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Growth and oscillation related to a second order linear differential equation

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Abstract. This paper is devoted to studying the growth and the oscillation of solutions of the second order non-homogeneous linear differential equation

$$f'' + A_1(z)\,e^{P(z)}f' + A_0(z)\,e^{Q(z)}f = F,$$

where $P(z)$, $Q(z)$ are nonconstant polynomials such that $\deg P = \deg Q = n$ and $A_j(z)$ $(\not\equiv 0)$ $(j = 0,1)$, $F(z)$ are entire functions with $\max\{\rho(A_j) : j = 0,1\} < n$. We also investigate the relationship between small functions and differential polynomials $g_f(z) = d_2 f'' + d_1 f' + d_0 f$, where $d_0(z), d_1(z), d_2(z)$ are entire functions such that at least one of d_0, d_1, d_2 does not vanish identically with $\rho(d_j) < n (j = 0,1,2)$ generated by solutions of the above equation.

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1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see, [9], [16]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f.

To give estimates of fixed points, we define:

Definition 1 (see [4, 11, 13])**.** *Let f be a meromorphic function and let $z_1, z_2, \cdots$, $(|z_j| = r_j,\ 0 < r_1 \leqslant r_2 \leqslant \cdots)$ be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by*

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$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f-z}\right)$ *is the counting function of distinct fixed points of* $f(z)$ *in* $\{z : |z| < r\}$.

In [6], Chen has investigated the second order linear differential equation

$$f'' + A_1(z)\, e^{P(z)} f' + A_0(z)\, e^{Q(z)} f = 0, \tag{1}$$

and has obtained the following result.

Theorem 1 (see [6]). *Let*

$$P(z) = \sum_{i=0}^{n} a_i z^i \text{ and } Q(z) = \sum_{i=0}^{n} b_i z^i$$

be nonconstant polynomials where a_i, b_i $(i = 0, 1, \cdots, n)$ *are complex numbers,* $a_n b_n \neq 0$, *let* $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ *be entire functions. Suppose that either (i) or (ii) below, holds:*

(*i*) $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$, $\rho(A_j) < n$ $(j = 0, 1)$;

(*ii*) $a_n = cb_n$ $(c > 1)$ *and* $deg(P - cQ) = m \geqslant 1$, $\rho(A_j) < m$ $(j = 0, 1)$.

Then every solution $f(z) \not\equiv 0$ *of* (1) *satisfies* $\rho_2(f) = n$.

In [1], the author and El Farissi have studied the relation between meromorphic functions of finite order and differential polynomials generated by meromorphic solutions of the second order linear differential equation (1) and have obtained the following result.

Theorem 2 (see [1]). *Let*

$$P(z) = \sum_{i=0}^{n} a_i z^i \text{ and } Q(z) = \sum_{i=0}^{n} b_i z^i$$

be nonconstant polynomials where a_i, b_i $(i = 0, 1, \cdots, n)$ *are complex numbers,* $a_n b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$ *and let* $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ *be meromorphic functions with* $\rho(A_j) < n$ $(j = 0, 1)$. *Let* $d_0(z), d_1(z), d_2(z)$ *be polynomials that are not all equal to zero,* $\varphi(z) \not\equiv 0$ *is a meromorphic function with finite order. If* $f(z) \not\equiv 0$ *is a meromorphic solution of* (1) *with* $\lambda(1/f) < \infty$, *then the differential polynomial* $g(z) = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\overline{\lambda}(g - \varphi) = \infty$.

Recently in [14], Wang and Laine have investigated the growth of solutions of some second order linear differential equations and have obtained.

Theorem 3 (see [14]). *Let $A_j(z)\ (\not\equiv 0)\ (j=0,1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j)\,(j=0,1),\rho(F)\}<1$, and let a,b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F$$

is of infinite order.

The present article may be understood as an extension and improvement of the recent article of the author and El Farissi [2]. The first main purpose of this paper is to study the growth and the oscillation of solutions of the second order non-homogeneous linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = F. \tag{2}$$

We obtain the following results.

Theorem 4. *Let*

$$P(z) = \sum_{i=0}^{n} a_i z^i \text{ and } Q(z) = \sum_{i=0}^{n} b_i z^i$$

be nonconstant polynomials where $a_i,b_i\ (i=0,1,\cdots,n)$ are complex numbers, $a_n b_n \times (a_n-b_n) \neq 0$. Let $A_j(z)\ (\not\equiv 0)\ (j=0,1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j)\ (j=0,1),\rho(F)\}<n$. Then every solution $f\not\equiv 0$ of equation (2) is of infinite order. Furthermore, if $F\not\equiv 0$, then every solution f of equation (2) satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty. \tag{3}$$

Remark 1. *If $\rho(F)\geqslant n$, then equation (2) can possess a solution of finite order. For instance, the equation*

$$f'' + e^{2z^n}f' - nz^{n-1}e^{z^n}f = \left(n^2z^{2n-2} - n(n-1)z^{n-2}\right)e^{-z^n} - nz^{n-1}$$

satisfies $\rho(F) = \rho\left(\left(n^2z^n - n(n-1)\right)z^{n-2}e^{-z^n} - nz^{n-1}\right) = n$ and has a finite order solution $f(z) = e^{-z^n} - 1$.

Theorem 5. *Let $P(z), Q(z), A_0(z), A_1(z)$ satisfy the hypotheses of Theorem 4, and let $F(z)$ be an entire function such that $\rho(F)\geqslant n$. Then every solution f of equation (2) satisfies (3) with at most one finite order solution f_0.*

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see, [17]). However, there are a few studies on the fixed points of solutions of differential equations. It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see, [4]). In [13], Wang and Yi investigated fixed points and hyper-order of differential polynomials generated by solutions of some second order linear differential equations. In [10], Laine and Rieppo gave an improvement of the results of [13] by considering fixed points and iterated order.

The second main purpose of this paper is to study the relation between small functions and some differential polynomials generated by solutions of second order linear differential equation (2). We obtain some estimates of their distinct fixed points.

Theorem 6. *Let*

$$P(z)=\sum_{i=0}^{n}a_i z^i \text{ and } Q(z)=\sum_{i=0}^{n}b_i z^i$$

be nonconstant polynomials where a_i, b_i $(i=0,1,\cdots,n)$ *are complex numbers,* $a_n b_n \times (a_n-b_n)\neq 0$. *Let* $A_j(z)$ $(\not\equiv 0)$ $(j=0,1)$ *and* $F(z)\not\equiv 0$ *be entire functions with* $\max\{\rho(A_j)\ (j=0,1),\rho(F)\}<n$. *Let* $d_0(z),d_1(z),d_2(z)$ *be entire functions such that at least one of* d_0,d_1,d_2 *does not vanish identically with* $\rho(d_j)<n$ $(j=0,1,2)$, $\varphi(z)$ *is an entire function with finite order. If* $f(z)$ *is a solution of* (2), *then the differential polynomial* $g_f(z)=d_2f''+d_1f'+d_0f$ *satisfies* $\overline{\lambda}(g_f-\varphi)=\lambda(g_f-\varphi)=\rho(f)=\infty$.

Corollary 1. *Let* $A_j(z)\ (j=0,1),F(z),d_j(z)\ (j=0,1,2),P(z),Q(z)$ *satisfy the additional hypotheses of Theorem 6. If* f *is a solution of* (2), *then the differential polynomial* $g_f=d_2f''+d_1f'+d_0f$ *has infinitely many fixed points and satisfies* $\overline{\tau}(g_f)=\tau(g_f)=\infty$.

Theorem 7. *Let* $A_j(z)\ (j=0,1),F(z),P(z),Q(z),\ \varphi(z)$ *satisfy the additional hypotheses of Theorem 6. If* f *is a solution of* (2), *then*

$$\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\rho(f)=+\infty.$$

Let us denote by

$$\alpha_1=d_1-d_2A_1e^P,\ \alpha_0=d_0-d_2A_0e^Q, \tag{4}$$

$$\beta_1=d_2A_1^2e^{2P}-\left((d_2A_1)'+P'd_2A_1+d_1A_1\right)e^P-d_2A_0e^Q+d_0+d_1', \tag{5}$$

$$\beta_0=d_2A_0A_1e^{P+Q}-((d_2A_0)'+Q'd_2A_0+d_1A_0)e^Q+d_0', \tag{6}$$

$$h=\alpha_1\beta_0-\alpha_0\beta_1 \tag{7}$$

and

$$\psi=\frac{\alpha_1\left(\varphi'-(d_2F)'-\alpha_1F\right)-\beta_1(\varphi-d_2F)}{h}. \tag{8}$$

Theorem 8. *Let* $P(z),\ Q(z),A_0(z),\ A_1(z),F(z)$ *satisfy the hypotheses of Theorem 5. Let* $d_0(z),d_1(z),d_2(z)$ *be entire functions such that at least one of* d_0,d_1,d_2 *does not vanish identically with* $\rho(d_j)<n$ $(j=0,1,2)$, $\varphi(z)$ *is an entire function with finite order such that* $\psi(z)$ *is not a solution of equation* (2). *If* $f(z)$ *is a solution of* (2), *then the differential polynomial* $g_f(z)=d_2f''+d_1f'+d_0f$ *satisfies* $\overline{\lambda}(g_f-\varphi)=\lambda(g_f-\varphi)=\infty$ *with at most one finite order solution* f_0.

Next, we investigate the relation between infinite order solutions of a pair of non-homogeneous linear differential equations and obtain the following result.

Theorem 9. *Let* $P(z)$, $Q(z), A_0(z)$, $A_1(z), d_j(z)$ $(j=0,1,2)$ *satisfy the hypotheses of Theorem 6. Let* $F_1 \not\equiv 0$ *and* $F_2 \not\equiv 0$ *be entire functions such that* $\max\{\rho(F_j) : j=1,2\} < n$ *and* $F_1 - CF_2 \not\equiv 0$ *for any constant* C, $\varphi(z)$ *is an entire function with finite order. If* f_1 *is a solution of the equation*

$$f'' + A_1(z)\, e^{P(z)} f' + A_0(z)\, e^{Q(z)} f = F_1 \tag{9}$$

and f_2 *is a solution of the equation*

$$f'' + A_1(z)\, e^{P(z)} f' + A_0(z)\, e^{Q(z)} f = F_2, \tag{10}$$

then the differential polynomial

$$g_{f_1 - Cf_2}(z) = d_2\left(f_1'' - Cf_2''\right) + d_1\left(f_1' - Cf_2'\right) + d_0\left(f_1 - Cf_2\right)$$

satisfies $\overline{\lambda}\left(g_{f_1-Cf_2} - \varphi\right) = \lambda\left(g_{f_1-Cf_2} - \varphi\right) = \infty$ *for any constant* C.

2. Preliminary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 1 (see [8])**.** *Let* f *be a transcendental meromorphic function of finite order* ρ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geqslant 0$ *for* $i = 1, \cdots, m$ *and let* $\varepsilon > 0$ *be a given constant. Then, there exists a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi \in [0, 2\pi) - E_1$, *then there is a constant* $R_1 = R_1(\psi) > 1$ *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geqslant R_1$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\rho - 1 + \varepsilon)}.$$

Lemma 2 (see [12, 5])**.** *Let* $P(z) = a_n z^n + \cdots + a_0$, $(a_n = \alpha + i\beta \neq 0)$ *be a polynomial with degree* $n \geqslant 1$ *and* $A(z)$ $(\not\equiv 0)$ *be an entire function with* $\rho(A) < n$. *Set* $f(z) = A(z)\, e^{P(z)}$,$z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3)$, *where* $E_3 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, then for sufficiently large* $|z| = r$, *we have*

(i) *If* $\delta(P, \theta) > 0$, *then*

$$\exp\left\{(1 - \varepsilon)\, \delta(P, \theta)\, r^n\right\} \leqslant |f(z)| \leqslant \exp\left\{(1 + \varepsilon)\, \delta(P, \theta)\, r^n\right\}. \tag{11}$$

(ii) *If* $\delta(P, \theta) < 0$, *then*

$$\exp\left\{(1 + \varepsilon)\, \delta(P, \theta)\, r^n\right\} \leqslant |f(z)| \leqslant \exp\left\{(1 - \varepsilon)\, \delta(P, \theta)\, r^n\right\}. \tag{12}$$

Lemma 3 (see [15]). *Let $f(z)$ be an entire function and suppose that*

$$G(z) := \frac{\log^{+}\left|f^{(s)}(z)\right|}{|z|^{\rho}}$$

is unbounded on some ray $\arg z = \theta$ *with constant* $\rho > 0$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, \cdots)$ *tending to infinity such that* $G(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right| \leqslant \frac{1}{(s-j)!}(1+o(1))|z_n|^{s-j} \quad (j = 0, \cdots, s-1) \quad as \quad n \to \infty.$$

Lemma 4 (see [15]). *Let $f(z)$ be an entire function with $\rho(f) < \infty$. Suppose that there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that* $\log^{+}\left|f\left(re^{i\theta}\right)\right| \leqslant Mr^{\sigma}$ *for any ray* $\arg z = \theta \in [0, 2\pi) \setminus E_4$, *where M is a positive constant depending on θ, while σ is a positive constant independent of θ. Then $\rho(f) \leqslant \sigma$.*

Lemma 5 (see [7, 16]). *Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ $(n \geqslant 2)$ are meromorphic functions and $g_1(z), g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0.$

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leqslant j < k \leqslant n$.

(iii) For $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$,

$$T(r, f_j) = o\left\{T\left(r, e^{g_h(z) - g_k(z)}\right)\right\} \quad (r \to \infty, r \notin E_5),$$

where E_5 is a set with finite linear measure.

Then $f_j(z) \equiv 0$ $(j = 1, \cdots, n)$.

Lemma 6. *Let*

$$P(z) = \sum_{i=0}^{n} a_i z^i \text{ and } Q(z) = \sum_{i=0}^{n} b_i z^i$$

be nonconstant polynomials where a_i, b_i $(i = 0, 1, \cdots, n)$ *are complex numbers,* $a_n b_n (a_n - b_n) \neq 0$. *Suppose that* $A_j(z) \not\equiv 0$ $(j = 0, 1)$ *are entire functions with* $\max\{\rho(A_j) : j = 0, 1\} < n$. *We denote*

$$L_f = f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f. \tag{13}$$

If $f \not\equiv 0$ is a finite order entire function, then we have $\rho(L_f) \geqslant n$.

Proof. First, if $f(z) \equiv C$, where C is a nonzero constant, then

$$L_f = A_0(z) e^{Q(z)} C.$$

Hence $\rho(L_f) = n$ and Lemma 6 holds. If f is a nonconstant entire function, we suppose that $\rho(L_f) < n$ and then we obtain a contradiction.
(i) If $\rho(f) = \rho < n$, then

$$\begin{aligned} & f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f - L_f \\ & \quad = f'' - L_f + A_1(z) f' e^{P(z)} + A_0(z) f e^{Q(z)} = 0. \end{aligned}$$

By Lemma 5, we have $A_j(z) \equiv 0$ $(j = 0, 1)$, and this is a contradiction. Hence $\rho(L_f) \geqslant n$.
(ii) If $\rho(f) = \rho \geqslant n$, we rewrite (13) as

$$A_0(z) e^{Q(z)} = \frac{L_f}{f} - \left(\frac{f''}{f} + A_1(z) e^{P(z)} \frac{f'}{f} \right). \tag{14}$$

Set

$$\max\{\rho(A_j)(j = 0, 1), \rho(L_f)\} = \sigma < n.$$

By Lemma 1, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geqslant R_1$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leqslant |z|^{2\rho}, \ 0 \leqslant i < j \leqslant 2. \tag{15}$$

By Lemma 2, there is a set $E_2 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$, where $E_3 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0, \delta(Q, \theta) = 0\} \cup \{\theta \in [0, 2\pi) : \delta(P, \theta) = \delta(Q, \theta)\}$ is a finite set, then for sufficiently large $|z| = r$, we have $\delta(P, \theta) \neq 0$, $\delta(Q, \theta) \neq 0$, $\delta(P, \theta) \neq \delta(Q, \theta)$ and $A_1(z) e^{P(z)}$, $A_0(z) e^{Q(z)}$ satisfies either inequality (11) or (12). Since $a_n \neq b_n$, then a_n, b_n satisfy either inequality $\delta(P, \theta) < \delta(Q, \theta)$ or $\delta(P, \theta) > \delta(Q, \theta)$.
Case 1: $\delta(P, \theta) < \delta(Q, \theta)$ and $\delta(Q, \theta) > 0$. Hence, there exists a positive number $\delta_1 > 0$ such that $\delta(P, \theta) \leqslant \delta_1 < \delta(Q, \theta)$. By Lemma 2, for any given ε

$$\left(0 < \varepsilon < \min\{\frac{\delta(Q, \theta) - \delta_1}{\delta(Q, \theta) + \delta_1}, n - \sigma\} \right),$$

we have

$$\exp\{(1 - \varepsilon)\delta(Q, \theta) r^n\} \leqslant \left| A_0(z) e^{Q(z)} \right|, \tag{16}$$

$$\left| A_1(z) e^{P(z)} \right| \leqslant \exp\{(1 + \varepsilon)\delta_1 r^n\} \tag{17}$$

provided that r is sufficiently large. We now proceed to show that

$$\frac{\log^+ |f(z)|}{|z|^{\sigma + \varepsilon}}$$

is bounded on the ray $\arg z=\theta$. Supposing that this is not the case, then by Lemma 3, there exists an infinite sequence of points $z_m=r_m e^{i\theta}$ $(m=1,2,\cdots)$ tending to infinity such that

$$\frac{\log^+|f(z_m)|}{|z_m|^{\sigma+\varepsilon}}\to\infty. \tag{18}$$

From (18) and the definition of the order, we get

$$\left|\frac{L_f(z_m)}{f(z_m)}\right|\to 0, \tag{19}$$

as $r_m\to\infty$. From equation (14), we obtain

$$\left|A_0(z_m)e^{Q(z_m)}\right|\leqslant\left|\frac{L_f(z_m)}{f(z_m)}\right|+\left|\frac{f''(z_m)}{f(z_m)}\right|+\left|A_1(z_m)e^{P(z_m)}\right|\left|\frac{f'(z_m)}{f(z_m)}\right|. \tag{20}$$

Using inequalities (15) -(17) and the limit (19), we conclude from inequality (20) that

$$\exp\{(1-\varepsilon)\delta(Q,\theta)r_m^n\}\leqslant r_m^{2\rho}+r_m^{2\rho}\exp\{(1+\varepsilon)\delta_1 r_m^n\}+o(1). \tag{21}$$

By ε $(0<\varepsilon<\min\{\frac{\delta(Q,\theta)-\delta_1}{\delta(Q,\theta)+\delta_1},n-\sigma\})$, we have as $r_m\to+\infty$

$$\frac{r_m^{2\rho}}{\exp\{(1-\varepsilon)\delta(Q,\theta)r_m^n\}}\to 0, \tag{22}$$

$$\frac{r_m^{2\rho}\exp\{(1+\varepsilon)\delta_1 r_m^n\}+o(1)}{\exp\{(1-\varepsilon)\delta(Q,\theta)r_m^n\}}\to 0. \tag{23}$$

By (22) and (23), we get from (21) that $1\leqslant 0$. This is a contradiction. Therefore, $\frac{\log^+|f(z)|}{|z|^{\sigma+\varepsilon}}$ is bounded on the ray $\arg z=\theta$, then there exists a bounded constant $M_1>0$ such that

$$|f(z)|\leqslant e^{M_1|z|^{\sigma+\varepsilon}}$$

on the ray $\arg z=\theta$.

Case 2: $\delta(P,\theta)<0$ and $\delta(Q,\theta)<0$. From (13), we get

$$1\leqslant\left|A_1(z)e^{P(z)}\right|\left|\frac{f'(z)}{f''(z)}\right|+\left|A_0(z)e^{Q(z)}\right|\left|\frac{f(z)}{f''(z)}\right|+\left|\frac{L_f(z)}{f''(z)}\right|. \tag{24}$$

By Lemma 2, for any given ε $(0<\varepsilon<\min\{1,n-\sigma\})$ we have

$$\left|A_0(z)e^{Q(z)}\right|\leqslant\exp\{(1-\varepsilon)\delta(Q,\theta)r^n\}, \tag{25}$$

$$\left|A_1(z)e^{P(z)}\right|\leqslant\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}. \tag{26}$$

We prove that

$$\frac{\log^+|f''(z)|}{|z|^{\sigma+\varepsilon}}$$

is bounded on the ray $\arg z = \theta$. Supposing that this is not the case, then by Lemma 3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \cdots)$ tending to infinity such that

$$\frac{\log^+ |f''(z_m)|}{|z_m|^{\sigma+\varepsilon}} \to \infty \tag{27}$$

and

$$\left|\frac{f'(z_m)}{f''(z_m)}\right| \leqslant (1 + o(1)) |z_m|, \quad \left|\frac{f(z_m)}{f''(z_m)}\right| \leqslant \frac{1}{2}(1 + o(1)) |z_m|^2 \text{ as } m \to \infty. \tag{28}$$

From (27) and the definition of the order, we get

$$\left|\frac{L_f(z_m)}{f''(z_m)}\right| \to 0 \tag{29}$$

as $r_m \to \infty$. Using inequalities (25), (26), (28) and the limit (29), we conclude from inequality (24) that

$$\begin{aligned} 1 \leqslant & \exp\{(1-\varepsilon)\delta(P,\theta) r_m^n\} r_m (1 + o(1)) \\ & + \frac{1}{2}\exp\{(1-\varepsilon)\delta(Q,\theta) r_m^n\} r_m^2 (1 + o(1)) + o(1). \end{aligned}$$

By $0 < \varepsilon < \min\{1, n - \sigma\}$, this is a contradiction, provided that r_m is sufficiently large enough. Therefore,

$$\frac{\log^+ |f''(z)|}{|z|^{\sigma+\varepsilon}}$$

is bounded on the ray $\arg z = \theta$, then there exists a bounded constant $M_2 > 0$ such that

$$|f''(z)| \leqslant e^{M_2 |z|^{\sigma+\varepsilon}} \tag{30}$$

on the ray $\arg z = \theta$. Hence, by two-fold iterated integration, along the line segment $[0, z]$, we conclude that

$$f(z) = f(0) + f'(0)\frac{z}{1!} + \int_0^z \int_0^t f''(u)\, du dt.$$

So, we get for a sufficiently large r

$$\begin{aligned} |f(z)| & \leqslant |f(0)| + |f'(0)| \frac{|z|}{1!} + \left| \int_0^z \int_0^t f''(u)\, du dt \right| \\ & \leqslant |f(0)| + |f'(0)| \frac{|z|}{1!} + |f''(z)| \frac{|z|^2}{2!} = \frac{1}{2}(1 + o(1)) r^2 |f''(z)| \end{aligned}$$

on the ray $\arg z = \theta$. Then by using (30) we obtain

$$|f(z)| \leqslant \frac{1}{2}(1 + o(1)) r^2 e^{M_2 |z|^{\sigma+\varepsilon}} \leqslant e^{M_2 r^{\sigma+2\varepsilon}}$$

on the ray $\arg z = \theta$.

Case 3 : $\delta(P,\theta) > \delta(Q,\theta)$ and $\delta(P,\theta) > 0$. Hence, there exists a positive number $\delta_1 > 0$ such that $\delta(Q,\theta) \leqslant \delta_1 < \delta(P,\theta)$. By Lemma 2, for any given

$$0 < \varepsilon < \min\{\frac{\delta(P,\theta) - \delta_1}{\delta(P,\theta) + \delta_1}, n - \sigma\},$$

we have

$$\exp\{(1-\varepsilon)\delta(P,\theta) r^n\} \leqslant \left|A_1(z) e^{P(z)}\right|, \tag{31}$$

$$\left|A_0(z) e^{Q(z)}\right| \leqslant \exp\{(1+\varepsilon)\delta_1 r^n\} \tag{32}$$

provided that r is sufficiently large. From (13), we get

$$\left|A_1(z) e^{P(z)}\right| \leqslant \left|\frac{f''(z)}{f'(z)}\right| + \left|A_0(z) e^{Q(z)}\right| \left|\frac{f(z)}{f'(z)}\right| + \left|\frac{L_f(z)}{f'(z)}\right|. \tag{33}$$

By the same reasoning as in Case 2, we prove that

$$\frac{\log^+ |f'(z)|}{|z|^{\sigma+\varepsilon}}$$

is bounded on the ray $\arg z = \theta$. Supposing that this is not the case, then by Lemma 3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \cdots)$ tending to infinity such that

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leqslant (1 + o(1)) |z_m| \text{ as } m \to \infty \tag{34}$$

and

$$\left|\frac{L_f(z_m)}{f'(z_m)}\right| \to 0 \text{ as } m \to \infty. \tag{35}$$

Using inequalities (15), (31), (32), (34) and the limit (35), we conclude from inequality (33) that

$$\exp\{(1-\varepsilon)\delta(P,\theta) r_m^n\} \leqslant r_m^{2\rho} + (1+o(1)) r_m \exp\{(1+\varepsilon)\delta_1 r_m^n\} + o(1).$$

Since

$$0 < \varepsilon < \min\{\frac{\delta(P,\theta) - \delta_1}{\delta(P,\theta) + \delta_1}, n - \sigma\},$$

this is a contradiction, provided that r_m is sufficiently large enough. Therefore,

$$\frac{\log^+ |f'(z)|}{|z|^{\sigma+\varepsilon}}$$

is bounded on the ray $\arg z = \theta$, then there exists a bounded constant $M_3 > 0$ such that

$$|f'(z)| \leqslant e^{M_3 |z|^{\sigma+\varepsilon}}$$

on the ray $\arg z = \theta$. Then, we get for a sufficiently large r

$$|f(z)| = \left| f(0) + \int_0^z f'(u)\,du \right| \leqslant |f(0)| + \left| \int_0^z f'(u)\,du \right| \leqslant |f(0)| + |z|\,|f'(z)|$$
$$= (1 + o(1))\, r\, |f'(z)| \leqslant e^{M_3 r^{\sigma+2\varepsilon}}$$

on the ray $\arg z = \theta$. Hence, in all cases, there exists a bounded positive constant $M > 0$ such that

$$|f(z)| \leqslant e^{M r^{\sigma+2\varepsilon}} \tag{36}$$

on the ray $\arg z = \theta$. Therefore, for any given $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$, where $(E_1 \cup E_2 \cup E_3) \subset [0, 2\pi)$ is a set of linear measure zero, we have (36), on the ray $\arg z = \theta$ for sufficiently large $|z| = r$. Then by Lemma 4 we have $\rho(f) \leqslant \sigma + 2\varepsilon < n$ for a small positive ε, a contradiction with $\rho(f) \geqslant n$. Hence $\rho(L_f) \geqslant n$. □

Lemma 7 (see [3]). *Let A_0, $A_1, \cdots$, A_{k-1}, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = \infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

3. Proof of Theorems

Proof of Theorem 4. Assume that $f \not\equiv 0$ is a solution of equation (2). We prove that f is of infinite order. We suppose the contrary $\rho(f) < \infty$. By Lemma 6, we have $n \leqslant \rho(L_f) = \rho(F) < n$ and this is a contradiction. Hence, every solution $f \not\equiv 0$ of equation (2) is of infinite order. Furthermore, if $F \not\equiv 0$, then by f is an infinite order solution of equation (2) and by using Lemma 7, every solution f satisfies (3). □

Proof of Theorem 5. Assume that f_0 is a solution of (2) with $\rho(f_0) = \rho < \infty$. If f_1 is a second finite order solution of (2), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation

$$f'' + A_1(z)\, e^{P(z)} f' + A_0(z)\, e^{Q(z)} f = 0,$$

but $\rho(f_1 - f_0) = \infty$ from Theorem 4, this is a contradiction. Hence (2) has at most one finite order solution f_0 and all other solutions f_1 of (2) satisfy (3) by Lemma 7. □

Proof of Theorem 6. Suppose that f is a solution of equation (2). Then by Theorem 4, we have $\rho(f) = \infty$. We prove $\rho(g_f) = \rho(d_2 f'' + d_1 f' + d_0 f) = \infty$.

First we suppose that $d_2 \not\equiv 0$. Substituting $f'' = F - A_1 e^P f' - A_0 e^Q f$ into g_f, we get

$$g_f - d_2 F = \left(d_1 - d_2 A_1 e^P\right) f' + \left(d_0 - d_2 A_0 e^Q\right) f. \tag{37}$$

Differentiating both sides of equation (37) and replacing f'' with $f''=F-A_1e^Pf'-A_0e^Qf$, we obtain

$$\begin{aligned} &g_f'-(d_2F)'-\left(d_1-d_2A_1e^P\right)F \\ &\quad=\left[d_2A_1^2e^{2P}-\left((d_2A_1)'+P'd_2A_1+d_1A_1\right)e^P-d_2A_0e^Q+d_0+d_1'\right]f' \\ &\qquad+\left[d_2A_0A_1e^{P+Q}-\left((d_2A_0)'+Q'd_2A_0+d_1A_0\right)e^Q+d_0'\right]f. \end{aligned} \tag{38}$$

Then, by (4)-(6), (37) and (38), we have

$$\alpha_1f'+\alpha_0f=g_f-d_2F, \tag{39}$$

$$\beta_1f'+\beta_0f=g_f'-(d_2F)'-\left(d_1-d_2A_1e^P\right)F. \tag{40}$$

Set

$$\begin{aligned} h=\alpha_1\beta_0-\alpha_0\beta_1&=\left(d_1-d_2A_1e^P\right)\left[d_2A_0A_1e^{P+Q}-((d_2A_0)'+Q'd_2A_0+d_1A_0)e^Q+d_0'\right] \\ &\quad-\left(d_0-d_2A_0e^Q\right)\left[d_2A_1^2e^{2P}-((d_2A_1)'+P'd_2A_1+d_1A_1)e^P-d_2A_0e^Q+d_0+d_1'\right] \\ &=H_0+H_Pe^{P(z)}+H_Qe^{Q(z)}+H_{P+Q}e^{P(z)+Q(z)}+H_{2P}e^{2P(z)}-d_2^2A_0^2e^{2Q(z)}, \end{aligned} \tag{41}$$

where $H_i(z)$ $(i\in\Lambda=\{0,P(z),Q(z),P(z)+Q(z),2P(z)\})$ are entire functions formed by A_0,A_1,d_0,d_1,d_2 and their derivatives, with order less than n, and Λ is a index set. Since any one of $P(z),Q(z)$, $P(z)+Q(z),2P(z)$ is not equal to $2Q(z)$, then by Lemma 5 we have $d_2^2A_0^2\equiv 0$. This is a contradiction. Thus, $h\not\equiv 0$.

Now suppose $d_2\equiv 0$, $d_1\not\equiv 0$. Using a similar reasoning as above we get $h\not\equiv 0$.

Finally, if $d_2\equiv 0$, $d_1\equiv 0$, $d_0\not\equiv 0$, then we have $h=-d_0^2\not\equiv 0$. Hence $h\not\equiv 0$. By (39), (40) and (41), we obtain

$$f=\frac{\alpha_1\left(g_f'-(d_2F)'-\alpha_1F\right)-\beta_1\left(g_f-d_2F\right)}{h}. \tag{42}$$

If $\rho(g_f)<\infty$, then by (42) we get $\rho(f)<\infty$ and this is a contradiction. Hence $\rho(g_f)=\infty$.

Set $w(z)=d_2f''+d_1f'+d_0f-\varphi$. Then, by $\rho(\varphi)<\infty$, we have $\rho(w)=\rho(g_f)=\rho(f)=\infty$. In order to prove $\overline{\lambda}(g_f-\varphi)=\lambda(g_f-\varphi)=\infty$, we need to prove only $\overline{\lambda}(w)=\lambda(w)=\infty$. Using $g_f=w+\varphi$, we get from (42)

$$f=\frac{\alpha_1\left(w'+\varphi'-(d_2F)'-\alpha_1F\right)-\beta_1\left(w+\varphi-d_2F\right)}{h}. \tag{43}$$

So,

$$f=\frac{\alpha_1w'-\beta_1w}{h}+\psi, \tag{44}$$

where ψ is defined in (8). Substituting (44) into equation (2), we obtain

$$\frac{\alpha_1}{h}w'''+\phi_2w''+\phi_1w'+\phi_0w=F-\left(\psi''+A_1(z)e^{P(z)}\psi'+A_0(z)e^{Q(z)}\psi\right)=A, \tag{45}$$

where ϕ_j $(j=0,1,2)$ are meromorphic functions with $\rho(\phi_j)<\infty$ $(j=0,1,2)$. Since $\rho(\psi)<\infty$, it follows that $A\not\equiv 0$ by Theorem 4. By $\alpha_1\not\equiv 0$, $h\not\equiv 0$ and Lemma 7, we obtain $\overline{\lambda}(w)=\lambda(w)=\rho(w)=\infty$, i.e., $\overline{\lambda}(g_f-\varphi)=\lambda(g_f-\varphi)=\infty$. □

Proof of Theorem 7. Suppose that $f(z)$ is a solution of equation (2). Then, by Theorem 4, we have $\rho(f) = \rho(f') = \rho(f'') = \infty$. Since $\rho(\varphi) < \infty$, then $\rho(f - \varphi) = \rho(f' - \varphi) = \rho(f'' - \varphi) = \infty$. By using a similar reasoning to that in the proof of Theorem 6, the proof of Theorem 7 can be completed. □

Proof of Theorem 8. By Theorem 5, we know that equation (2) has at most one finite order solution f_0 and all other solutions f_1 of (2) satisfy $\rho(f_1) = \infty$. By hypothesis of Theorem 8, $\psi(z)$ is not a solution of equation (2). Then

$$F - \left(\psi'' + A_1(z) e^{P(z)} \psi' + A_0(z) e^{Q(z)} \psi\right) \not\equiv 0. \tag{46}$$

By reasoning similar to that in the proof of Theorem 6, we can prove Theorem 8. □

Proof of Theorem 9. Suppose that f_1 is a solution of equation (9) and f_2 is a solution of equation (10). Set $w = f_1 - Cf_2$. Then w is a solution of the equation

$$w'' + A_1(z) e^{P(z)} w' + A_0(z) e^{Q(z)} w = F_1 - CF_2. \tag{47}$$

By $\rho(F_1 - CF_2) < n$, $F_1 - CF_2 \not\equiv 0$ and Theorem 4, we have $\rho(w) = \infty$. Thus, by Theorem 6, we obtain that

$$\overline{\lambda}(g_{f_1 - Cf_2} - \varphi) = \lambda(g_{f_1 - Cf_2} - \varphi) = \infty.$$

□

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