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The ruin probabilities of a multidimensional perturbed risk model

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Abstract. In this paper we consider the ruin probabilities of a multidimensional insurance risk model perturbed by Brownian motion. A Lundberg-type upper bound is derived for the infinite-time ruin probability when claims are light-tailed. The proof is based on the theory of martingales. An explicit asymptotic estimate is obtained for the finite-time ruin probability in the heavy-tailed claims case.

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1. The model

Multidimensional models with a common arrival process describe situations where each claim event usually produces more than one type of claim. One common example is natural catastrophe insurance where an accident could cause claims for different types of bodily injuries and property damages. The same situations exist in motor insurance.

We consider a multidimensional insurance risk process $\overrightarrow{R}(t) = (R_1(t), \ldots, R_n(t))^T$ perturbed by a multidimensional Brownian motion

$$\begin{pmatrix} R_1(t) \\ \cdots \\ R_n(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ \cdots \\ u_n \end{pmatrix} + t \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix} - \sum_{i=1}^{N(t)} \begin{pmatrix} X_{1i} \\ \cdots \\ X_{ni} \end{pmatrix} + \begin{pmatrix} \sigma_1 B_1(t) \\ \cdots \\ \sigma_n B_n(t) \end{pmatrix}, t \geq 0. \quad (1)$$

Here $\overrightarrow{u} = (u_1, \ldots, u_n)^T$ stands for the initial surplus vector, $\overrightarrow{c} = (c_1, \ldots, c_n)^T$ for the premium rate vector, while $\overrightarrow{X}_i = (X_{1i}, \ldots, X_{ni})^T$, $i = 1, 2, \ldots$ denote n-tuples of claims whose common arrival times constitute a counting process $\{N(t), t \geq 0\}$. The process $\{N(t), t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ and $\{\overrightarrow{X}_i, i = 1, 2, \ldots\}$ is a sequence of independent copies of the random n-tuple $\overrightarrow{X} = (X_1, \ldots, X_n)^T$ with a joint distribution function $F(x_1, \ldots, x_n)$ and marginal distribution functions $F_1(x_1), \ldots, F_n(x_n)$. The vector $\overrightarrow{B}(t) = (B_1(t), \ldots, B_n(t))^T$ denotes a standard multidimensional Brownian motion with constant correlation coefficients $r_{ij} \in [-1, 1], i = 1, \ldots, n-1, j = i+1, \ldots, n$, while $\sigma_i \geq 0, i = 1, \ldots, n$ denote the

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marginal volatility coefficients of $\overrightarrow{B}(t)$. All vectors $\overrightarrow{X}_i$ for $i = 1, 2, \dots,$ $\overrightarrow{u}$ and $\overrightarrow{c}$ consist of only nonnegative components. The random processes $\{\overrightarrow{X}_i, i = 1, 2, \dots\}$, $\{N(t), t \geq 0\}$ and $\{\overrightarrow{B}(t), t \geq 0\}$ are all mutually independent.

Let $\overrightarrow{x} = (x_1, \dots, x_n)^T$ and $\overrightarrow{y} = (y_1, \dots, y_n)^T$ be two n-dimensional vectors. Then we write $\overrightarrow{x} < \overrightarrow{y}$ if $x_i < y_i, i = 1, \dots, n$ and in the same way we define other inequalities.

The ruin time of the model (1) can be defined in two different ways:

$$T_{min} = \inf\{t > 0 | \min\{R_1(t), \dots, R_n(t)\} < 0\}$$

or

$$T_{max} = \inf\{t > 0 | \overrightarrow{R}(t) < \overrightarrow{0}\} = \inf\{t > 0 | \max\{R_1(t), \dots, R_n(t)\} < 0\}.$$

Here we assume that $\inf \emptyset = \infty$. T_{max} is the first time when all $R_i(t), i = 1, \dots, n$ go below zero. At time T_{min} the insurance company may be able to survive more easily because probably only one of its subsidiary companies gets ruined. That means that T_{max} represents a more critical time than T_{min}. We also define the infinite-time ruin probability of the model (1) in two ways:

$$\psi(\overrightarrow{u}) = P(T_{min} < \infty | \overrightarrow{R}(0) = \overrightarrow{u}) \tag{2}$$

or

$$\psi(\overrightarrow{u}) = P(T_{max} < \infty | \overrightarrow{R}(0) = \overrightarrow{u}),$$

respectively.

Finally, we define the finite-time ruin probability

$$\psi(\overrightarrow{u}; T) = P(T_{max} \leq T | \overrightarrow{R}(0) = \overrightarrow{u}),\ T > 0. \tag{3}$$

In Section 2, we derive a Lundberg-type upper bound for the case of light-tailed claims and for the infinite-time ruin probability $\psi(\overrightarrow{u})$. We use the techniques from martingale theory with no restrictions on the dependence structure of the process $\overrightarrow{X}$. In Section 3, we derive an explicit asymptotic estimate for the finite-time ruin probability $\psi(\overrightarrow{u}; T)$ for the case of heavy-tailed claims, where we do assume that $X_1, \dots, X_n$ are independent.

2. A Lundberg-type upper bound for the ruin probability of light-tailed claims

Throughout this section we consider only the claims with light tails. We also assume that the claim vector $\overrightarrow{X}$ has a finite mean vector $\overrightarrow{\mu} = (\mu_1, \dots, \mu_n)^T$ and that the safety loading condition $\overrightarrow{c} > \lambda \overrightarrow{\mu}$ holds.

Our main result - an upper bound for the infinite-time ruin probability is given by the following theorem:

Theorem 1. *Let*

(*i*) $\hat{m}(s_1,\dots,s_n)=\mathbf{E}[exp\{s_1X_1+\dots+s_nX_n\}]$;

(*ii*) $f(s_1,\dots,s_n)=\lambda\hat{m}(s_1,\dots,s_n)-\lambda-\sum_{i=1}^n c_is_i$
$\quad+\frac{1}{2}[\sum_{i=1}^n\sigma_i^2s_i^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^n r_{ij}\sigma_i\sigma_js_is_j]$;

(*iii*) $s_1^0=\sup\{s_1|\hat{m}(s_1,0,\dots,0)<\infty\},\dots,s_n^0=\sup\{s_n|\hat{m}(0,\dots,0,s_n)<\infty\}$;

(*iv*) $G^0=\{(s_1,\dots,s_n)|s_1\ge 0,\dots,s_n\ge 0,\hat{m}(s_1,\dots,s_n)<\infty\}\setminus(0,\dots,0)$;

(*v*) $\Delta^0=\{(s_1,\dots,s_n)\in G^0|f(s_1,\dots,s_n)=0\}$.

If $s_1^0>0,\dots,s_n^0>0$ *and* $\sup_{(s_1,\dots,s_n)\in G^0}f(s_1,\dots,s_n)>0$, *then*

$$\psi(\overrightarrow{u})\le\inf_{(s_1,\dots,s_n)\in\Delta^0}\exp\Big\{-\sum_{i=1}^n s_iu_i\Big\}. \tag{4}$$

Hölder inequality gives that the set G^0 is non-empty provided that $s_1^0>0,\dots,s_n^0>0$. First we will prove a proposition:

Proposition 1. *Let* $s_1^0>0,\dots,s_n^0>0$ *and* $\sup_{(s_1,\dots,s_n)\in G^0}f(s_1,\dots,s_n)>0$. *Then the following statements hold:*

(*a*) *The set* Δ^0 *is non-empty.*

(*b*) *If* $v>0$ *solves the equation* $f(s_1,\dots,ls_1,\dots)=0$ *for given* $l\ge 0$, *then* $f(s_1,\dots,ls_1,\dots)>0$ *for every* $s_1>v$ *and* $f(s_1,\dots,ls_1,\dots)<0$ *for every* $0<s_1<v$. *Here* ls_1 *comes in the* i*-th position and* $s_j=s_1$ *for* $j\ne i$, $i=1,\dots,n$.

Proof. (a): For some given $l\ge 0$ and $s_i=ls_1$, $i=1,\dots,n$, we calculate

$$\begin{aligned}\frac{\mathrm{d}f(s_1,\dots,ls_1,\dots)}{\mathrm{d}s_1}=&\lambda\Big[\sum_{j=1,j\ne i}^n\frac{\partial\hat{m}(s_1,\dots,s_n)}{\partial s_j}+l\frac{\partial\hat{m}(s_1,\dots,s_i,\dots)}{\partial s_i}\Big]_{s_i=ls_1,s_j=s_1,j\ne i}\\&-\sum_{j=1,j\ne i}^n c_j-lc_i+\sum_{j=1,j\ne i}^n\sigma_j^2s_1+2l\sum_{j=1}^{i-1}r_{ji}\sigma_i\sigma_js_1\\&+2l\sum_{j=i+1}^n r_{ij}\sigma_i\sigma_js_1+2\sum_{j=1,j\ne i}^{n-1}\sum_{k=j+1,k\ne i}^n r_{jk}\sigma_j\sigma_ks_1+l^2\sigma_i^2s_1,\end{aligned}$$

so that

$$\frac{\mathrm{d}f(s_1,\dots,ls_1,\dots)}{\mathrm{d}s_1}\Big|_{s_1=0}=-\sum_{j=1,j\ne i}^n(c_j-\lambda\mu_j)-l(c_i-\lambda\mu_i)<0,$$

because of the safety loading conditions. This means that the function $s_1\to f(s_1,\dots,ls_1,\dots)$ decreases when $s_1>0$ is sufficiently close to the point $s_1=0$. For $l=\infty$, the equation $s_i=ls_1$ represents the line $s_1=0$ and in this case we can easily show that the function $f(0,\dots,s_i,\dots)$ takes smaller values than $f(0,\dots,0)$

when $s_i > 0$. Now we conclude that $f(s_1, \dots, s_i, \dots) < 0$ holds for all $(s_1, \dots, s_n)$ sufficiently close to the origin, because $f(0, \dots, 0) = 0$ and $0 \le l \le \infty$ can be arbitrary. By this and the condition $\sup_{(s_1,\dots,s_n)\in G^0} f(s_1, \dots, s_n) > 0$ we see that (a) holds.

(b): Let $s_1 > 0$ and $l \ge 0$. We have

$$\begin{aligned}
\frac{\mathrm{d}^2 f(s_1, \dots, l s_1, \dots)}{\mathrm{d}s_1^2} &= \lambda \bigg[\sum_{j=1, j\neq i}^{n} \frac{\partial^2 \hat{m}(s_1, \dots, s_n)}{\partial s_j^2} \\
&\qquad + 2l \sum_{j=1, j\neq i}^{n} \frac{\partial^2 \hat{m}(s_1, \dots, s_n)}{\partial s_i \partial s_j} + l^2 \frac{\partial^2 \hat{m}(s_1, \dots, s_n)}{\partial s_i^2} \bigg]_{s_i = l s_1, s_j = s_1, j \neq i} \\
&\qquad + \sum_{j=1, j\neq i}^{n} \sigma_j^2 + 2l \sum_{j=1}^{i-1} r_{ji}\sigma_i\sigma_j + 2l \sum_{j=i+1}^{n} r_{ij}\sigma_i\sigma_j \\
&\qquad + 2 \sum_{j=1, j\neq i}^{n-1} \sum_{k=j+1, k\neq i}^{n} r_{jk}\sigma_j\sigma_k + l^2\sigma_i^2 \\
&\ge \lambda \sum_{j=1, j\neq i}^{n} \mathbf{E}[(X_j + l X_i)^2] + \sum_{j=1, j\neq i}^{n} [\sigma_j - l\sigma_i]^2 > 0,
\end{aligned}$$

where $s_i = l s_1$, $i = 1, \dots, n$. We conclude that the function $s_1 \to f(s_1, \dots, l s_1, \dots)$ is convex on $(0, s_1^0)$, so the equation $f(s_1, \dots, l s_1, \dots) = 0$ can have only one root in $(0, s_1^0)$ and the result (b) obviously follows. $\square$

Now we will prove the theorem using Proposition 1.

Proof of the theorem. First we are going to construct a martingale based on the surplus process $\{\overrightarrow{R}(t), t \ge 0\}$. This martingale is needed for establishing a Lundberg-type upper bound for the ruin probability. Let $s_1, \dots, s_n$ be real numbers such that $\hat{m}(s_1, \dots, s_n) < \infty$. We will show that the process

$$M(\overrightarrow{R}(t)) = \exp\bigg\{ -\sum_{i=1}^{n} s_i R_i(t) - f(s_1, \dots, s_n) t \bigg\},\ t \ge 0,$$

is an $\mathcal{F}$-martingale, where $\mathcal{F} = \{\mathcal{F}_t, t \ge 0\}$ represents the natural filtration of $\{\overrightarrow{R}(t), t \ge 0\}$.

For every $t, h \ge 0$ we have

$$\begin{aligned}
&\mathbf{E}\bigg[\exp\bigg\{ -\sum_{i=1}^{n} s_i (R_i(t+h) - R_i(t)) \bigg\} \bigg] \\
&\quad = \exp\bigg\{ -h \sum_{i=1}^{n} s_i c_i \bigg\} \exp\{\lambda \hat{m}(s_1, \dots, s_n) h - \lambda h\} \\
&\qquad \times \exp\bigg\{ \frac{1}{2} \bigg[\sum_{i=1}^{n} \sigma_i^2 s_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}\sigma_i\sigma_j s_i s_j \bigg] h \bigg\} \\
&\quad = \exp\{f(s_1, \dots, s_n) h\},
\end{aligned}$$

since the Poisson process $\{N(t), t \geq 0\}$ has stationary independent increments. This gives that

$$\begin{aligned}\mathbf{E}[M(\overrightarrow{R}(t+h))|\mathcal{F}_t] &= \mathbf{E}\Big[\exp\Big\{-\sum_{i=1}^{n} s_i R_i(t+h) - f(s_1,\dots,s_n)(t+h)\Big\}\Big|\mathcal{F}_t\Big] \\ &= \exp\Big\{-\sum_{i=1}^{n} s_i R_i(t) - f(s_1,\dots,s_n)t\Big\} = M(\overrightarrow{R}(t))\end{aligned}$$

and we conclude that $M(\overrightarrow{R}(t))$ is a martingale with respect to $\mathcal{F}$. Now the equation

$$\mathbf{E}[M(\overrightarrow{R}(t))] = \exp\Big\{-\sum_{i=1}^{n} s_i u_i\Big\},\ t \geq 0 \tag{5}$$

follows from $M(\overrightarrow{R}(0)) = \exp\Big\{-\sum_{i=1}^{n} s_i u_i\Big\}$ and the definition of a martingale.

Next we will show that T_{min} and $M(\overrightarrow{R}(t))$ are a stopping time and a martingale, respectively, with respect to a common filtration $\mathcal{F}' = \{\mathcal{F}'_t, t \geq 0\}$.

Let $\{\overline{\mathcal{F}}_t, t \geq 0\}$ be a complete σ-algebra of $\{\mathcal{F}_t, t \geq 0\}$ with respect to P and let $\overline{\mathcal{F}}_{t+} = \cap_{s>t}\overline{\mathcal{F}}_s$. $M(\overrightarrow{R}(t))$ is an right-continuous $\mathcal{F}$-martingale, so it is also a martingale with respect to $\{\overline{\mathcal{F}}_{t+}, t \geq 0\}$ (see [1, Theorem VI.1.3]). The definition of T_{min} and the fact that $\{\overrightarrow{R}(t), t \geq 0\}$ is a *cádlág* process, gives that T_{min} is an $\{\mathcal{F}_{t+}, t \geq 0\}$-stopping time, hence an $\{\overline{\mathcal{F}}_{t+}, t \geq 0\}$-stopping time since $\mathcal{F}_{t+} \subset \overline{\mathcal{F}}_{t+}$ (see [3, I.1.28 Proposition]). So, if we select $\mathcal{F}' = \{\overline{\mathcal{F}}_{t+}, t \geq 0\}$, we get that T_{min} is an $\mathcal{F}'$-stopping time and $M(\overrightarrow{R}(t))$ is an $\mathcal{F}'$-martingale, respectively.

Let 1_A be the indicator function of an event A. By equality (5) and by the fact that T_{min} and $M(\overrightarrow{R}(t))$ are a stopping time and a martingale, for every $(s_1,\dots,s_n)$ such that $\hat{m}(s_1,\dots,s_n) < \infty$ we have

$$\begin{aligned}\exp\Big\{-\sum_{i=1}^{n} s_i u_i\Big\} &= \mathbf{E}[M(\overrightarrow{R}(t))] \geq \mathbf{E}[M(\overrightarrow{R}(t))1_{(T_{min}\leq t)}] \\ &= \mathbf{E}\{\mathbf{E}[M(\overrightarrow{R}(t))|\overline{\mathcal{F}}_{T_{min}+}]1_{(T_{min}\leq t)}\} \\ &= \mathbf{E}\{M(\overrightarrow{R}(T_{min}))|T_{min} \leq t\}P(T_{min} \leq t).\end{aligned} \tag{6}$$

Since there is at least one $i \in \{1,\dots,n\}$ such that $R_i(T_{min}) < 0$, we can find $(s_1,\dots,s_n) \in G^0$ for which

$$\exp\Big\{-\sum_{i=1}^{n} s_i R_i(T_{min})\Big\} \geq 1.$$

By rearranging inequality (6) using the definition of $M(\overrightarrow{R}(t))$ and the above inequality we get

$$P(T_{min} \leq t) \leq \exp\Big\{-\sum_{i=1}^{n} s_i u_i\Big\} \sup_{0<h<t} \exp\{f(s_1,\dots,s_n)h\}.$$

We define $\Delta^- = \{(s_1, \ldots, s_n) \in G^0 | f(s_1, \ldots, s_n) < 0\}$ and $\Delta^+ = \{(s_1, \ldots, s_n) \in G^0 | f(s_1, \ldots, s_n) > 0\}$. For $(s_1, \ldots, s_n) \in \Delta^+$, the right-hand side of the above relation tends to ∞ as $t \to \infty$ so this case makes no sense. We conclude that

$$P(T_{min} \le t) \le \inf_{(s_1,\ldots,s_n) \in \Delta^- \cup \Delta^0} \exp\left\{ -\sum_{i=1}^{n} s_i u_i \right\}.$$

By Proposition 1(a) we know that the equation $f(s_1, \ldots, s_n) = 0$ has at least one root in G^0. Applying Proposition 1(b) it is easy to see that the infimum in the above inequality can be attained on Δ^0. It follows that

$$P(T_{min} \le t) \le \inf_{(s_1,\ldots,s_n) \in \Delta^0} \exp\left\{ -\sum_{i=1}^{n} s_i u_i \right\}$$

and for $t \to \infty$ we get a Lundberg-type upper bound for the infinite-time ruin probability (4) when the ruin time equals T_{min}. In view of the obvious inequality

$$P(T_{max} < t) \le P(T_{min} < t)$$

we can see that the relation (4) also holds when we take T_{max} to be the ruin time of the process. $\square$

3. Asymptotics for the finite-time ruin probability

In this section we consider the risk process (1) with heavy-tailed claims. We further assume that the claim vector $\overrightarrow{X}$ and the multidimensional Brownian motion $\overrightarrow{B}(t)$ consist of independent components. Here we do not assume the safety loading condition.

A well-known class of heavy-tailed distribution functions is the subexponential class. A distribution function F on $[0, \infty)$ is said to be subexponential if for some (or, equivalently, for all) $n = 2, 3, \ldots$ the relation

$$\overline{F^{*n}}(x) \sim n\overline{F}(x),\ x \to \infty \tag{7}$$

holds, where F^{*n} denotes the n-fold convolution of F and if $\overline{F}(x) > 0$ for all $x \ge 0$. Here $\sim$ means that the quotient of the left-hand and the right-hand side tends to 1 according to the indicated limit procedure. We write $F \in \mathcal{S}$. More on subexponential distributions can be found in [6, 2.5].

In the following theorem we derive an asymptotic estimate for the finite-time ruin probability $\psi(\overrightarrow{u}; T)$ defined in (3). The limit procedure used in this theorem is always $(u_1, \ldots, u_n) \to (\infty, \ldots, \infty)$.

Theorem 2. *Let $F_1, \ldots, F_n$ be in $\mathcal{S}$. Then*

$$\psi(\overrightarrow{u}; T) \sim f(n)\overline{F}_1(u_1) \ldots \overline{F}_n(u_n), \tag{8}$$

for every fixed time $T > 0$ and for each positive integer n, where $f(0) = 1$, $f(1) = \lambda T$ and $f(n) = \lambda T\left(\sum_{i=0}^{n-1} \binom{n-1}{i} f(i) \right)$.

Proof. First we define

$$\underline{B}_j(T) = \inf_{0\le t\le T} B_j(t),\ \overline{B}_j(T) = \sup_{0\le t\le T} B_j(t), j = 1,\dots,n.$$

By the reflection principle (see [4, 2.6]) there is

$$P(\underline{B}_j(T) < -x) = P(\overline{B}_j(T) > x) = 2P(B_j(T) > x),$$

so because $F_j \in \mathcal{S}$

$$P(\underline{B}_j(T) < -x) = P(\overline{B}_j(T) > x) = o(\overline{F}_j(x))$$

for every $x > 0$ and $j = 1,\dots,n$. Obviously,

$$\begin{aligned}\psi(\overrightarrow{u};T) &= P(\overrightarrow{R}(t) < \overrightarrow{0} \text{ for some } 0 < t \le T | \overrightarrow{R}(0) = \overrightarrow{u}) \\ &= P\bigg(\sum_{i=1}^{N(t)} \overrightarrow{X}_i - t\overrightarrow{c} - (\sigma_1 B_1(t),\dots,\sigma_n B_n(t))^T > \overrightarrow{u} \text{ for some } 0<t\le T\bigg).\end{aligned} \tag{9}$$

First we will find an asymptotic upper bound for $\psi(\overrightarrow{u};T)$. From the assumed independence of random vector components we get

$$\begin{aligned}\psi(\overrightarrow{u};T) &\le P\bigg(\sum_{i=1}^{N(T)} \overrightarrow{X}_i - (\sigma_1 \underline{B}_1(T),\dots,\sigma_n \underline{B}_n(T))^T > \overrightarrow{u}\bigg) \\ &= \sum_{k=0}^{\infty} P(N(T)=k) \prod_{j=1}^{n} P\bigg(\sum_{i=1}^{k} X_{ji} - \sigma_j \underline{B}_j(T) > u_j\bigg).\end{aligned} \tag{10}$$

Now we need the result from [2, Lemma 1.3.5]:

- If F is a subexponential distribution, then for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\overline{F^{*n}}(x) \le C_\epsilon (1+\epsilon)^n \overline{F}(x) \tag{11}$$

  holds for all $n = 1, 2, \dots$ and all $x \ge 0$.

By inequality (11) for every $\epsilon > 0$ there exist constants $C_\epsilon^{(1)}, C_\epsilon^{(2)} > 0$ such that for all $k = 1, 2, \dots,$

$$\begin{aligned}&P\bigg(\sum_{i=1}^{k} X_{1i} - \sigma_1 \underline{B}_1(T) > u_1\bigg) \\ &\quad= \int_{-u_1}^{0} P\bigg(\sum_{i=1}^{k} X_{1i} - x > u_1\bigg) P(\sigma_1 \underline{B}_1(T) = dx) + P(\sigma_1 \underline{B}_1(T) < -u_1) \\ &\quad\le C_\epsilon^{(1)} (1+\epsilon)^k \int_{-u_1}^{0} P(X_1 - x > u_1) P(\sigma_1 \underline{B}_1(T) = dx) + P(\sigma_1 \underline{B}_1(T) < -u_1) \\ &\quad\le C_\epsilon^{(1)} (1+\epsilon)^k P(X_1 - \sigma_1 \underline{B}_1(T) > u_1) + P(\sigma_1 \underline{B}_1(T) < -u_1) \\ &\quad\le C_\epsilon^{(1)} C_\epsilon^{(2)} (1+\epsilon)^k \overline{F}_1(u_1).\end{aligned}$$

In the last step we used $P(X_1-\sigma_1\underline{B}_1(T)>u_1)\sim\overline{F}_1(u_1)$ which follows from the fact that for nonnegative independent random variables X and Y with X distributed by $F\in\mathcal{S}$ it holds that

$$P(X-Y>x)\sim\overline{F}(x),\ x\to\infty \tag{12}$$

[7, Lemma 4.2]. We also used $P(\sigma_1\underline{B}_1(T)<-u_1)=o(1)\overline{F}_1(u_1)$. For every fixed $k=1,2,\dots,$ by (7) and from the fact that for distribution functions $F,G\in\mathcal{S}$ on $[0,\infty)$ satisfying $\overline{G}(x)=o(\overline{F}(x))$ it holds that

$$\overline{F*G}(x)\sim\overline{F}(x); \tag{13}$$

[6, Lemma 2.5.2], we have

$$P\bigg(\sum_{i=1}^{k}X_{1i}-\sigma_1\underline{B}_1(T)>u_1\bigg)\sim k\overline{F}_1(u_1).$$

The same relations also hold for $P\bigg(\sum_{i=1}^{k}X_{ji}-\sigma_j\underline{B}_j(T)>u_j\bigg)$, where $k=1,2\dots$ and $j=1,\dots,n$. Now using the dominated convergence theorem, we can see that the right-hand side of (10) is asymptotic to

$$\sum_{k=0}^{\infty}P(N(T)=k)k^n\overline{F}_1(u_1)\dots\overline{F}_n(u_n)=f(n)\overline{F}_1(u_1)\dots\overline{F}_n(u_n),$$

where $f(0)=1$, $f(1)=\lambda T$ and $f(n)=\lambda T\bigg(\sum_{i=0}^{n-1}\binom{n-1}{i}f(i)\bigg)$.

This proves that

$$\psi(\overrightarrow{u};T)\le(1+o(1))f(n)\overline{F}_1(u_1)\dots\overline{F}_n(u_n). \tag{14}$$

Next, we derive asymptotic lower bound for the ruin probability $\psi(\overrightarrow{u};T)$. From relation (9) we have

$$\begin{aligned}\psi(\overrightarrow{u};T)&\ge P\bigg(\sum_{i=1}^{N(T)}\overrightarrow{X}_i-T\overrightarrow{c}-(\sigma_1\overline{B}_1(T),\dots,\sigma_n\overline{B}_n(t))^T>\overrightarrow{u}\bigg)\\&=\sum_{k=0}^{\infty}P(N(T)=k)\prod_{j=1}^{n}P\bigg(\sum_{i=1}^{k}X_{ji}-\sigma_j\overline{B}_j(T)>u_j+c_jT\bigg).\end{aligned} \tag{15}$$

As in the first part of the proof we can see that

$$P\bigg(\sum_{i=1}^{k}X_{ji}-\sigma_j\overline{B}_j(T)>u_j+c_jT\bigg)\sim k\overline{F}_j(u_j)$$

for every $j=1,\dots,n$ and for each fixed $k=1,2\dots.$. Therefore, using the dominated convergence theorem, the right-hand side of (16) is also asymptotic to

$$\sum_{k=0}^{\infty}P(N(T)=k)k^n\overline{F}_1(u_1)\dots\overline{F}_n(u_n)=f(n)\overline{F}_1(u_1)\dots\overline{F}_n(u_n),$$

where $f(0)=1$, $f(1)=\lambda T$ i $f(n)=\lambda T\Bigg(\sum_{i=0}^{n-1}\binom{n-1}{i}f(i)\Bigg)$. This proves that

$$\psi(\overrightarrow{u};T)\geq(1+o(1))f(n)\overline{F}_1(u_1)\dots\overline{F}_n(u_n). \tag{16}$$

Finaly, using inequalities (14) and (16) we obtain the required relation (8). □

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