MATHEMATICAL COMMUNICATIONS
Math. Commun. **18**(2013), 241–256

A new approach on helices in Euclidean n-space

Ali Şenol[1], Evren Ziplar[2*], Yusuf Yayli[2] and İsmail Gök[2]

[1] *Department of Mathematics, Faculty of Science, Çankırı Karatekin University, TR-18 000 Çankırı, Turkey*
[2] *Department of Mathematics, Faculty of Science, University of Ankara, TR-06 100 Tandoğan-Ankara, Turkey*

Received August 12, 2012; accepted January 31, 2013

Abstract. In this work, we give some new characterizations for inclined curves and slant helices in n-dimensional Euclidean space E^n. Morever, we consider the pre-characterizations about inclined curves and slant helices and restructure them.
AMS subject classifications: 14H45, 14H50, 53A04
Key words: Inclined curve, slant helices, harmonic curvature

1. Introduction

The helices share common origins in the geometries of the platonic solids, with inherent hierarchical potential that is typical of biological structures. The helices provide an energy-efficient solution to close-packing in molecular biology, a common motif in protein construction, and a readily observable pattern at many size levels throughout the body. The helices are described in a variety of anatomical structures, suggesting their importance to structural biology and manual therapy [10].

In [9], Özdamar and Hacısalihoğlu defined harmonic curvature functions H_i $(1 \leq i \leq n-2)$ of a curve α in n-dimensional Euclidean space E^n. They generalized inclined curves in E^3 to E^n and then gave a characterization for the inclined curves in E^n :

$$\text{“A curve } \alpha \text{ is an inclined curve if and only if } \sum_{i=1}^{n-2} H_i^2 = \text{constant”}. \tag{1}$$

Harmonic curvature functions have an important role in characterizations of general helices in higher dimensions. Because the notion of a general helix can be generalized to higher dimension in different ways. But, these ways are not easy to show which curves are general helices and finding the axis of a general helix is complicated in higher dimension. Thanks to harmonic curvature functions, we can easily obtain the axis of such curves. Moreover, this way is confirmed in 3-dimensional spaces.

Then, Izumiya and Takeuchi defined a new kind of helix (slant helix) and gave a characterization of slant helices in Euclidean 3-space E^3 [7]. In 2008, Önder *et al.*

*Corresponding author. *Email addresses:* `asenol@karatekin.edu.tr` (A. Şenol), `evrenziplar@yahoo.com` (E. Zıplar), `yayli@science.ankara.edu.tr` (Y. Yayli), `igok@science.ankara.edu.tr` (İ. Gök)

defined a new kind of slant helix in Euclidean 4-space E^4 which is called a B_2-slant helix and they gave some characterizations of these slant helices in Euclidean 4-space E^4 [8]. And then in 2009, Gök *et al.* defined a new kind of slant helix in Euclidean n-space E^n, $n > 3$, which they called a V_n-slant helix and they gave some characterizations of these slant helices in Euclidean n-space [5]. The new kind of helix is a generalization of a B_2-slant helix to Euclidean 4-space E^4. On the other hand, Camcı *et al.* give some characterizations for a non-degenerate curve to be a generalized helix by using its harmonic curvatures in Euclidean n-space [3].

Since Özdamar and Hacısalihoğlu defined harmonic curvature functions, lots of authors have used them in their papers to characterize inclined curves and slant helices. In these studies, they gave some characterizations similar to (1) for inclined curves and slant helices. But, Camcı *et al.* see for the first time that the characterization of inclined curves in (1) is true for the case necessity but not true for the case sufficiency and gave an example of inclined curve in order to show why the case sufficiency is not true [3]. Also, they gave a property of inclined curves [3, Theorem 3.3, p. 2594]. But, they did not obtain when the condition of curves to be inclined. And then, Gök *et al.* [5] corrected the characterization of a B_2-slant helix ([8, Theorem 3.1, p. 1436]) like the characterization in (1). But, they also did not give the answer of the question: When the condition of curves to be inclined? After them, Ahmad and Lopez [1] and Ahmad and Melih [2] gave the definition of G_i $(1 \leq i \leq n)$ functions and obtain a characterization of inclined curves and slant helices, that is, V_1 and V_2-slant helix, respectively.

In this paper, we investigate the answer of the following question by the similar method in Theorem 4.1 in [4]:

When do the conditions of inclined curves and slant helices in Euclidean n-space E^n which are similar to (1) *turn to be necessary and sufficient?*

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be an arbitrary smooth curve in E^n. Recall that the curve α is said to be a curve of unit speed (or parameterized by its arclength) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ where $\langle , \rangle$ denotes the standart inner product of $\mathbb{R}^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_{2,...,} x_n), Y = (y_1, y_{2,...,} y_n) \in \mathbb{R}^n$. In particular, the norm of a vector $X \in \mathbb{R}^n$ is given by

$$\|X\|^2 = \langle X, X \rangle .$$

Let $\{V_1, V_2, \ldots, V_n\}$ be the moving Frenet frame along the unit speed curve α. Here V_i $(i = 1, 2, \ldots, n)$ denotes the ith Frenet vector field. Then the Frenet formulas are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-2}' \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{bmatrix},$$

where $k_i (i = 1, 2, \dots, n-1)$ denotes the ith curvature function of the curve [6]. If all of the curvatures k_i $(i = 1, 2, \dots, n-1)$ of the curve vanish nowhere in $I \subset \mathbb{R}$, the curve is said to be non-degenerate and of order n.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \to E^n$ be a curve in E^n with arc-length parameter s and let X be a unit constant vector of E^n. For all $s \in I$, if*

$$\langle V_1, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = constant,$$

then the curve α is called a general helix or inclined curve (V_1-slant helix) in E^n where V_1 is the unit tangent vector of α at its point $\alpha(s)$ [9].

Definition 2. *Let $\alpha : I \subset \mathbb{R} \to E^n$ be a curve in E^n with arc-length parameter s and let X be a unit constant vector of E^n. For all $s \in I$, if*

$$\langle V_2, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = constant,$$

then the curve α is called a slant helix or a V_2-slant helix in E^n where V_2 is the 2nd vector field of α and φ is a constant angle between the vector fields V_2 and X [1].

Definition 3. *Let $\alpha : I \subset \mathbb{R} \to E^n$ be a unit speed curve with nonzero curvatures k_i ($1 \leq i \leq n-1$) in E^n and let $\{V_1, V_2, \dots, V_n\}$ denote the Frenet frame of the curve α. We call α a V_n-slant helix if the n-th unit vector field V_n makes a constant angle φ with a fixed direction X, that is,*

$$\langle V_n, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = constant,$$

along the curve α. [5].

From now on, a smooth curve α is supposed to be non-degenerate throughout of this paper.

3. Inclined curves and their harmonic curvature functions

In this section, we restructure some known characterizations by using harmonic curvatures for inclined curves.

Definition 4. *Let α be a unit curve in E^n. The harmonic curvatures $H_i : I \to \mathbb{R}$, $i = 0, 1, \ldots, n-2$, of α are defined inductively by*

$$H_0 = 0, H_1 = \frac{k_1}{k_2}, H_i = \left\{ H'_{i-1} + k_i H_{i-2} \right\} \frac{1}{k_{i+1}}$$

for $2 \leq i \leq n-2$, where $k_i \neq 0$ for $i = 1, 2, \ldots, n-1$ [9] .

Lemma 1. *Let α be a unit curve in $E^n (n \geq 3)$. Suppose $H_{n-2} \neq 0$. Then, $H_1^2 + H_2^2 + \cdots + H_{n-2}^2$ is a nonzero constant if and only if $H'_{n-2} = -k_{n-1} H_{n-3}$.*

Proof. First, we assume that $H_1^2 + H_2^2 + \cdots + H_{n-2}^2$ is a nonzero constant. By the definition of H_i, we can write

$$k_{i+1} H_i = H'_{i-1} + k_i H_{i-2},\ 3 \leq i \leq n-2. \tag{2}$$

Hence, in (2), if we take $i+1$ instead of i, we get

$$H'_i = k_{i+2} H_{i+1} - k_{i+1} H_{i-1},\ 2 \leq i \leq n-3 \tag{3}$$

together with

$$H'_1 = k_3 H_2. \tag{4}$$

On the other hand, since $H_1^2 + H_2^2 + \cdots + H_{n-2}^2$ is constant, we have

$$H_1 H'_1 + H_2 H_2' + \cdots + H_{n-2} H'_{n-2} = 0$$

and so,

$$H_{n-2} H'_{n-2} = -H_1 H'_1 - H_2 H'_2 - \cdots - H_{n-3} H'_{n-3}. \tag{5}$$

By using (3) and (4), we obtain

$$H_1 H'_1 = k_3 H_1 H_2 \tag{6}$$

and

$$H_i H'_i = k_{i+2} H_i H_{i+1} - k_{i+1} H_{i-1} H_i,\ 2 \leq i \leq n-3. \tag{7}$$

Therefore, by using (5), (6) and (7), algebraic calculus shows that

$$H_{n-2} H'_{n-2} = -k_{n-1} H_{n-3} H_{n-2}.$$

Since $H_{n-2} \neq 0$, we get the relation $H'_{n-2} = -k_{n-1} H_{n-3}$.

Conversely, we assume that

$$H'_{n-2} = -k_{n-1} H_{n-3}. \tag{8}$$

By using (8) and $H_{n-2} \neq 0$, we can write

$$H_{n-2} H'_{n-2} = -k_{n-1} H_{n-2} H_{n-3}. \tag{9}$$

From (7), we have

$$\begin{aligned}
&\text{for } i = n-3,\ H_{n-3}H'_{n-3} = k_{n-1}H_{n-3}H_{n-2} - k_{n-2}H_{n-4}H_{n-3},\\
&\text{for } i = n-4,\ H_{n-4}H'_{n-4} = k_{n-2}H_{n-4}H_{n-3} - k_{n-3}H_{n-5}H_{n-4},\\
&\text{for } i = n-5,\ H_{n-5}H'_{n-5} = k_{n-3}H_{n-5}H_{n-4} - k_{n-4}H_{n-6}H_{n-5},\\
&\qquad \vdots\\
&\text{for } i = 2,\ H_2H'_2 = k_4H_2H_3 - k_3H_1H_2
\end{aligned}$$

and from (6), we have

$$H_1H'_1 = k_3H_1H_2.$$

So, algebraic calculus shows that

$$H_1H'_1 + H_2H'_2 + \cdots + H_{n-5}H'_{n-5} + H_{n-4}H'_{n-4} + H_{n-3}H'_{n-3} + H_{n-2}H'_{n-2} = 0. \tag{10}$$

And, by integrating (10), we can easily obtain that

$$H_1^2 + H_2^2 + \cdots + H_{n-2}^2$$

is a non-zero constant. This completes the proof. □

Theorem 1. *Let α be an inclined curve and X the axis of α . Then,*

$$\langle V_{i+2}, X\rangle = H_i \langle V_1, X\rangle,\ 1 \leq i \leq n-2,$$

where $\{V_1, V_2, \ldots, V_n\}$ denote the Frenet frame of a curve α of order $n \geq 3$ and $\{H_1, H_2, \ldots, H_{n-2}\}$ denote the harmonic curvature functions of α [6] or [9].

Theorem 2. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$ and let $\{H_1, H_2, \ldots, H_{n-2}\}$ be the harmonic curvature functions of α. Then, α is an inclined curve (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n-1$) in E^n if and only if its harmonic curvatures satisfy that*

$$\sum_{i=1}^{n-2} H_i^2$$

is equal to the constant and $H_{n-2} \neq 0$.

Proof. Let α be an inclined curve. According to Definition 1,

$$\langle V_1, X\rangle = \cos(\varphi) = \text{constant} \tag{11}$$

where X is the axis of α. And, from Theorem (1),

$$\langle V_{i+2}, X\rangle = H_i \langle V_1, X\rangle \tag{12}$$

for $1 \leq i \leq n-2$. Moreover, from (11) and Frenet equations, we can write $\langle V_2, X\rangle = 0$. Since the orthonormal system$\{V_1, V_2, \ldots, V_n\}$ is a basis of $\chi(E^n)$ (tangent bundle), X can be expressed in the form

$$X = \sum_{i=1}^{n} \langle V_i, X\rangle V_i. \tag{13}$$

Hence, by using the equations (11), (12) and (13), we obtain

$$X = \cos(\varphi)V_1 + \sum_{i=1}^{n-2} H_i \cos(\varphi)V_{i+2}.$$

Since X is a unit vector field (see Definition 1),

$$\cos^2(\varphi) + \sum_{i=1}^{n-2} H_i^2 \cos^2(\varphi) = 1$$

and so

$$\sum_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant}.$$

Now, we show that $H_{n-2} \neq 0$. We assume that $H_{n-2} = 0$. Then, for $i = n-2$ in Theorem 1,

$$\langle V_n, X\rangle = H_{n-2} \langle V_1, X\rangle = 0.$$

So, $\langle D_T V_n, X\rangle = \langle -k_{n-1}V_{n-1}, X\rangle = 0$. We deduce that $\langle V_{n-1}, X\rangle = 0$. On the other hand, for $i = n-3$ in Theorem 1,

$$\langle V_{n-1}, X\rangle = H_{n-3} \langle V_1, X\rangle.$$

And, since $\langle V_{n-1}, X\rangle = 0$, $H_{n-3} = 0$. Continuing this process, we get that $H_1 = 0$. Recalling that $H_1 = \dfrac{k_1}{k_2}$, we find it is a contradiction because all the curvatures are nowhere zero. Consequently, $H_{n-2} \neq 0$.

Conversely, we assume that $\sum\limits_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant}$ and $H_{n-2} \neq 0$. We consider the vector field

$$X = \cos(\varphi)V_1 + \sum_{i=3}^{n} H_{i-2} \cos(\varphi)V_i.$$

We shall verify that X is parallel along α, i.e. $D_{V_1}X = 0$. We have,

$$\begin{aligned}
D_{V_1}X &= D_{V_1}\left(\cos(\varphi)V_1\right) + \sum_{i=3}^{n} D_{V_1}\left(H_{i-2}\cos(\varphi)V_i\right) \\
&= \cos(\varphi)D_{V_1}V_1 + \sum_{i=3}^{n}\left(H'_{i-2}\cos(\varphi)V_i + H_{i-2}\cos(\varphi)D_{V_1}V_i\right) \\
&= \cos(\varphi)(k_1V_2 + \sum_{i=3}^{n-1}\left(H'_{i-2}V_i - k_{i-1}H_{i-2}V_{i-1} + k_iH_{i-2}V_{i+1}\right) \\
&\quad + H'_{n-2}V_n - k_{n-1}H_{n-2}V_{n-1}).
\end{aligned}$$

On the other hand, by using (3), we can write

$$H'_{i-2} = k_iH_{i-1} - k_{i-1}H_{i-3} \tag{14}$$

for $4 \leq i \leq n-1$ together with (4). Moreover, from Lemma 1, we know that

$$H'_{n-2} = -k_{n-1}H_{n-3} \tag{15}$$

Therefore, by using (4), (14) and (15), an algebraic calculus shows that $D_{V_1}X = 0$. Since

$$\begin{aligned}
\|X\| &= \cos^2(\varphi) + \sum_{i=3}^{n} H_{i-2}^2\cos^2(\varphi) \\
&= \cos^2(\varphi)\left(1 + \sum_{i=1}^{n-2} H_i^2\right) \\
&= \cos^2(\varphi)\left(1 + \tan^2(\varphi)\right) \\
&= 1\ ,
\end{aligned}$$

X is a unit vector field. Furthermore, $\langle V_1, X\rangle = \cos(\varphi)$ =constant. Hence, we deduce that α is an inclined curve. □

Remark 1. *The following corollary is the restructuring of Theorem 3.4 in [3].*

Corollary 1. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$ and let $\{H_1, H_2, \ldots, H_{n-2}\}$ be the harmonic curvature functions of α. Then, α is an inclined curve (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n-1$) in E^n if and only if $H'_{n-2} = -k_{n-1}H_{n-3}$ and $H_{n-2} \neq 0$.*

Proof. It is obvious by using Lemma 1 and Theorem 2. □

4. V_n-slant helices and their harmonic curvature functions

In this section, we restructure some known characterizations by using harmonic curvatures for V_n-slant helices.

Definition 5. *Let $\alpha : I \subset \mathbb{R} \to E^n$ be a unit speed curve with nonzero curvatures k_i $(i = 1, 2, \ldots, n-1)$ in E^n. Harmonic curvature functions $H_i^* : I \subset \mathbb{R} \to \mathbb{R}$ of α are defined inductively by*

$$H_0^* = 0, H_1^* = \frac{k_{n-1}}{k_{n-2}}, H_i^* = \left\{k_{n-i}H_{i-2}^* - H_{i-1}^{*\prime}\right\} \frac{1}{k_{n-(i+1)}}$$

for $2 \le i \le n-2$ [5].

Lemma 2. *Let α be a unit curve in E^n of order $n \ge 3$. When $H_{n-2}^* \neq 0$, the sum $H_1^{*2} + H_2^{*2} + \cdots + H_{n-2}^{*2}$ is a nonzero constant if and only if $H_{n-2}^{*\prime} = k_1 H_{n-3}^*$.*

Proof. First, we assume that $H_1^{*2} + H_2^{*2} + \cdots + H_{n-2}^{*2}$ is a nonzero constant. By the definition of H_i^*, we can write

$$k_{n-(i+1)}H_i^* = k_{n-i}H_{i-2}^* - H_{i-1}^{*\prime}, \; 3 \le i \le n-2. \tag{16}$$

Hence, in (16), if we take $i+1$ instead of i, we get

$$H_i^{*\prime} = k_{n-(i+1)}H_{i-1}^* - k_{n-(i+2)}H_{i+1}^*, 2 \le i \le n-3 \tag{17}$$

together with

$$H_1^{*\prime} = -k_{n-3}H_2^*. \tag{18}$$

On the other hand, since $H_1^{*2} + H_2^{*2} + \cdots + H_{n-2}^{*2}$ is constant, we have

$$H_1^* H_1^{*\prime} + H_2^* H_2^{*\prime} + \cdots + H_{n-2}^* H_{n-2}^{*\prime} = 0$$

and so,

$$H_{n-2}^* H_{n-2}^{*\prime} = -H_1^* H_1^{*\prime} - H_2^* H_2^{*\prime} - \cdots - H_{n-3}^* H_{n-3}^{*\prime}. \tag{19}$$

By using (17) and (18), we obtain

$$H_1^* H_1^{*\prime} = -k_{n-3} H_1^* H_2^* \tag{20}$$

and

$$H_i^* H_i^{*\prime} = k_{n-(i+1)} H_{i-1}^* H_i^* - k_{n-(i+2)} H_i^* H_{i+1}^*, \; 2 \le i \le n-3. \tag{21}$$

Therefore, by using (19), (20) and (21), algebraic calculus shows that

$$H_{n-2}^* H_{n-2}^{*\prime} = k_1 H_{n-3}^* H_{n-2}^*.$$

Since $H_{n-2}^* \neq 0$, we get the relation $H_{n-2}^{*\prime} = k_1 H_{n-3}^*$.

Conversely, we assume that

$$H_{n-2}^{*\prime} = k_1 H_{n-3}^*. \tag{22}$$

By using (22) and $H_{n-2}^* \neq 0$, we can write

$$H_{n-2}^* H_{n-2}^{*\prime} = k_1 H_{n-2}^* H_{n-3}^*. \tag{23}$$

From (21), we have

$$\begin{aligned}
&\text{for } i = n-3,\ H_{n-3}^{*}H_{n-3}^{*\prime} = k_2 H_{n-4}^{*}H_{n-3}^{*} - k_1 H_{n-3}^{*}H_{n-2}^{*},\\
&\text{for } i = n-4,\ H_{n-4}^{*}H_{n-4}^{*\prime} = k_3 H_{n-5}^{*}H_{n-4}^{*} - k_2 H_{n-4}^{*}H_{n-3}^{*},\\
&\text{for } i = n-5,\ H_{n-5}^{*}H_{n-5}^{*\prime} = k_4 H_{n-6}^{*}H_{n-5}^{*} - k_3 H_{n-5}^{*}H_{n-4}^{*},\\
&\qquad\vdots\\
&\text{for } i = 2,\ H_{2}^{*}H_{2}^{*\prime} = k_{n-3} H_{1}^{*}H_{2}^{*} - k_{n-4} H_{2}^{*}H_{3}^{*}
\end{aligned}$$

and from (20), we have

$$H_1^{*}H_1^{*\prime} = -k_{n-3}H_1^{*}H_2^{*}.$$

So, algebraic calculus shows that

$$H_1^{*}H_1^{*\prime} + H_2^{*}H_2^{*\prime} + \cdots + H_{n-5}^{*}H_{n-5}^{*\prime} + H_{n-4}^{*}H_{n-4}^{*\prime} + H_{n-3}^{*}H_{n-3}^{*\prime} + H_{n-2}^{*}H_{n-2}^{*\prime} = 0. \quad (24)$$

And, by integrating (24), we can easily say that

$$H_1^{*2} + H_2^{*2} + \cdots + H_{n-2}^{*2}$$

is a nonzero constant. This completes the proof. □

Proposition 1 (see [5]). *Let $\alpha : I \subset \mathbb{R} \to E^n$ be an arc-lengthed parametrized curve in E^n and X a unit constant vector field of $\mathbb{R}^n$. We denote by $\{V_1, V_2, \ldots, V_n\}$ the Frenet frame of the curve α and by $\left\{H_1^{*}, H_2^{*}, \ldots, H_{n-2}^{*}\right\}$ the harmonic curvature functions of the curve α. If $\alpha : I \subset \mathbb{R} \to E^n$ is an V_n-slant helix with X as its axis, then we have for all $i = 0, 1, \ldots, n-2$*

$$\left\langle V_{n-(i+1)}, X\right\rangle = H_i^{*}\left\langle V_n, X\right\rangle.$$

Remark 2. *The following Theorem is new version of Theorem 4 in [5] which adds the sufficiency case.*

Theorem 3. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$ and let $\left\{H_1^{*}, H_2^{*}, \ldots, H_{n-2}^{*}\right\}$ be the harmonic curvature functions of α. Then, α is a V_n-slant helix (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n-1$) in E^n if and only if it satisfies that*

$$\sum_{i=1}^{n-2} H_i^{*2}$$

is equal to constant and $H_{n-2}^{*} \neq 0$.

Proof. Suppose α is a V_n-slant helix. According to Definition 3,

$$\langle V_n, X\rangle = \cos(\varphi) = \text{constant} \quad (25)$$

where X is the axis of α. From Proposition 1 we have

$$\left\langle V_{n-(i+1)}, X\right\rangle = H_i^{*}\left\langle V_n, X\right\rangle \quad (26)$$

for $1 \leq i \leq n-2$. Moreover, from (25) and Frenet equations, we can write $\langle V_{n-1}, X \rangle = 0$. Since the orthonormal system$\{V_1, V_2, \ldots, V_n\}$ is a basis of $\varkappa(E^n)$ (tangent bundle), X can be expressed in the form

$$X = \sum_{i=1}^{n} \langle V_i, X \rangle V_i. \tag{27}$$

Hence, by using the equations (25), (26) and (27), we obtain

$$X = \cos(\varphi)V_n + \sum_{i=1}^{n-2} H_i^* \cos(\varphi) V_{n-(i+1)}.$$

Since X is a unit vector field (see Definition 3),

$$\cos^2(\varphi) + \sum_{i=1}^{n-2} H_i^{*2} \cos^2(\varphi) = 1$$

and so

$$\sum_{i=1}^{n-2} H_i^{*2} = \tan^2(\varphi) = \text{constant}.$$

Now, we show that $H_{n-2}^* \neq 0$. We assume that $H_{n-2}^* = 0$. Then, for $i = n-2$ in (26),

$$\langle V_1, X \rangle = H_{n-2}^* \langle V_n, X \rangle = 0.$$

So, $\langle D_T T, X \rangle = \langle k_1 V_2, X \rangle = 0$. We deduce that $\langle V_2, X \rangle = 0$. On the other hand, for $i = n-3$ in (26),

$$\langle V_2, X \rangle = H_{n-3}^* \langle V_n, X \rangle.$$

And since $\langle V_2, X \rangle = 0$ and $H_{n-3}^* = 0$. Continuing this process, we get that $H_1^* = 0$. Let us recall that $H_1^* = \dfrac{k_{n-1}}{k_{n-2}}$, thus we have a contradiction because all the curvatures are nowhere zero. Consequently, $H_{n-2}^* \neq 0$.

Conversely, we assume that $\sum\limits_{i=1}^{n-2} H_i^{*2} = \tan^2(\varphi)$ =constant and $H_{n-2}^* \neq 0$. We take the vector field

$$X = \cos(\varphi)V_n + \sum_{i=3}^{n} H_{i-2}^* \cos(\varphi) V_{n-(i-1)},$$

and it is parallel along α, i.e. $D_{V_1}X = 0$. By direct calculation, we have

$$\begin{aligned} D_{V_1}X &= D_{V_1}\left(\cos(\varphi)V_n\right) + \sum_{i=3}^{n} D_{V_1}\left(H^*_{i-2}\cos(\varphi)V_{n-(i-1)}\right) \\ &= \cos(\varphi)D_{V_1}V_n + \sum_{i=3}^{n}\left(H^{*\prime}_{i-2}\cos(\varphi)V_{n-(i-1)} + H^*_{i-2}\cos(\varphi)D_{V_1}V_{n-(i-1)}\right) \\ &= \cos(\varphi)(-k_{n-1}V_{n-1} \\ &\quad + \sum_{i=3}^{n-1}\left(H^{*\prime}_{i-2}V_{n-(i-1)} - k_{n-i}H^*_{i-2}V_{n-i} + k_{n-(i-1)}H^*_{i-2}V_{n-(i-2)}\right) \\ &\quad + H^{*\prime}_{n-2}V_1 + k_1H^*_{n-2}V_2). \end{aligned}$$

Here, in the case $n = 3$ we omit the term of sum.

On the other hand, by using (17), we can write

$$H^{*\prime}_{i-2} = k_{n-(i-1)}H^*_{i-3} - k_{n-i}H^*_{i-1} \tag{28}$$

for $4 \leq i \leq n-1$ together with (18). Moreover, from Lemma 2, we know that

$$H^{*\prime}_{n-2} = k_1H^*_{n-3}. \tag{29}$$

Therefore, by using (18), (28), (29) and by the definition of H^*_1, and the Einstein tensor, algebraic calculus shows that $D_{V_1}X = 0$. Since

$$\begin{aligned} \|X\| &= \cos^2(\varphi) + \sum_{i=3}^{n} H^{*2}_{i-2}\cos^2(\varphi) \\ &= \cos^2(\varphi)\left(1 + \sum_{i=1}^{n-2} H^{*2}_i\right) \\ &= \cos^2(\varphi)\left(1 + \tan^2(\varphi)\right) \\ &= 1\ , \end{aligned}$$

X is a unit vector field. Furthermore, $\langle V_n, X\rangle = \cos(\varphi)$ =constant. Hence, we deduce that α is a V_n.-slant helix. $\square$

Remark 3. *The following corollary is a restructuring of Theorem 2 in [5].*

Corollary 2. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$ and let $\left\{H^*_1, H^*_2, \ldots, H^*_{n-2}\right\}$ be the harmonic curvature functions of α. Then, α is a V_n-slant helix (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n-1$) in E^n if and only if $H^{*\prime}_{n-2} = k_1H^*_{n-3}$ and $H^*_{n-2} \neq 0$.*

Proof. It is obvious by using Lemma 2 and Theorem 3. $\square$

5. Slant helices and their G_i differentiable functions

In this section, we restructure some known characterizations of slant helices by using differentiable functions which are similar to harmonic curvature functions.

Definition 6. *Let $\alpha : I \to E^n$ be a unit speed curve (with the curvatures $k_i \neq 0$, $i = 1, 2, \dots, n-1$) in E^n. Define the functions G_i inductively by*

$$G_1 = \int k_1(s)ds \ , \ G_2 = 1 \ , \ G_3 = \frac{k_1}{k_2} G_1 \ , \ G_i = \frac{1}{k_{i-1}} \left[k_{i-2} G_{i-2} + G'_{i-1} \right] \tag{30}$$

where $4 \leq i \leq n$ [1].

Lemma 3. *Let α be a unit curve in E^n. Suppose $G_n \neq 0$. Then, $G_1^2 + G_2^2 + \cdots + G_n^2$ is a nonzero constant if and only if $G'_n = -k_{n-1} G_{n-1}$.*

Proof. First, we assume that $G_1^2 + G_2^2 + \cdots + G_n^2$ is a nonzero constant. By the definition of G_i, we can write

$$k_{i-1} G_i = G'_{i-1} + k_{i-2} G_{i-2}, \ 5 \leq i \leq n. \tag{31}$$

Hence, in (31), if we take $i+1$ instead of i, we get

$$G'_i = k_i G_{i+1} - k_{i-1} G_{i-1}, \ 4 \leq i \leq n-1. \tag{32}$$

together with

$$G_1 = \int k_1(s)ds \ , \ G_2 = 1 \ , \ G_3 = \frac{k_1}{k_2} G_1. \tag{33}$$

On the other hand, since $G_1^2 + G_2^2 + \cdots + G_n^2$ is constant, we have

$$G_1 G'_1 + G_2 G_2' + \cdots + G_n G'_n = 0$$

and so,

$$G_n G'_n = -G_1 G'_1 - G_2 G_2' - \cdots - G_{n-1} G'_{n-1}. \tag{34}$$

By using (32) and (33), we obtain

$$G_2 G'_2 = 0 \ \text{ and } \ k_3 G_3 G_4 = G_1 G'_1 + G_3 G'_3 \tag{35}$$

and

$$G_i G'_i = k_i G_i G_{i+1} - k_{i-1} G_{i-1} G_i \ , \ 4 \leq i \leq n-1. \tag{36}$$

Therefore, by using (34), (35) and (36), algebraic calculus shows that

$$G_n G'_n = -k_{n-1} G_{n-1} G_n \ .$$

Since $G_n \neq 0$, we get the relation $G'_n = -k_{n-1} G_{n-1}$.

Conversely, we assume that

$$G'_n = -k_{n-1} G_{n-1} \ . \tag{37}$$

By using (37) and $G_n \neq 0$, we can write

$$G_n \acute{G_n} = -k_{n-1} G_{n-1} G_n \ . \tag{38}$$

From (36), we have

$$\begin{aligned}
&\text{for } i = n-1,\ G_{n-1}\acute{G}_{n-1} = k_{n-1}G_{n-1}G_n - k_{n-2}G_{n-2}G_{n-1},\\
&\text{for } i = n-2,\ G_{n-2}\acute{G}_{n-2} = k_{n-2}G_{n-2}G_{n-1} - k_{n-3}G_{n-3}G_{n-2},\\
&\text{for } i = n-3,\ G_{n-3}\acute{G}_{n-3} = k_{n-3}G_{n-3}G_{n-2} - k_{n-4}G_{n-4}G_{n-3},\\
&\qquad \vdots\\
&\text{for } i = 4,\ G_4\acute{G_4} = k_4G_4G_5 - k_3G_3G_4
\end{aligned}$$

and so, from (38) and the last system, we have

$$G_4G_4' + G_5G_5' + \cdots + G_nG_n' = -k_3G_3G_4 \tag{39}$$

by doing algebraic calculus. On the other hand, from (35), we know that

$$G_2G_2' = 0 \ \text{ and } \ k_3G_3G_4 = G_1G_1' + G_3G_3' \ . \tag{40}$$

Finally, from (39) and (40) we obtain

$$G_1\acute{G_1} + G_2\acute{G_2} + \cdots + G_n\acute{G_n} = 0 \ . \tag{41}$$

And by integrating (41) we can easily say that

$$G_1^2 + G_2^2 + \cdots + G_n^2$$

is a nonzero constant. This completes the proof. □

Corollary 3. *Let $\alpha : I \subset \mathbb{R} \to E^n$ be an arc-lengthed parametrized curve with nonzero curvatures k_i $(1 \leq i \leq n-1)$ in E^n and X a unit constant vector field of $\mathbb{R}^n$. $\{V_1, V_2, \ldots, V_n\}$ denote the Frenet frame of the curve α. If $\alpha : I \subset \mathbb{R} \to E^n$ is a V_2-slant helix with X as its axis, then we have for all $i = 1, \ldots, n$*

$$\langle V_i, X\rangle = G_i \langle V_2, X\rangle \ .$$

Proof. It is obvious by using the proof of Theorem 1.2 in [1]. □

Remark 4. *The following Theorem is new version of Theorem 1.2 in [1].*

Theorem 4. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$. Then, α is a V_2-slant helix (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n-1$) in E^n if and only if*

$$\sum_{i=1}^{n} G_i^2$$

is equal to constant and $G_n \neq 0$. Here,

$$G_1 = \int k_1(s)ds \ , \ G_2 = 1 \ , \ G_3 = \frac{k_1}{k_2}G_1 \ , \ G_i = \frac{1}{k_{i-1}}\left[k_{i-2}G_{i-2} + G_{i-1}'\right]$$

where $4 \leq i \leq n$.

Proof. Let α be a V_2-slant helix. According to Definition 2,

$$\langle V_2, X\rangle = \cos(\varphi) = \text{constant}, \tag{42}$$

where X is the axis of α. And from Corollary 3

$$\langle V_i, X\rangle = G_i \langle V_2, X\rangle \tag{43}$$

for $1 \leq i \leq n$. Since the orthonormal system$\{V_1, V_2, \ldots, V_n\}$ is a basis of $\varkappa(E^n)$ (tangent bundle), X can be expressed in the form

$$X = \sum_{i=1}^{n} \langle V_i, X\rangle V_i. \tag{44}$$

Hence, by using the equations (42), (43) and (44), we obtain

$$X = \sum_{i=1}^{n} G_i \cos(\varphi) V_i.$$

Since X is a unit vector field (see Definition 2),

$$\cos^2(\varphi)\left(\sum_{i=1}^{n} G_i^2\right) = 1$$

and so

$$\sum_{i=1}^{n} G_i^2 = \frac{1}{\cos^2(\varphi)} = \text{constant} .$$

Now, we are going to show that $G_n \neq 0$. We assume that $G_n = 0$. Then, for $i = n$ in (43),

$$\langle V_n, X\rangle = G_n \langle V_2, X\rangle = 0.$$

So, $\langle D_T V_n, X\rangle = \langle -k_{n-1} V_{n-1}, X\rangle = 0$. We deduce that $\langle V_{n-1}, X\rangle = 0$. On the other hand, for $i = n-1$ in (43),

$$\langle V_{n-1}, X\rangle = G_{n-1} \langle V_2, X\rangle .$$

And since $\langle V_{n-1}, X\rangle = 0$, we obtain $G_{n-1} = 0$. Continuing this process, we get that $G_3 = 0$. Let us recall that $G_3 = \frac{k_1}{k_2} \int k_1(s)\, ds$, thus we have a contradiction because all the curvatures are nowhere zero. Consequently $G_n \neq 0$.

Conversely, we assume that $\sum\limits_{i=1}^{n} G_i^2 = \frac{1}{\cos^2(\varphi)}$ =constant and $G_n \neq 0$. We consider the vector field

$$X = \sum_{i=1}^{n} G_i \cos(\varphi) V_i.$$

Then, by taking account

$$G_1 = \int k_1(s)ds \ , \ G_2 = 1 \ , \ G_3 = \frac{k_1}{k_2} G_1 \ , \ G_i = \frac{1}{k_{i-1}} \left[k_{i-2} G_{i-2} + G'_{i-1} \right] \ , \ 4 \leq i \leq n$$

and Frenet equations, an algebraic calculus shows that $D_{V_1} X = 0$. That is, X is a constant along α. Also, since

$$\begin{aligned} \|X\| &= \sum_{i=1}^{n} G_i^2 \cos^2(\varphi) \\ &= \cos^2(\varphi) \left(\sum_{i=1}^{n} G_i^2 \right) \\ &= \cos^2(\varphi) \frac{1}{\cos^2(\varphi)} \\ &= 1 \ , \end{aligned}$$

X is a unit vector field. Furthermore, $\langle V_2, X \rangle = \cos(\varphi)$ =constant. Hence, we deduce that α is a V_2.-slant helix. □

Remark 5. *The following corollary is a restructuring of Theorem 3.1 in [1].*

Corollary 4. *Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame of a curve α of order $n \geq 3$. Then, α is a V_2-slant helix in E^n if and only if $G'_n = -k_{n-1} G_{n-1}$ and $G_n \neq 0$, where the functions $\{G_1, G_2, \ldots, G_n\}$ are defined in (30).*

Proof. It is obvious by using Lemma 3 and Theorem 4. □

Acknowledgement

The authors would like to express their sincere gratitude to the referees and Professor H. H.Hacısalihoğlu for valuable suggestions to improve the paper. Also, the last author would like to thank Tubitak-Bideb for their financial support during his PhD studies.

References

[1] A. T. ALI, R. LOPEZ, *Some characterizations of inclined curves in Euclidean E^n space*, Novi Sad J. Math. **40**(2010), 9–17.
[2] A. T. ALI, T. MELIH, *Some characterizations of slant helices in the Euclidean space E^n*, Hacet. J. Math. Stat. **39**(2010), 327–336.
[3] Ç. CAMCI, K. İLARSLAN, L. KULA, H. H. HACISALIHOĞLU, *Harmonic curvatures and generalized helices in E^n*, Chaos, Solitons and Fractals **40**(2009), 2590–2596.
[4] A. J. DI SCALA, G. RUIZ-HERNÁNDEZ, *Higher codimensional euclidean helix submanifolds*, Kodai Math. J. **33**(2010), 192–210.
[5] İ. GÖK, Ç. CAMCI, H. H. HACISALIHOĞLU, *V_n-slant helices in Euclidean n-space E^n*, Math. Commun. **14**(2009), 317–329.
[6] H. H. HACISALIHOĞLU, *Differential Geometry*, Faculty of Sciences and Arts, University of İnönü Press, 1983.

[7] S. Izumiya, N. Takeuchi, *New special curves and developable surfaces*, Turk J. Math. **28**(2004), 531–537.

[8] M. Önder, M. Kazaz, H. Kocayiğit, O. Kiliç, *B_2-slant helix in Euclidean 4-space* E^4, Int. J. Cont. Math. Sci. **3**(2008), 1433–1440.

[9] E. Özdamar, H. H. Hacisalihoğlu, *A characterization of inclined curves in Euclidean n-space*, Communication de la faculte des sciences de L' Universite d' Ankara series A1 **24AA**(1975), 15–22.

[10] G. Scarr, *Helical tensegrity as a structural mechanism in human anatomy*, International Journal of Osteopathic Medicine **14**(2011), 24–32.