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Minimization of the trace of the solution of Lyapunov equation connected with damped vibrational systems

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Abstract. Our aim is to optimize the damping of a linear vibrating system. As the optimality criterion we use the one where the penalty function is given as the average total energy over all initial states of unit energy, which is equal to the trace of the corresponding Lyapunov solution multiplied by a matrix corresponding to the chosen measure on the set of initial states. We solve this optimization problem and show that the optimal damping matrix corresponds to the so-called modal critical damping.
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1. Introduction

We consider a damped linear vibrational system described by the differential equation

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{1a}$$

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0, \tag{1b}$$

where M, D and K (called mass, damping and stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite, and D positive semi–definite matrices. In some important applications (e.g. with so–called lumped masses in vibrating structures) M, too, is only semi–definite. This can be easily reduced to the case with a non–singular M at least if the null-space of M is contained in the one of D.

Systems of the form (1) have been extensively studied in the context of the stability of mechanical structures, but they also have applications in other fields. For the basic introduction to these systems we refer the reader to [10].

To (1) there corresponds the eigenvalue equation

$$(\lambda^2 M + \lambda D + K)x = 0. \tag{2}$$

Obviously, all eigenvalues of (2) lie in the left complex half–plane. Equation (2) can be written as a $2n$–dimensional linear eigenvalue problem. This can be done by introducing

$$y_1 = L_1^* x, \quad y_2 = L_2^* \dot{x}, \tag{3}$$

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where

$$K = L_1 L_1^*, \quad M = L_2 L_2^*. \tag{4}$$

It can be easily seen that (1) is equivalent to

$$\dot{y} = Ay, \tag{5a}$$

$$y(0) = y_0, \tag{5b}$$

where $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, $y_0 = \left(\begin{smallmatrix} L_1^* x_0 \\ L_2^* \dot{x}_0 \end{smallmatrix}\right)$, and

$$A = \begin{pmatrix} 0 & L_1^* L_2^{-*} \\ -L_2^{-1} L_1 & -L_2^{-1} D L_2^{-*} \end{pmatrix}, \tag{6}$$

with the solution $y(t) = e^{At} y_0$. The eigenvalue problem $Ay = \lambda y$ is obviously equivalent to (2).

Our aim is to optimize the vibrational system described by (1) in the sense of finding an optimal damping matrix D so as to insure an optimal evanescence.

There exists a number of optimality criteria. The most popular is the spectral abscissa criterion, which requires that the (penalty) function

$$D \mapsto s(A) := \max_k \Re \lambda_k$$

is minimized, where λ_k are eigenvalues of A (so they are the phase space complex eigenfrequencies of the system). This criterion concerns the asymptotic behavior of the system and it is not a priori clear that it will favorably influence the behavior of the system in finite times, too.

Another criterion is given by the requirement for the minimization of the total energy of the system. The energy of the system at time t (as a sum of kinetic and potential energy) is given by the formula

$$E(t; x_0, \dot{x}_0) \; (= E(t; y_0)) = \frac{1}{2} \dot{x}(t)^* M \dot{x}(t) + \frac{1}{2} x(t)^* K x(t).$$

Note that using 3 and 4 it is easy to show that:

$$y(t; y_0)^* y(t; y_0) = \|y(t; y_0)\|^2 = 2E(t; y_0).$$

In other words, the Euclidean norm of this phase-space representation equals twice the total energy of the system. The total energy of the system is given by

$$\int_0^\infty E(t; y_0) \mathrm{d}t. \tag{7}$$

Note that this criterion, in the contrast to the criteria mentioned above, does depend on the initial conditions. The two most popular ways to correct this defect are:

(i) maximizing (7) over all initial states of unit energy, i.e.

$$\max_{\|y_0\|=1} \int_0^\infty E(t; y_0) \mathrm{d}t, \tag{8}$$

(ii) taking the average of (7) over all initial states of unit energy, i.e.

$$\int_{\|y_0\|=1} \int_0^\infty E(t;y_0)\mathrm{d}t\mathrm{d}\sigma, \tag{9}$$

where σ is some probability measure on the unit sphere in $\mathbb{R}^{2n}$.

In some simple cases all these criteria lead to the same optimal matrix D, but in general, they lead to different optimal matrices.

The criterion with the penalty function (9), introduced in [11], will be used in the sequel. The advantage of this criterion is that we can, by choosing the appropriate measure σ, implement our knowledge about the most dangerous input frequencies.

To make this criteria more applicable we proceed as follows.

$$\begin{aligned}\int_0^\infty E(t;y_0)\mathrm{d}t &= \frac{1}{2}\int_0^\infty y(t;y_0)^* y(t;y_0)\mathrm{d}t = \frac{1}{2}\int_0^\infty y_0^* e^{A^*t} e^{At} y_0 \mathrm{d}t \\ &= \frac{1}{2} y_0^* X y_0,\end{aligned}$$

where

$$X = \int_0^\infty e^{A^*t} e^{At}\mathrm{d}t. \tag{10}$$

The matrix X is obviously positive definite. By the well–known result (see, for example [7]) the matrix X is the solution of the Lyapunov equation

$$A^*X + XA = -I. \tag{11}$$

Expression (9) now can be written as

$$\frac{1}{2}\int_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma.$$

Since the map

$$X \mapsto \int_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma$$

is a linear functional on the space of the symmetric matrices, by Riesz representation theorem applied on the space of symmetric matrices with the trace inner product, there exists a symmetric matrix Z such that

$$\int_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma = \mathrm{tr}(XZ), \text{ for all symmetric } X.$$

Let $w \in \mathbb{R}^{2n}$ be arbitrary. Set $X = ww^*$. Then

$$0 \leq \int_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma = \mathrm{tr}(XZ) = \mathrm{tr}(ww^*Z) = \mathrm{tr}(w^*Zw),$$

hence Z is always positive semi–definite.

Hence the criterion given by the penalty function (9) can be written as

$$\operatorname{tr}(XZ) \to \min, \tag{12}$$

where X solves (11), and the matrix Z depends on the measure σ.

Since A is J–symmetric, where $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, it is easy to see that

$$\operatorname{tr}(XZ) = \operatorname{tr}(Y), \tag{13}$$

where Y is the solution of another, so–called "dual Lyapunov equation"

$$AY + YA^* = -Z. \tag{14}$$

Indeed, from the integral representation of the solution of the Lyapunov equation

$$X = \int_0^\infty e^{A^*t} e^{At} d\,t$$

it follows

$$\operatorname{tr}(XZ) = \int_0^\infty \operatorname{tr}(e^{A^*t} e^{At} Z) d\,t = \int_0^\infty \operatorname{tr}(e^{At} Z e^{A^*t}) d\,t = \operatorname{tr} Y.$$

For the surface measure σ generated by the Lebesgue measure on $\mathbb{R}^{2n}$, we obtain $Z = \frac{1}{2n} I$ (see, for example, [4]).

Natural choice for σ are surface measures generated by a Gaussian measure on $\mathbb{R}^{2n}$ (for more details about surface measures, see, for example, [4]), where the corresponding covariance matrix K is of the form $K = \begin{pmatrix} \tilde{K} & 0 \\ 0 & \tilde{K} \end{pmatrix}$. Here $\tilde{K}$ is defined by $\tilde{K}x = \sum \lambda_i x_i e_i$, where e_i's are eigenvectors of the corresponding undamped system (i.e. resonant frequencies of the system) and λ_i's are weights chosen in such a manner as to implement our knowledge of the most dangerous resonant frequencies for the system (more dangerous i–th eigenfrequency $\Rightarrow$ greater λ_i). If we take that some λ_i's are zero, that means that the frequencies which correspond to these λ_i's need not be damped. Given a measure of the above form, one can explicitly calculate the corresponding matrix Z. The explicit procedure for the calculation of the matrix Z is given in [8]. In particular, it is shown that if the matrix K is diagonal, so is the matrix Z. In the rest of the paper, we assume that the measure σ is constructed in a manner as described above.

Let $L_2^{-1} L_1 = U_2 \Omega U_1^*$ be SVD of the matrix $L_2^{-1} L_1$, with $\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_n) > 0$. We can assume $\omega_1 \le \omega_2 \le \cdots \le \omega_n$. Set $U = \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$. Then

$$\widehat{A} = U^* A U = \begin{bmatrix} 0 & \Omega \\ -\Omega & -C \end{bmatrix}, \tag{15}$$

where $C = U_2^* L_2^{-1} D L_2^{-*} U_2$ is positive semi–definite. If we denote $F = L_2^{-*} U_2$, then $F^* M F = I$, $F^* K F = \Omega^2$. Thus we have obtained a particularly convenient, the so–called *modal representation* of the problem (2). In the following we assume that the matrix A has the form given in (15).

In this basis, one can easily see that the corresponding matrix Z has the form $Z = \begin{pmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$, where $\widetilde{Z}$ is a diagonal matrix with nonnegative entries.

Modally damped system are characterized by the generalized commutativity property

$$DK^{-1}M = MK^{-1}D.$$

In the modal representation (15) this implies that C and Ω^{-2} commute, hence C and Ω commute. It has been shown in [2] that

$$X = \begin{bmatrix} \frac{1}{2}C\Omega^{-2} + C^{-1} & \frac{1}{2}\Omega^{-1} \\ \frac{1}{2}\Omega^{-1} & C^{-1} \end{bmatrix}. \tag{16}$$

Hence, the optimal matrix C, for the criterion with the penalty function (8) with $Z = I$, as well as for the criterion given by (12), is $C = 2\Omega$. This can be easily seen in the case when $\omega_i \neq \omega_j$, $i \neq j$, since then the matrix C must be diagonal.

This result can be generalized to the case when the matrix Z has the form $Z = \begin{pmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$, where $\widetilde{Z}$ is diagonal with zeros and ones on the diagonal.

The case of the friction damping i.e. when $D = 2aM$, $a > 0$ was considered in [2], where it was shown that the optimal parameter for the criterion with the penalty function (8) is $a = \sqrt{\omega_1}\sqrt{\frac{\sqrt{5}-1}{2}}$. Recently, in [6] has been considered a problem of finding optimal parameters for modally damped systems with respect to three minimization criteria: minimization of the trace of X, $\|X\|_2$, and $\|X\|_F$.

The set of damping matrices over which we optimize the system is determined by the physical properties of the system. The maximal admissible set is the set of all symmetric matrices C for which the corresponding matrix A is stable. Usually, the admissible matrices must be positive semi–definite. The important case is when the admissible set consists of all positive semi–definite matrices C for which the corresponding matrix A is stable. For this case Brabender [1] (see also [12]) had shown that the following theorem holds.

Theorem 1. *Let the matrix Z be of the form $Z = \begin{pmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$, where $\widetilde{Z} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$, $1 \leq s \leq n$. Denote by $\mathcal{M}$ the set of all matrices of the form*

$$2\begin{bmatrix} \Omega_s & 0 \\ 0 & H \end{bmatrix}, \Omega_s = \mathrm{diag}(\omega_1, \ldots, \omega_s),$$

where H varies over the set of all symmetric positive semi–definite matrices of order $n - s$ such that the corresponding matrix A is stable. On the set $\mathcal{M}$ the function $X \mapsto \mathrm{tr}(XZ)$, where X solves (11), achieves a strict local minimum. In particular, this function is constant on $\mathcal{M}$.

For $s = n$ the set $\mathcal{M}$ reduces to a single matrix 2Ω, hence in this case, the function $X \mapsto \mathrm{tr}(XZ)$ attains in $C = 2\Omega$ local minimum.

In [3] it was shown that in the case $Z = I$ (i.e. all resonant frequencies are equally dangerous), under the assumptions of Theorem 1, the function $X \mapsto \mathrm{tr}(XZ)$ achieves a unique global minimum. In the next section we will generalize this result to the case of a general Z.

2. Main result

The Lyapunov equation (11) can be written as

$$(A_0 - BCB^*)^* X + X(A_0 - BCB^*) = -I, \tag{17}$$

where

$$A_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Let

$$D_s = \{C \in \mathbb{R}^{n\times n} : C \geq 0, A_0 - BCB^* \text{ is stable}\}.$$

To emphasize the dependence of X to the parameter C we write $X(C)$. We are interested in the following optimization problem:

$$\text{(OD)} \qquad \text{minimize } \operatorname{tr}(X(C)Z) \text{ subject to } C \in D_s \text{ and (17).}$$

Theorem 2. *Let* $\widetilde{Z} = \operatorname{diag}(\alpha_1, \ldots, \alpha_s, 0, \ldots, 0)$*, where* $1 \leq s \leq n$ *and* $\alpha_i > 0$, $i = 1, \ldots, s$. *Set* $Z = \left(\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right)$.

The problem (OD) has a solution, and the set on which the minimum is attained is

$$\mathcal{C}_{\min} = \left\{ C = \begin{bmatrix} 2\Omega_s & 0 \\ 0 & H \end{bmatrix} : H \geq 0 \right\}.$$

Proof. Let $C \in D_s$ be arbitrary. Since Z commutes with $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$ and $A_0 - BCB^*$ is J-symmetric,

$$\operatorname{tr}(X(C)Z) = \operatorname{tr}(\tilde{X}(C)) \tag{18}$$

holds, where $\tilde{X}(C)$ solves the dual Lyapunov equation

$$(A_0 - BCB^*)X + X(A_0 - BCB^*)^* = -Z. \tag{19}$$

Let $\widetilde{Z}_i$ be a diagonal matrix with all entries zero except the i-th which is α_i. Set $Z_i = \left(\begin{smallmatrix} \widetilde{Z}_i & 0 \\ 0 & \widetilde{Z}_i \end{smallmatrix}\right)$. Let X_i be the solution of the Lyapunov equation

$$(A_0 - BCB^*)X + X(A_0 - BCB^*)^* = -Z_i. \tag{20}$$

Then it is easy to see that the solution of the Lyapunov equation (19) is

$$\tilde{X} = \sum_{i=1}^{s} X_i. \tag{21}$$

Our aim is to show

$$\min\{\operatorname{tr}(X) : X \text{ solves (20)}, \; C \in D_s\} \geq \frac{2\alpha_i}{\omega_i}, \quad i = 1, \ldots, s. \tag{22}$$

First observe that by simple permutation argument we can assume $i = 1$. Secondly, we can assume $\alpha_i = 1$ (just multiply (20) by $1/\alpha_i$; then the solution of (20) becomes $\frac{1}{\alpha_i}X$). Let us decompose a matrix $X \in \mathbb{R}^{2n\times 2n}$ in the following way:

$$X = \begin{bmatrix} x_{11} & X_{12} & x_{13} & X_{14} \\ X_{12}^* & X_{22} & X_{23} & X_{24} \\ x_{13} & X_{23}^* & x_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}, \tag{23}$$

where $x_{11}, x_{33}, x_{13} \in \mathbb{R}$, $X_{12}, X_{14}, X_{34} \in \mathbb{R}^{1\times(n-1)}$, $X_{22}, X_{24}, X_{44} \in \mathbb{R}^{(n-1)\times(n-1)}$, and $X_{23} \in \mathbb{R}^{(n-1)\times 1}$. Next we partition the Lyapunov equation

$$(A_0 - BCB^*)X + X(A_0 - BCB^*)^* = -Z_1$$

in the same way as we did with X. We obtain

$$x_{13}\omega_1 + \omega_1 x_{13}^* + 1 = 0 \tag{1,1}$$
$$\omega_1 X_{23}^* + X_{14}\Omega_{n-1} = 0 \tag{1,2}$$
$$\omega_1 x_{33} - x_{11}\omega_1 - x_{13}c_{11} - X_{14}C_{12}^* = 0 \tag{1,3}$$
$$\omega_1 X_{34} - X_{12}\Omega_{n-1} - x_{13}C_{12} - X_{14}C_{22} = 0 \tag{1,4}$$

$$\Omega_{n-1}X_{24}^* + X_{24}\Omega_{n-1} = 0 \tag{2,2}$$
$$\Omega_{n-1}X_{34}^* - X_{12}^*\omega_1 - X_{23}c_{11} - X_{24}C_{12}^* = 0 \tag{2,3}$$
$$\Omega_{n-1}X_{44} - X_{22}\Omega_{n-1} - X_{23}C_{12} - X_{24}C_{22} = 0 \tag{2,4}$$
$$-\omega_1 x_{13} - c_{11}x_{33} - C_{12}X_{34}^* - x_{13}^*\omega_1 - x_{33}c_{11} - X_{34}C_{12}^* + 1 = 0 \tag{3,3}$$
$$-\omega_1 X_{14} - c_{11}X_{34} - C_{12}X_{44} - X_{23}^*\Omega_{n-1} - x_{33}C_{12} - X_{34}C_{22} = 0 \tag{3,4}$$
$$-\Omega_{n-1}X_{24} - C_{12}^*X_{34} - C_{22}X_{44} - X_{24}^*\Omega_{n-1} - X_{34}^*C_{12} - X_{44}C_{22} = 0, \tag{4,4}$$

where $\omega_1, c_{11} \in \mathbb{R}$, $C_{12} \in \mathbb{R}^{1\times(n-1)}$, and $C_{22}, \Omega_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$.

From (1,1) we obtain $x_{13} = -\frac{1}{2\omega_1}$. Since $C \geq 0$, one can easily see that $c_{11} = 0$ implies $C_{12} = 0$, hence (3,3) reads $2 = 0$, a contradiction. Hence, $c_{11} > 0$. From (3,3) we now get

$$x_{33} = \frac{1 - X_{34}C_{12}^*}{c_{11}}. \tag{24}$$

The relation (4,4), together with the facts $X_{44} \geq 0$, $C_{22} \geq 0$, implies

$$\operatorname{tr}(C_{12}^*X_{34} + X_{34}^*C_{12}) \leq -\operatorname{tr}(\Omega_{n-1}X_{24} + X_{24}^*\Omega_{n-1}),$$

and the relation (2,2) implies $\operatorname{tr}(X_{24}\Omega_{n-1}) = 0$, hence we obtain

$$\operatorname{tr}(X_{34}^*C_{12}) = \operatorname{tr}(X_{34}C_{12}^*) \leq 0. \tag{25}$$

From the relation (1,3) we obtain

$$x_{11} = x_{33} - x_{13}c_{11}\omega_1^{-1} - \omega_1^{-1}X_{14}C_{12}^*.$$

From relation (2,4) we obtain

$$X_{22} = \Omega_{n-1} X_{44} \Omega_{n-1}^{-1} - X_{23} C_{12} \Omega_{n-1}^{-1} - X_{24} C_{22} \Omega_{n-1}^{-1},$$

hence

$$\operatorname{tr} X_{22} = \operatorname{tr} X_{44} - \operatorname{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}) - \operatorname{tr}(X_{24} C_{22} \Omega_{n-1}^{-1}).$$

From the relation (2,2) we obtain

$$X_{24} = \frac{1}{2} S \Omega_{n-1}^{-1},$$

where $S \in \mathbb{R}^{(n-1)\times(n-1)}$ is a skew–symmetric matrix.

Hence, from the formulas given above, we have

$$\begin{aligned} \operatorname{tr} X &= x_{11} + \operatorname{tr} X_{22} + x_{33} + \operatorname{tr} X_{44} \\ &= 2x_{33} + 2 \operatorname{tr} X_{44} + \frac{c_{11}}{2\omega_1^2} - \frac{1}{\omega_1} X_{14} C_{12}^* - \operatorname{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}) \\ &\quad - \frac{1}{2} \operatorname{tr}(S \Omega_{n-1}^{-1} C_{22} \Omega_{n-1}^{-1}) \\ &= 2x_{33} + 2 \operatorname{tr} X_{44} + \frac{c_{11}}{2\omega_1^2} - \frac{1}{\omega_1} X_{14} C_{12}^* - \operatorname{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}), \end{aligned}$$

where we used the fact $\operatorname{tr}(SH) = 0$ for S skew–symmetric and H symmetric.

From the relation (1,2) it follows $X_{23} = -\frac{1}{\omega_1} \Omega_{n-1} X_{14}^*$, hence

$$\operatorname{tr} X = 2x_{33} + 2 \operatorname{tr} X_{44} + \frac{c_{11}}{2\omega_1^2}.$$

Now (24) and (25) imply

$$\begin{aligned} \operatorname{tr} X &= 2 \frac{1 - X_{34} C_{12}^*}{c_{11}} + \frac{c_{11}}{2\omega_1^2} + 2 \operatorname{tr} X_{44} \geq \\ &\geq \frac{2}{c_{11}} + \frac{c_{11}}{2\omega_1^2} \geq \frac{2}{\omega_1}. \end{aligned} \tag{26}$$

The last inequality follows from the following observation. Let us define the function $g(x) = \frac{2}{x} + \frac{x}{2\omega_1^2}$. Then the function g attains its unique minimum $\frac{2}{\omega_1}$ in $x = 2\omega_1$.

Hence, we have shown (22). Now (21) and (18), together with the permutation argument and due to the assumption $\alpha_i = 1$, imply

$$\operatorname{tr}(X(C)Z) \geq 2 \sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}.$$

Since $\operatorname{tr}(X(2\Omega)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$, this is indeed the global minimum.

Assume that $C \in D_s$ is such that $\operatorname{tr}(X(C)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$. Then (26) and (21) imply $\operatorname{tr} X_i = \frac{2}{\omega_i}$. By multiplying with $\frac{1}{\alpha_i}$ we can again assume $\alpha_i = 1$. Also, we can again, by the permutation argument, assume $i = 1$. Let us decompose the matrix X_1 as in (23). Then (26) implies $X_{44} = 0$. Since $X_1 \geq 0$, it follows $X_{14} = X_{24} = X_{34} = 0$. From the relation (1,2) follows immediately $X_{23} = 0$. The relation (1,3) implies $x_{11} = \frac{3}{2}\frac{1}{\omega_1}$, which implies $\operatorname{tr} X_{22} = 0$. Hence $X_{22} = 0$. This implies $X_{12} = 0$. Finally, from (1,4) now follows $C_{12} = 0$.

By repeating this procedure for $i = 2, \dots, s$ we obtain that for each i, corresponding C_{12} part of the matrix C is zero, hence $C \in \mathcal{C}_{\min}$.

On the other hand, it is easy to see that $\operatorname{tr}(X(C)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$, for all $C \in \mathcal{C}_{\min}$. □

3. Examples

Example 1. *This example is taken from [5], Example 2.1. The vibrational system is a simple three–mass system. We use the following parameters: $m_1 = 10$, $m_2 = m_3 = 1$, $k_1 = k_3 = 0.1$, $k_2 = 0$, $k_4 = 1$. We want to minimize the vibrations due to the smallest resonant frequency $\omega_3 = 0.1$. We take $Z = \operatorname{diag}(1, 0, \dots 0, 1, 0 \dots, 0)$. We obtain that an optimal damping matrix is*

$$D_{\text{opt}} = \begin{bmatrix} 0.022 & -0.002 & 0 \\ -0.002 & 0.002 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

which can be implemented by using $d_1 = 0.02$, $d_2 = 0.002$, $d_3 = 0$, $d_4 = 2$. The optimal damping matrix 2Ω for the average total energy with respect to all resonant frequencies cannot be implemented by the given dampers. Indeed, in this case the optimal damping matrix is

$$\begin{bmatrix} 2.0924 & -0.1477 & 0 \\ -0.1477 & 0.6150 & 0 \\ 0 & 0 & 2.0000 \end{bmatrix},$$

and since damping matrices have the form

$$\begin{bmatrix} d_1 + d_2 & -d_2 & 0 \\ -d_2 & d_2 + d_3 & -d_3 \\ 0 & -d_3 & d_3 + d_4 \end{bmatrix},$$

the optimal damping is not physically realizable.

Example 2. *This example is taken from [9], problem P.8.6 on page 184. The vibrational system we want to optimize is cubic hexapod isolator with the fixed base and the payload which is an axisymmetrical rigid body. We use the following parameters: $Z_c = 0.5$, $R_x = 1.5$, $R_z = 2$, $m = 2.5$, $k = 1$ and $L = 4$. We also assume that the undamped system has an internal Rayleigh damping $D_0 = \alpha M + \beta K$, with $\alpha = \beta = 0.04$, which in modal representation is given by $C_0 = \alpha I + \beta \Omega^2$. We want to minimize the vibrations due to the two largest resonant frequencies $\omega_1 = 1.7889$ and $\omega_2 = 1.2649$, but with ω_1 twice as dangerous as ω_2. Hence we can take $Z = \operatorname{diag}(1, 0.5, 0, \dots 0, 1, 0.5, 0 \dots, 0)$. The average total energy due to the two largest frequencies is 21.5626.*

By choosing an optimal damping matrix

$$C_{\text{opt}} = C_0 + \begin{bmatrix} 2\omega_1 - C_0(1,1) & 0 & 0 \\ 0 & 2\omega_2 - C_0(2,2) & 0 \\ 0 & & 0 \end{bmatrix},$$

we obtain that we need to put the damper with viscosity 0.6065 *on the node* 2*, damper with viscosity* 9.7033 *on the node* 4*, damper with viscosity* 34.0971 *on the node* 6 *and damper with viscosity* 4.8516 *on the payload axis connecting nodes* 2 *and* 4*. The average total energy of the corresponding damped system due to the largest frequency drops to* 1.9086*, a tenfold decrease.*

The optimal damping matrix 2Ω *for the average total energy with respect to all resonant frequencies is not constructible by the use of dampers on the active nodes since the corresponding damping matrix (in original coordinates) is*

$$\begin{bmatrix} 0.1800 & 0 & 0 & 0 & 0.0500 & 0 \\ 0 & 0.7865 & 0 & 2.3758 & 0 & 0 \\ 0 & 0 & 0.1800 & 0 & 0 & 0 \\ 0 & 2.3758 & 0 & 10.2733 & 0 & 0 \\ 0.0500 & 0 & 0 & 0 & 0.5700 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35.7771 \end{bmatrix},$$

and hence we would need to put dampers on all three fixed nodes.

The main conclusion of the paper is that for a class of optimality criteria with the penalty function given as an average total energy, the optimal damping matrix corresponds to the so–called modal critical damping, thus generalizing the known results to a wider class of optimality criteria which can incorporate our knowledge of the most dangerous resonances.

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