_{13}\eta = 0, \text{ or} \tag{71}$$

$$k_\omega \equiv a_{22}\xi^2 + a_{11} = 0, \tag{72}$$

depending on Δ_0 being different from or equal to zero. Following the earlier described procedure, all quadrics in $I_3^{(2)}$ related to k_ω given by equation (71) or (72) in the affine space coordinates, are of the form

$$a_{22}y^2 + 2a_{13}xz + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0, \text{ or} \tag{73}$$

$$a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0. \tag{74}$$

Ad 13) k_ω is a parabolic circle. With

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 \neq 0, \tag{75}$$

(71) is an equation of a parabolic circle. Using translations of the coordinate system in the direction of the x, y, and $z-$axes, the quadric equation (73) can be reduced to

$$a_{22}y^2 + 2a_{13}xz + a_{00} = 0, \tag{76}$$

where $a_{22} = -\frac{\Delta_0}{\sqrt{-\Delta_{22}}}$, $a_{13} = \sqrt{-\Delta_{22}}$, and $a_{00} = \frac{\Delta}{\Delta_0}$. The surface defined by:

13.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_{22} < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta > 0$, will be called a ***nonisotropic one-sheet hyperboloid of the third family***;

13.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_{22} < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta < 0$, will be called a textbf*nonisotropic two-sheet hyperboloid of the third family*;

13.3. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 \neq 0$, and $\Delta_{22} < 0$, $(\Gamma_1 < 0 \vee \Delta_0\alpha_0 < 0)$, $\Delta = 0$, will be called a ***real nonisotropic cone of the third family***.

Ad 14) $\mathbf{k}_\omega$ is a pair of isotropic lines. With

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{33} < 0, \tag{77}$$

it follows that (72) represents an equation of the conic from the title. Under the above conditions the quadric equation (75) can be reduced to

$$|a_{11}|x^2 - |a_{22}|y^2 + 2a_{03}z + a_{00} = 0, \tag{78}$$

wherefrom we derive two possibilities:

14.1.

$$a_{03} \neq 0 \Rightarrow |a_{11}|x^2 - |a_{22}|y^2 + 2a_{03}z = 0, \quad a_{03} = \sqrt{-\frac{\Delta}{\Delta_{33}}}, \quad \Delta_{33} = a_{11}a_{22}. \tag{79}$$

Surface (80) defined by:

14.1.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} < 0$, and $\Delta > 0$, will be called a ***double isotropic hyperbolical paraboloid of the third family***.

14.2.

$$a_{03} = 0 \Rightarrow |a_{11}|x^2 - |a_{22}|y^2 + a_{00} = 0, a_{00} = \frac{D_{33}}{\Delta_{33}}, \Delta_{33} = a_{11}a_{22}. \tag{80}$$

The possibilities are:

14.2.1 $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} < 0$, and $\Delta = 0$, $D_{33} \neq 0$, will be called a ***double isotropic hyperbolical cylinder of the third family***;

14.2.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} < 0$, and $\Delta = 0$, $D_{33} = 0$, will be called a ***pair of planes with a double isotropic cross-section of the third family***.

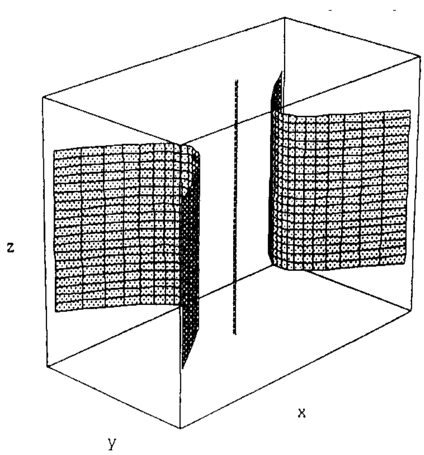

Figure 6. *Double isotropic hyperbolical cylinder of the third family*

Ad 15) $\mathbf{k}_\omega$ is an imaginary pair of isotropic lines With

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{33} > 0, \tag{81}$$

it follows that (72) represents an equation of the conic from the title. Similarly to the above case, the quadric equation (75) can be reduced to

$$a_{11}x^2 + a_{22}y^2 + 2a_{03}z + a_{00} = 0, \tag{82}$$

and the possibilities are:

15.1.

$$a_{03} \neq 0 \Rightarrow a_{11}x^2 + a_{22}y^2 + 2a_{03}z = 0, a_{03} = \sqrt{-\frac{\Delta}{\Delta_{33}}}, \Delta_{33} = a_{11}a_{22}. \tag{83}$$

A surface defined by:

15.1.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} > 0$, and $\Delta < 0$, will be called a ***double isotropic elliptical paraboloid of the third family***.

15.2.

$$a_{03} = 0 \Rightarrow a_{11}x^2 + a_{22}y^2 + a_{00} = 0, a_{00} = \frac{D_{33}}{\Delta_{33}}, \Delta_{33} = a_{11}a_{22}. \tag{84}$$

On the assumption of (81) D_{33} is invariant.

15.2.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} > 0$, and $\Delta = 0$, $(\Gamma < 0 \vee \alpha_1 D_{33} < 0)$, will be called a ***double isotropic elliptical cylinder of the third family***;

15.2.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} > 0$, and $\Delta = 0$, $a_{22}D_{33} > 0$, will be called a ***double isotropic imaginary cylinder of the third family***;

15.2.3. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} > 0$, and $\Delta = 0$, $a_{22}D_{33} = 0$, will be called a ***pair of imaginary planes with a double isotropic cross-section of the third family***.

On case 15.2.2 we are going to demonstrate that the conditions on the invariants characterize the types unequally.

First, let us prove that $D_{33} = D_{11} + D_{22} + D_{33} = \overline{D}_{11} + \overline{D}_{22} + \overline{D}_{33}$:

$$\overline{D}_{11} = \begin{vmatrix} \overline{a}_{00} & \overline{a}_{02} & \overline{a}_{03} \\ \overline{a}_{02} & \overline{a}_{22} & \overline{a}_{23} \\ \overline{a}_{03} & \overline{a}_{23} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} = D_{11} = 0,$$

$$\overline{D}_{22} = \begin{vmatrix} \overline{a}_{00} & \overline{a}_{01} & \overline{a}_{03} \\ \overline{a}_{01} & \overline{a}_{11} & \overline{a}_{13} \\ \overline{a}_{03} & \overline{a}_{13} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} + a^2 a_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0, D_{22} = \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$\overline{D}_{33} = \begin{vmatrix} \overline{a}_{00} & \overline{a}_{01} & \overline{a}_{02} \\ \overline{a}_{01} & \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{02} & \overline{a}_{12} & \overline{a}_{22} \end{vmatrix} = \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} + a^2 a_{22} & a a_{22} \\ 0 & a a_{22} & a_{22} \end{vmatrix} = a_{00}a_{11}a_{22},$$

$$D_{33} = \begin{vmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{00}a_{11}a_{22}.$$

We will also need $\Delta_{33} > 0 \Rightarrow \Delta_{11} + \Delta_{22} + \Delta_{33} > 0$, i.e. $\overline{\Delta}_{11} + \overline{\Delta}_{22} + \overline{\Delta}_{33} > 0$:

$$\overline{\Delta}_{11} = \begin{vmatrix} \overline{a}_{22} & \overline{a}_{23} \\ \overline{a}_{23} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & 0 \\ 0 & 0 \end{vmatrix} = \Delta_{11} = 0,$$

$$\overline{\Delta}_{22} = \begin{vmatrix} \overline{a}_{11} & \overline{a}_{13} \\ \overline{a}_{13} & \overline{a}_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + a^2 a_{22} & 0 \\ 0 & 0 \end{vmatrix} = 0, \Delta_{22} = \begin{vmatrix} a_{11} & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$\overline{\Delta}_{33} = \begin{vmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{12} & \overline{a}_{22} \end{vmatrix} = \begin{vmatrix} a_{11} + a^2 a_{22} & a a_{22} \\ a a_{22} & a_{22} \end{vmatrix} = a_{11}a_{22}, \Delta_{33} = \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}.$$

Finally, $\Delta_2 = 0 \& \Delta_{33} > 0 \& a_{22}D_{33} > 0 \Rightarrow (D_{11}+D_{22}+D_{33})(a_{11}+a_{22}+a_{00}) > 0$, i.e. $(\overline{D}_{11} + \overline{D}_{22} + \overline{D}_{33})(\overline{a}_{11} + \overline{a}_{22} + \overline{a}_{00}) > 0$:

With $\Delta_{33} = a_{11}a_{22} > 0 \Rightarrow \begin{cases} a_{11}, a_{22} > 0, \text{ or} \\ a_{11}, a_{22} < 0 \end{cases}$, $\Delta_2 = a_{33} = 0$, $a_{22}D_{33} > 0$, we obtain

$$(D_{11} + D_{22} + D_{33})(a_{11} + a_{22} + a_{33}) = D_{33}(a_{11} + a_{22} + a_{00}) > 0, \text{ and}$$
$$(\overline{D}_{11} + \overline{D}_{22} + \overline{D}_{33})(\overline{a}_{11} + \overline{a}_{22} + \overline{a}_{33}) = D_{33}(a_{11} + a_{22}(a^2 + 1) + a_{00}) > 0.$$

Presuming now conditions 15.2.2, and taking into consideration what has been proved above, we derive:

(i) With $\Delta_2 = \Delta_1 = 0$, a surface belongs to the third family of quadrics;

(ii) With $\Delta = \Delta_0 = 0$, $\Delta_{33} > 0$, and $a_{22}D_{33} > 0$, according to the affine classification of second order surfaces, it is an imaginary cylinder;

(iii) (i) & (ii) $\Rightarrow$ the longitudinal axis is a double isotropic straight line.

All other cases can be checked in a similar way.

Ad 16) $\mathbf{k}_\omega$ consists of two coinciding isotropic lines. With

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{33} = 0, a_{22} \neq 0, \tag{85}$$

(72) represents an equation of a conic from the title. After being translated in the direction of the y axis the equation of the associated quadrics becomes

$$a_{22}y^2 + 2a_{01}x + 2a_{03}z + a_{00} = 0, a_{01} = \sqrt{-\frac{D_{33}}{a_{22}}}, a_{03} = \sqrt{-\frac{D_{11}}{a_{22}}}. \tag{86}$$

On the other hand, on the assumption that in this case the surface equation disjoints in two linear factors, compared with (86), $a_{01} = a_{03} = 0$ is derived. Therefore, it is of the form

$$a_{22}y^2 + a_{00} = 0, \quad a_{22}a_{00} = \gamma_{02}. \tag{87}$$

Three surfaces can be distinguished:

16.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} = 0$, $a_{22} \neq 0$, and, $\Delta = D_{11} = D_{33} = 0$, $\gamma_{02} < 0$, will be called a ***real pair of isotropic parallel planes of the third family***;

16.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} = 0$, $a_{22} \neq 0$, and, $\Delta = D_{11} = D_{33} = 0$, $\gamma_{02} > 0$, will be called an ***imaginary pair of isotropic parallel planes of the third family***;

16.3. $\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{33} = 0, a_{22} \neq 0$, and, $\Delta = D_{11} = D_{33} = 0, \gamma_{02} = 0$, will be called ***two coinciding isotropic planes of the third family***.

6.4. k_ω belongs to the 4^{th} family

k_ω being of the 4^{th} family means that the following conditions are fulfilled:

$$\Delta_2 = 0, \Delta_1 = 0. \tag{88}$$

Besides, on the assumption that $a_{22} = 0$, using isotropic motions, it is always possible to reduce the conic equation (21) to the form

$$k_\omega \equiv 2a_{12}\xi + 2a_{13}\eta + a_{11} = 0, \tag{89}$$

wherefrom it follows that the conic consists of two straight lines, one being an absolute line f. In the affine space coordinates we have

$$k_\omega \equiv x\,(a_{11}x + 2a_{12}y + 2a_{13}z) = a_{11}x^2 + 2a_{12}xy + 2a_{13}xz = 0. \tag{90}$$

All quadrics having (90) as the cross-section with the plane of infinity ω are of the following form:

$$a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + 2a_{01}x + 2a_{02}y + 2a_{03}z + a_{00} = 0. \qquad (91)$$

Ad 17) $\mathbf{k}_\omega$ consists of a nonisotropic line + an absolute line f. Choosing

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} < 0, \qquad (92)$$

we obtain that (89) represents the conic from the title. Using translations in the direction of the x, and $z-$axes, (91) can be reduced to

$$2a_{13}xz + 2a_{02}y + a_{00} = 0. \qquad (93)$$

There are two possibilities:

17.1

$$a_{02} \neq 0 \Rightarrow a_{13}xz + a_{02}y = 0, \quad a_{13} = \sqrt{-\Delta_{22}}, \quad a_{02} = \sqrt{-\frac{\Delta}{\Delta_{22}}}. \qquad (94)$$

Surface (94) defined by:

17.1.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta > 0$, will be called an ***isotropic hyperbolical paraboloid of the fourth family***.

17.2.

$$a_{02} = 0 \Rightarrow 2a_{13}xz + a_{00} = 0, \quad a_{13} = \sqrt{-\Delta_{22}}, \quad a_{00} = \frac{D_{22}}{\Delta_{22}}. \qquad (95)$$

From (95) two surfaces can be distinguished:

17.2.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta = 0$, $D_{22} \neq 0$, will be called an ***isotropic hyperbolical cylinder of the fourth family***;

17.2.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} < 0$, and $\Delta = 0$, $D_{22} = 0$, will be called a ***pair of planes, one being a 2-isotropic plane, with an isotropic cross-section of the fourth family***.

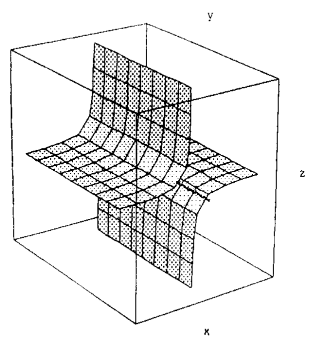

Figure 7. *Isotropic hyperbolical cylinder of the fourth family*

Ad 18) k_ω consists of an isotropic line + an absolute line f. With

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} = 0, a_{22} = 0, \Delta_{33} < 0, \tag{96}$$

it follows that (89) represents an equation of the conic from the title. Under the above conditions, and after translations in the direction of the x and $y-$axes have been performed, the quadric equation (91) becomes as follows:

$$2a_{12}xy + 2a_{03}z + a_{00} = 0. \tag{97}$$

There are two possibilities:

18.1.

$$a_{03} \neq 0 \Rightarrow a_{12}xy + a_{03}z = 0, \quad a_{13} = \sqrt{-\Delta_{33}}, \quad a_{03} = \sqrt{-\frac{\Delta}{\Delta_{33}}}. \tag{98}$$

Surface (98) defined by:

18.1.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, $a_{22} = 0$, $\Delta_{33} < 0$, and $\Delta > 0$, will be called a ***double isotropic hyperbolical paraboloid of the fourth family***.

18.2.

$$a_{03} = 0 \Rightarrow 2a_{12}xy + a_{00} = 0, \quad a_{13} = \sqrt{-\Delta_{33}}, \quad a_{00} = \frac{D_{33}}{\Delta_{33}}. \tag{99}$$

We have:

18.2.1. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, $a_{22} = 0$, $\Delta_{33} < 0$, and $\Delta = 0$, $D_{33} \neq 0$, will be called a ***double isotropic hyperbolical cylinder of the fourth family***;

18.2.2. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, $a_{22} = 0$, $\Delta_{33} < 0$, and $\Delta = 0$, $D_{33} = 0$, will be called a ***pair of planes,*** 1 ***and*** 2***-isotropic, with a double isotropic cross-section of the fourth family***.

Ad 19) k_ω is a double absolute line f. Choosing

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} = 0, \Delta_{33} = 0, a_{22} = 0, a_{11} \neq 0, \tag{100}$$

k_ω is a double absolute line f. After being translated in the direction of the $x-$axis, the equation of the associated quadrics becomes of the following form:

$$a_{11}x^2 + 2a_{02}y + 2a_{03}z + a_{00} = 0, a_{02} = \sqrt{-\frac{D_{33}}{a_{11}}}, a_{03} = \sqrt{-\frac{D_{22}}{a_{11}}}. \tag{101}$$

On the other hand, on the assumption that in this case, similarly to cases 10) and 16), the surface equation disjoints in two linear factors, and therefore $a_{02} = a_{03} = 0$ must be fulfilled. The surface equation becomes:

$$a_{11}x^2 + a_{00} = 0, \quad a_{11}a_{00} = \gamma_{01}. \tag{102}$$

Three surfaces can be distinguished:

19.1. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, and $\Delta = D_{22} = D_{33} = 0$, $\gamma_{01} < 0$, will be called a ***real pair of 2-isotropic planes of the fourth family***;

19.2. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, and $\Delta = D_{22} = D_{33} = 0$, $\gamma_{01} > 0$, will be called an ***imaginary pair of 2-isotropic planes of the fourth family***;

19.3. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, and $\Delta = D_{22} = D_{33} = 0$, $\gamma_{01} = 0$, will be called ***two coinciding* 2*-isotropic planes of the fourth family***.

Ad 20) All points in the isotropic plane. Choosing

$$\Delta_2 = 0, \Delta_1 = 0, \Delta_0 = 0, \Delta_{22} = 0, \Delta_{33} = 0, a_{22} = 0, a_{11} = 0, \tag{103}$$

it follows that k_ω consists of all points in I_2. In the homogenous coordinates and under the above conditions the quadric equation (91) is of the form

$$x_0(a_{00}x_0 + 2a_{01}x_1 + 2a_{02}x_2 + 2a_{03}x_3) = 0. \tag{104}$$

There are four possibilities for this type of surfaces:

20.1. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = a_{11} = 0$, and $\Delta = 0$, $\gamma_{03} \neq 0$, will be called a ***pair of planes:*** ω ***+ a nonisotropic plane***;

20.2. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = a_{11} = 0$, and $\Delta = 0$, $\gamma_{03} = 0$, $\gamma_{02} \neq 0$, will be called a ***pair of planes:*** ω ***+ a* 1*-isotropic plane***;

20.3. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = a_{11} = 0$, and $\Delta = 0$, $\gamma_{03} = 0$, $\gamma_{02} = 0$, $\gamma_{01} \neq 0$, will be called a ***pair of planes:*** ω ***+ a* 2*-isotropic plane***;

20.4. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = a_{11} = 0$, and $\Delta = 0$, $\Gamma = 0$, $a_{00} \neq 0$, will be called a ***double absolute plane*** ω.

6.5. k_ω is a pair $\{S, G\}$, $S \in g$

We must also take into consideration the case when k_ω is of the type $\{S, g\}$, where g is a double straight line in the isotropic plane I_2, and S a point on it. Second order surfaces having such a cross-section with the plane of infinity ω are parabolic cylinders.

Ad 1) k_ω **is a double nonisotropic line** g. It follows from *Section 6.1* that choosing $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, and $\Delta_{22} = 0$, relation $k_\omega \equiv \eta^2 = 0$ represents an equation of a double nonisotropic line. All quadrics having such a cross-section with the plane of infinity could be written as $z^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0$.
For the point $S \in g$ there are two possibilities:

a) $S \notin f$; and

b) $S \in f \setminus \{F\}$.

Ad a)

$$S \notin f, S(0:1:0:0) \Rightarrow a_{01} = 0, a_{02} \neq 0 \Rightarrow z^2 + 2a_{02}y = 0. \tag{105}$$

The surface defined by:

1. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, and $\Delta = 0$, $D_{11} \neq 0$, will be called a ***nonisotropic parabolic cylinder of the β subfamily***.

Ad b)

$$S \in f \setminus \{F\}, S(0:1:0:0) \Rightarrow a_{01} \neq 0, a_{02} = 0 \Rightarrow z^2 + 2a_{01}x = 0. \tag{106}$$

The surface defined by:

2. $\Delta_2 \neq 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{22} = 0$, and $\Delta = 0$, $D_{11} = 0$, $D_{22} \neq 0$ will be called an ***isotropic parabolic cylinder of the β subfamily***.

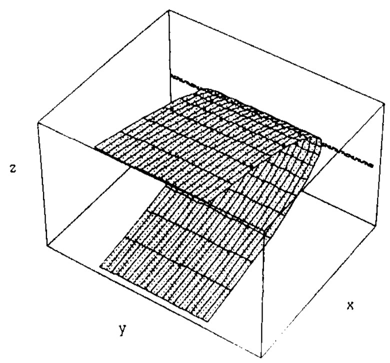

Figure 8. *Isotropic parabolic cylinder of the β subfamily*

Ad 2) k_ω is a double isotropic line g. It follows from *Section 6.3* that choosing $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$,$\Delta_{33} = 0$, and $a_{22} \neq 0$ relation $k_\omega \equiv \xi^2 = 0$ represents an equation of a double isotropic line. All quadrics having such a cross-section with the plane of infinity could be written in the form $y^2 + 2a_{01}x + 2a_{03}z + a_{00} = 0$.

For the point $S \in g$ there are two possibilities:

a) $S \notin f$; and

b) $S = F(0:0:0:1)$.

Ad a)

$$S \notin f, S(0:1:0:0) \Rightarrow a_{01} = 0, a_{03} \neq 0 \Rightarrow y^2 + 2a_{03}z = 0. \tag{107}$$

Surface (107) defined by:

3. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} = 0$, $a_{22} \neq 0$, and $\Delta = 0$, $D_{11} \neq 0$, will be called a ***nonisotropic parabolic cylinder of the third family***.

Ad b)

$$S \in f \setminus \{F\}, S(0:0:0:1) \Rightarrow a_{01} \neq 0, a_{03} = 0 \Rightarrow y^2 + 2a_{01}x = 0. \tag{108}$$

Surface (88) defined by:

4. $\Delta_2 = 0$, $\Delta_1 = 0$, $\Delta_0 = 0$, $\Delta_{33} = 0$, $a_{22} \neq 0$, and $\Delta = 0$, $D_{11} = 0$, $D_{33} \neq 0$ will be called a ***double isotropic parabolic cylinder of the third family***.

Ad 3) k_ω is a double absolute line $f = g$. It follows from *Section 6.3* that choosing $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, relation $k_\omega \equiv x_1^2 = 0$ represents an equation of a double absolute line f. All quadrics having such a cross-section with the plane of infinity could be written as $x^2 + 2a_{02}y + 2a_{03}z + a_{00} = 0$. For the point $S \in g$ there are two possibilities:

a) $S \in f \setminus \{F\}$; and

b) $S = F(0:0:0:1)$.

Ad a)

$$S \in f \setminus \{F\}, S(0:0:1:0) \Rightarrow a_{02} = 0, a_{03} \neq 0 \Rightarrow x^2 + 2a_{03}z = 0. \tag{109}$$

Surface (109) defined by:

5. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, and $\Delta = 0$, $D_{22} \neq 0$, will be called an ***isotropic parabolic cylinder of the fourth family***.

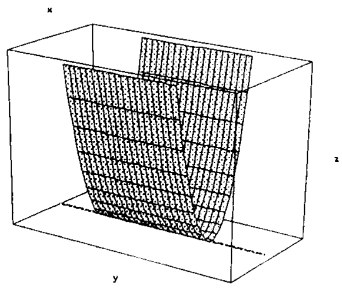

Figure 9. *Isotropic parabolic cylinder of the fourth family*

Ad b)

$$S = F(0:0:0:1) \Rightarrow a_{02} \neq 0, a_{03} = 0 \Rightarrow x^2 + 2a_{02}y = 0. \tag{110}$$

Surface (110) defined by:

6. $\Delta_2 = \Delta_1 = \Delta_0 = \Delta_{22} = \Delta_{33} = a_{22} = 0$, $a_{11} \neq 0$, and $\Delta = 0$, $D_{22} = 0$, $D_{33} \neq 0$, will be called a ***double isotropic parabolic cylinder of the fourth family***.

Finally, we have

Proposition 4. *In the double isotropic space* $I_3^{(2)}$ *there are* 70 *different second order surfaces.*

In the conclusion we should point out that the given approach could be understood as an example of classifying quadratic forms in the spaces of various dimensions having no regular metric.

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