On completely equidistributed numbers*

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Abstract. *We analyzed a new method for proving equidistribution of numbers. The proposed method is simple and can be used to prove equidistribution of all known classes of numbers such as Weyl's numbers and Koksma's numbers. Emphasis of this approach is put on the complete equidistribution of numbers and a non-trivial result in this direction is obtained.*

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1. Introduction

Points of the cube $D = [0,1]^d$ are denoted by $\boldsymbol{\beta}_k$. A denumerable sequence of functions $\mathbf{x}_k : G \mapsto \mathbb{R}^d$, where $G \subset \mathbb{R}^r$, defines a denumerable sequence of points $\boldsymbol{\beta}_k \in D$, by the expression:

$$\boldsymbol{\beta}_k \;=\; \mathbf{x}_k(\mathbf{t}) \;[\text{mod } \mathbf{1}], \quad \mathbf{t} \in G.$$

The set G is called the set of seeds. Let $\mathbb{1}_S$ be the indicator of a set $S = \prod_j [a_j, b_j] \subset D$, $0 \leq a_j < b_j \leq 1$. We say that a sequence $\boldsymbol{\beta} = \{\boldsymbol{\beta}_k : k \in \mathbb{N}\}$ is equidistributed in D in Weyl's sense if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_S(\boldsymbol{\beta}_k) \;=\; \text{mess}\,(S)$$

for each $S \subset D$.

Let I be a bounded interval of $\mathbb{R}$. Familiar examples of functions $x_k : I \mapsto \mathbb{R}$, defining equidistributed sequences in $[0,1]$, are as follows. The sequence of functions,

$$x_k(t) \;=\; n^p\, t, \quad t \in I \;=\; (0,1), \tag{1}$$

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where $p \in \mathbb{N}$, defines Weyl's numbers $\beta_k \in [0,1]$ of parameter p. The sequence of functions

$$x_k(t) \;=\; M^k\, t, \quad t \in I \;=\; (0,1), \tag{2}$$

defines multiplicatively generated numbers in $[0,1]$ for each $M \in \{2,3,\dots,\}$. The sequence of functions

$$x_k(t) \;=\; t^k, \quad t \in I \;=\; (1,a), \tag{3}$$

with any $a > 1$, defines Koksma's numbers in $[0,1]$. All these numbers are equidistributed for almost all seeds [We, F1, Ko] (see also [Ca]). This means that there exists a set $T \subset I, \operatorname{mess}(T) = \operatorname{mess}(I)$, such that $\beta_k = x_k(t)$ [mod 1] is equidistributed in $[0,1]$ for $t \in T$.

The functions (1) - (3) can be used to generate points $\boldsymbol{\beta}_k = (\beta_{k1}, \beta_{k2}, \dots, \beta_{kd})$ in $D = [0,1]^d$ by two different possibilities:

$$\beta_{kj} \;=\; x_{j+(k-1)d}(t_j)\ [\text{mod } 1], \quad \mathbf{t} \;=\; (t_1, t_2, \dots, t_d) \;\in\; I^d, \tag{4}$$

$$\beta_{kj} \;=\; x_{k+j-1}(t)\ [\text{mod } 1], \quad t \;\in\; I. \tag{5}$$

We say that the numbers $\{\beta_k : k \in \mathbb{N}\} \subset [0,1]$ are d-dimensionally equidistributed in $[0,1]$ if the sequence of points $\boldsymbol{\beta}_k$ defined by (5) is equidistributed in $[0,1]^d$ for a set $T \subset I, \operatorname{mess}(T) = \operatorname{mess}(I)$. In case this property holds for any $d > 1$, we say that the sequence $\{\beta_k : k \in \mathbb{N}\}$ is *completely* equidistributed. Koksma's numbers are completely equidistributed [F2].

The objective of this work is to present a unique method by which the encountered results can be obtained. This method is an alternative to the known ones [We, F1, F2, Ko]. It is based on the Weyl's criterion and a.e. convergence of a sequence of measurable functions. Hence, this new method is based on a tool which is extensively used in the theory of probability. The proposed method is very simple for the case of points $\boldsymbol{\beta}_k$ which are defined in terms of linear functions $\mathbf{x}_k$. This includes numbers defined by the functions (1), (2), and p-dimensionally equidistribution of numbers defined by (1). The method is described in *Section 2*. Nonlinear functions x_k are considered in *Section 3*, where the complete equidistribution of Koksma's numbers is proved by the proposed method. In addition, to illustrate an efficiency of the method, a new class of completely equidistributed numbers is presented at the end of the section. Concluding remarks are given in *Section 4.*.

2. Linear case

Let $\mathbb{Z}^d$ be the set of multi-indices $\mathbf{m} = (m_1, m_2, \dots, m_d), m_k \in \mathbb{Z}$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence of linear functions from $I = [0,1]$ to $\mathbb{R}$ and let points $\boldsymbol{\beta}_k \in D$ be defined by (4). A brief notation is $\boldsymbol{\beta}_k = \boldsymbol{x}_k(\mathbf{t})$ [mod $\mathbf{1}$].

In accordance with Weyl criterion we consider the quantities

$$W_N(\boldsymbol{\beta}, \mathbf{m}) \;=\; \frac{1}{N} \sum_{k=1}^{N} \exp\big(2\pi i\, \mathbf{m} \cdot \boldsymbol{\beta}_k\big),$$

where $\mathbf{a} \cdot \mathbf{b}$ is the scalar product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$. Now we use the fact that $\boldsymbol{\beta}_k \in D$ are equidistributed iff $W_N(\boldsymbol{\beta}, \mathbf{m}) \mapsto 0$ for $N \mapsto \infty$ and each $\mathbf{m} \neq \mathbf{0}$.

Let us use the notation $\mathbf{z} \mapsto e(\mathbf{m}, \mathbf{z}) = \exp(2\pi i \mathbf{m} \cdot \mathbf{z})$ and let $\mathbf{x}_k(\cdot)$ be defined by (1), (2), and (4). Then the sequence of functions $\mathbf{t} \mapsto e(\mathbf{m}, \mathbf{x}_k(\mathbf{t}))$ defines an orthonormal sequence of elements in $L_2(I^d)$.

The functions

$$D \ni \mathbf{t} \ \mapsto \ f_N(\mathbf{t}) \ = \ \frac{1}{N} \sum_{k=1}^{N} e\big(\mathbf{m}, \mathbf{x}_k(\mathbf{t})\big), \tag{6}$$

$\mathbf{m}$ is fixed, are the main object in our proof of equidistribution. Apparently, their $L_2(D)$-norms are $\|f_N\|_2 = 1/\sqrt{N}$ and certain subsequences converge a.e. to zero on D. Hence, certain subsequence of $W_N(\boldsymbol{\beta}, \mathbf{m})$ converges to zero. In order to pass from a subsequence to the whole sequence $\{W_N(\boldsymbol{\beta}, \mathbf{m}) : N \in \mathbb{N}\}$, we need an auxiliary result [Do, Ha]:

If $\alpha > 1$, then the sequence

$$\mathcal{I}(\alpha) \ = \ \{N(1), N(2), \dots, N(k), \dots\}, \quad N(k) \ \in \ [k^\alpha, k^\alpha + 1)$$

is an infinite subsequence of $\mathbb{N}$.

Lemma 1. *Let for each $\mathbf{m} \neq \mathbf{0}$ there exist $\alpha(\mathbf{m}) > 1$ such that a sequence of points $\{\boldsymbol{\beta}_k : k \in \mathbb{N}\} \subset D$ satisfies the conditions*

$$\lim_{N \in \mathcal{I}(\alpha(\mathbf{m}))} W_N(\boldsymbol{\beta}, \mathbf{m}) \ = \ 0, \quad \mathbf{m} \in \mathbb{Z}^d, \quad \mathbf{m} \neq \mathbf{0}.$$

Then $\{\boldsymbol{\beta}_k : k \in \mathbb{N}\}$ is equidistributed in D.

Now we can prove the main result of this section.

Proposition 1. *Let a sequence of linear functions $x_j : [0,1] \mapsto \mathbb{R}$ define points $\boldsymbol{\beta}_k \in D$ by (4). If for each $\mathbf{m} \in \mathbb{Z}^d, \mathbf{m} \neq \mathbf{0}$, the sequence $e(\mathbf{m}, \mathbf{x}_k) \in L_2(D)$ is orthonormal,*

$$\big(\, e(\mathbf{m}, \mathbf{x}_k) \,|\, e(\mathbf{m}, \mathbf{x}_l) \,\big) \ = \ \delta_{kl},$$

then the sequence of points $\boldsymbol{\beta}_k$ is equidistributed in D.

Proof. From Chebyshev's inequality

$$\operatorname{mess}\big(\big|f_N(\mathbf{t})\big| > \varepsilon\big) \ \le \ \frac{\|f_N\|_2^2}{\varepsilon^2} \ = \ \frac{1}{\varepsilon^2 N},$$

one concludes that $f_N(\mathbf{t}) \mapsto 0$ for almost all seeds $\mathbf{t} \in D$ if $N \mapsto \infty, N \in \mathcal{I}(\alpha)$. Then the statement follows from *Lemma 1*. □

Example 1. *Let $q_j, j \in \mathbb{N}$, be real numbers, $x_j(t) = j^p\, t + q_j$, and let $\mathbf{x}_k : D \mapsto \mathbb{R}^d$ be defined by components $x_{j+(k-1)d}(t_j), j = 1, 2, \dots, d$. Then the points (4) are equidistributed in D for almost all seeds $\mathbf{t} \in D$.*

Example 2. *Let d pairs (M_j, q_j), $M_j \in \{2, 3, \dots\}$, $q_j \in \mathbb{R}$, define the functions $x_k : [0,1] \mapsto \mathbb{R}$ by:*

$$t_j \ \mapsto \ x_{j+(k-1)d}(t_j) \ = \ M_j^k t_j \ + \ q_j, \quad j \ = \ 1, 2, \dots, d, \quad k \in \mathbb{N}.$$

Columns with elements $M_j^k t_j \ + \ q_j, j = 1, 2, \dots, d$, are denoted by $\mathbf{M}^k \mathbf{t} + \boldsymbol{q}$. To show that the sequence of points $\boldsymbol{\beta}_k = \mathbf{x}_k(\mathbf{t})$ [mod $\mathbf{1}$] is equidistributed in D for almost all seeds $\mathbf{t} \in D$ we need an auxiliary result.

Let $M_j \in \{2,3,\dots\}, j = 1,2,\dots,d$, and $\boldsymbol{q},\boldsymbol{\sigma} \in \mathbb{R}^d$ with components q_j,σ_j related by the equations $\sigma_j(M_j-1)+q_j = 0$. Two sequences of points in $\overline{D}$:

$$\boldsymbol{\beta}_{k+1} \;=\; \mathbf{M}\boldsymbol{\beta}_k \;[\text{mod }\mathbf{1}], \quad \boldsymbol{\beta}_0 \in D,$$
$$\boldsymbol{\gamma}_{k+1} \;=\; \mathbf{M}\boldsymbol{\gamma}_k \;+\; \boldsymbol{q}\;[\text{mod }\mathbf{1}], \quad \boldsymbol{\gamma}_0 \;=\; \boldsymbol{\beta}_0 \;+\; \boldsymbol{\sigma}\;[\text{mod }\mathbf{1}],$$

are simultaneously either equidistributed or not. This statement can be easily proved. By a straightforward calculation one gets $\boldsymbol{\gamma}_k = \boldsymbol{\beta}_k + \boldsymbol{\sigma}$ [mod $\mathbf{1}$], $k \in \mathbb{N}$, and the statement follows from $W_N(\boldsymbol{\gamma},\mathbf{m}) = \exp(2\pi i\mathbf{m}\cdot\boldsymbol{\rho})W_N(\boldsymbol{\beta},\mathbf{m})$.

Example 3. *Let us define $\beta_j = j^p t + q_j$ where $q_k \in \mathbb{R}$. For each $\mathbf{m} \in \mathbb{Z}^p$, $\mathbf{m} \neq \mathbf{0}$, the functions:*

$$x_k(t) \;=\; t\sum_{j=1}^{p} m_j\,(k+j-1)^p \;+\; z_k,$$

where $z_k = \sum_j m_j q_j$, fulfil the conditions of Proposition 1. *This means that $\gamma_k = x_k(t)$ [mod 1] are equidistributed in [0,1] for almost all seeds. Thus, β_j are p-dimensionaly equidistributed in* $[0,1]$ *for almost all seeds.*

3. Nonlinear case

A function f on $\mathbb{R}^d$ is said to be piecewise continuous if there exists a finite collection of disjoint subsets $S_k \subset \mathbb{R}^d, k = 1,2,\dots,L$, $\mathbb{R}^d = \cup_k S_k$, and bounded, uniformly continuous functions on $\mathbb{R}^d$, $f_1, f_2, \dots, f_L$, such that $f = \sum_{j=1}^{L} f_j \mathbb{1}_{S_j}$. A function f_D on $D \subset \mathbb{R}^d$ is said to be piecewise continuous if there exists a piecewise continuous f on $\mathbb{R}^d$ such that $f_D = f|D$.

Let $\{\boldsymbol{\beta}_k : k \in \mathbb{N}\} \subset D$ and $\mathcal{L}$ be a subspace of the linear space of piecewise continuous functions from $\overline{D}$ into $\mathbb{R}$ such that the equality

$$\frac{1}{N}\sum_{k=1}^{N} f(\boldsymbol{\beta}_k) \;=\; \int_D f(\mathbf{x})\,d\mathbf{x}, \quad f \;\in\; \mathcal{L}, \tag{7}$$

is valid iff $\{\boldsymbol{\beta}_k : k \in \mathbb{N}\}$ is equidistributed in D. For instance, $\mathcal{L}$ can be the linear space of continuous functions on $\overline{D}$, the linear space spanned by indicators of cubes in D, spanned by piecewise continuous functions on $\overline{D}$.

An orhonormal basis $\{e_m, m \in \mathbb{N}_0\}$ in $L_2(D)$, such that e_m are piecewise continuous on $\overline{D}$, $e_0(\mathbf{x}) = 1$, is denoted by $\mathcal{E}$. Let $G \subset \mathbb{R}^r$ be a closed, bounded set and let a sequence of piecewise continuous functions $\mathbf{x}_k : G \mapsto D$, $k \in \mathbb{N}$, be given. Instead of functions (6) we consider now

$$f_N(\mathbf{t}) \;=\; \frac{1}{N}\sum_{k=1}^{N} e_m\big(\mathbf{x}_k(\mathbf{t})\big). \tag{8}$$

Each e_m is bounded on $\overline{D}$ so that *Lemma 1* has a straightforward generalization:

Corollary 1. *Let $m \in \mathbb{N}$ be fixed and for some $\alpha > 1$:*

$$\lim_{N\in\mathcal{I}(\alpha)} f_N(\mathbf{t}) \;=\; 0 \quad \text{a.e. on G.}$$

Then the sequence $\{f_N : N \in \mathbb{N}\}$ *converges to zero a.e. on* G.

In the previous sections the quantities

$$K(k,l) \;=\; \big(\, e_m(\mathbf{x}_k) \,\big|\, e_m(\mathbf{x}_l) \,\big),$$

had values δ_{kl}. In the present generalization they may have nonzero values for $k \neq l$.

Theorem 1. *Let the orthonormal basis* $\mathcal{E} \subset L_2(D)$*, and a sequence of piecewise continuous functions* $\mathbf{x}_k(\cdot)$*, from a bounded closed subset* $G \subset \mathbb{R}^r$ *into* D*, have the following properties:*

(i) For each $m \in \mathbb{N}$ *there exist two positive numbers* $\kappa(m), \rho(m)$ *such that*

$$\big|K(k,l)\big| \;\leq\; \kappa(m)\,|k-l|^{-\rho(m)};$$

(ii) Let $\mathcal{K}$ *be the linear space spanned by elements of* $\mathcal{E}$*. For each* $f \in \mathcal{L}$ *and* $\varepsilon > 0$ *there exist* $f^-, f^+ \in \mathcal{K}$ *such that*

$$\begin{aligned} f^-(\mathbf{x}) \;\leq\; f(\mathbf{x}) \;\leq\; f^+(\mathbf{x}), &\quad \mathbf{x} \in \overline{D}, \\ \|f^+ \;-\; f^-\|_1 \;<\; \varepsilon. & \end{aligned}$$

Then the sequence of points $\boldsymbol{\beta}_k = \mathbf{x}_k(\mathbf{t})$ *is equidistributed in* D *for almost all* $\mathbf{t} \in G$.

Proof. The functions (8) have their $L_2(D)$-norms:

$$\|f_N\|_2 \;\leq\; \frac{1}{N^2} \sum_{k,l=1}^{N} \big|K(k,l)\big|.$$

Because of (i), the right-hand side can be bounded from above by a number $\omega N^{-\gamma}, \gamma > 0$ [Ha], where ω depends on m and does not depend on N. Thus, any subsequence of functions $\{f_N(\cdot) : N \in \mathcal{I}(\alpha)\}$, with an α such that $\alpha\gamma > 1$, converges a.e. to zero on G. By *Corollary 1* the sequence $\{f_N(\cdot) : N \in \mathbb{N}\}$ converges to zero for almost all $\mathbf{t} \in G$.

Let us consider

$$\overline{f} \;=\; \limsup_N \frac{1}{N} \sum_{k=1}^{N} f\big(\boldsymbol{\beta}_k\big), \quad \underline{f} \;=\; \liminf_N \frac{1}{N} \sum_{k=1}^{N} f\big(\boldsymbol{\beta}_k\big).$$

Because of (ii) $\overline{f} - \underline{f} \leq \|f^+ - f^-\|_1 < \varepsilon$, implying the equidistribution of numbers $\boldsymbol{\beta}_k$. □

Example 4. *Koksma's numbers, defined by functions (3), can be considered now by specifying* $G = [1,a]$*,* $e_m(x) = \exp(\pm 2\pi i m x)$ *and* $\mathcal{L} = C([0,1])$*. In accordance with (i) of* Theorem 1 *we consider the functions* $t \mapsto g(m,k,l,t) = 2\pi m(t^k - t^l), k > l$*, and integrals*

$$I(m,k,l) \;=\; \int_1^a \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} \big(g(m,k,l,t)\big)\,dt.$$

Numbers $K(k,l)$ *of* Theorem 1 *are linear combinations of* $I(m,k,l)$. *It suffices to consider the case with* sin-*function. The quantity* $I(m,k,l)$ *is a finite series of terms obtained by dividing the interval* $[1,a]$ *into subintervals on which the integrand does not change its sign. Terms of such obtained series have strictly decresing magnitudes and alternating signs. Hence, the value of* $I(m,k,l)$ *may be estimated by the first term:*

$$I(m,k,l) \ < \ \int_1^{1+\tau} \sin\big(g(m,k,l,t)\big)\,dt,$$

where $1, 1+\tau$ *are the first two zeros of the integrand, i.e.* $g(m,k,l,1) = g(m,k,l,1+\tau) = 0$. *It is easy to check* $\tau < 1/(2m(k-l))$. *Hence,* Theorem 1 *can be applied.*

In the proof of equidistribution of the previous example two properties of the functions $g(m,k,l), m \in \mathbb{N}$, are used.

Corollary 2. *Let* $x_k : I \mapsto \mathbb{R}$, *and* $g(k,l,t) = x_k(t) - x_l(t)$, *have the following properties:*

P1) For sufficiently large k,l, *the functions* $g(k,l), g'(k,l)$ *are continuous, positive and strictly increasing on* I;

P2) For sufficiently large k,l, *there exist positive numbers* c_1, c_2 *such that*

$$g'(k,l) \ \geq \ c_1\,|k-l|^{c_2}, \quad k,l \in \mathbb{N}.$$

Then the sequence of numbers $\beta_k = x_k(t)$ [mod 1] *is equidistributed in* $[0,1]$ *for almost all seeds* $t \in I$.

Example 5. *The complete equidistribution of Koksma's numbers follows from their d-dimensional equidistribution for each* $d \in \mathbb{N}$. *The corresponding set of seeds is* $T = \cap T_d$. *Let* $\mathbf{m} \in \mathbb{Z}^d, \mathbf{m} \neq \mathbf{0}$, *and*

$$g(\mathbf{m},k,l,t) \ = \ 2\pi \sum_{j=1}^{d} m_j\left(t^{k+j-1} - t^{l+j-1}\right) \ = \ p(t)\,(t^k - t^l),$$

where p *is a polynomial of degree* $d-1$ *with coefficients depending on* $\mathbf{m}$. *Zeros of* p, $t_1, t_2, \dots, t_{d-1}$, *are surrounded by intervals* $I_r = (-\varepsilon + t_r, t_r + \varepsilon)$, *where* $\varepsilon = k^{-1/d}$. *There are disjoint closed intervals* J_r *such that* $J = [1,a] \setminus \big(\cup_r I_r\big) = \cup_r J_r$. *Hence,*

$$\left|I(m,k,l) \ - \sum_r \int_{J_r} \sin\big(g(\mathbf{m},k,l,t)\big)\,dt\right| \ < \ \sum_r \operatorname{mess}(I_r) \ < \ 2(d-1)k^{-1/d}.$$

It suffices to demonstrate that the functions $g(\mathbf{m},k,l)$ *(or* $-g(\mathbf{m},k,l)$*) on each* J_r *have the properties P1), P2) of* Corollary 2. *From the expression*

$$g'(\mathbf{m},k,l,t) \ = \ \frac{p(t)}{t}\big(k\,t^k - l\,t^l\big)\left[1 \ + \ \frac{tp'(t)}{p(t)}\,\frac{t^k - t^l}{k\,t^k - l\,t^l}\right],$$

we have the following estimate:

$$\sup_{t\in J_r} \left|g'(\mathbf{m},k,l,t)\right| \ \geq \ c_0(\mathbf{m},d)\,(k-l)^{1/d}\left[1 \ - \ O\left(\frac{1}{k-l}\right)\right].$$

Thus, for sufficiently large $k-l$ we have $|g'(\mathbf{m},k,l,t)| > c_1(\mathbf{m},d)(k-l)^{1/d}$, i.e. the properties P1), P2) are valid either for $g(\mathbf{m},k,l)$ or $-g(\mathbf{m},k,l)$.

Now we present a class of functions x_k, generating completely equidistributed numbers, and generalizing Koksma's numbers defined in terms of functions (3), $x_k(t) = \exp(k \ln t)$. One expects that the functions $x_k(t) = \exp(k^\rho \ln t)$ on $(1,\infty)$ also generate completely equidistributed numbers. First, the functions $\exp(k^\rho \ln t), \rho > 1$, are replaced by a more general sequence of functions $u_k(\cdot)^{w(k)}$.

Theorem 2. *Let*

$$x_k(t) \;=\; u_k(t)^{w(k)}, \quad t \in I, \quad k \in \mathbb{N},$$

where

(i) w is strictly increasing from $\mathbb{N}$ into $[b,\infty), b>0$, with the following properties;

$$w(k+r) \;\le\; \phi_+\, w(\max\{k,r\}),$$
$$w(k+r) \,-\, w(k) \;\ge\; \phi_-\, w(k)^\alpha\, w(r)^\beta, \quad k,r \in \mathbb{N},$$

for some positive numbers $\phi_\pm, \alpha, \beta$.

(ii) u_k, u_k', $k \in \mathbb{N}$, are continuous, strictly increasing from a closed interval I into $[a,\infty), a>1$, with the following additional properties:

$$\frac{u_{k+1}}{u_k} \;\ge\; 1, \qquad c_0 \;\le\; \frac{u_{k-1}'(t)}{u_{k-1}(t)} \;\le\; \frac{u_k'(t)}{u_k(t)} \;\le\; a^{\gamma w(k)^\delta},$$

for some $c_0>0$ and $\gamma \ge 0, \delta \ge 0, \delta < \alpha$.

Then the functions x_k generate numbers $\beta_k = x_k(t)$ [mod 1], completly equidistributed in $[0,1]$ for almost all $t \in I$.

Proof. It can be assumed $w(1)=1$. The functions $g_k = x_k - x_l$ fulfil conditions P1), P2) of *Corollary 2*, so that the numbers β_k are equidistributed for almost all $t \in I$. It has to be proved that they are d-dimensionaly equidistributed for all $d>1$. Let $\mathbf{m} \in \mathbb{Z}^d$ and the corresponding function

$$g(\mathbf{m},k,l,t) \;=\; \sum_{j=1}^{d} m_j \left(x_{k+j-1}(t) \,-\, x_{l+j-1}(t)\right).$$

It suffices to prove that $g(\mathbf{m},k,l)$ satisfy P1) and P2) for sufficiently large $k-l$. Let us define the functions

$$h_k(\mathbf{m},t) \;=\; \sum_{j=1}^{d} m_j\, \frac{x_{k+j-1}(t)}{x_{k+d-1}(t)}.$$

Then

$$g(\mathbf{m},k,l,t) \;=\; h_k(\mathbf{m},t)\, x_{k+d-1}(t) \,-\, h_l(\mathbf{m},t)\, x_{l+d-1}(t).$$

From properties (i), (ii), we have

$$\left|\frac{x_{k+j-1}(t)}{x_{k+d-1}(t)}\right| \;\le\; a^{-\phi_- w(k-j+1)^\alpha} \;\le\; a^{-\phi_- w(k)^\alpha},$$

$$\left|\left(\frac{x_{k+j-1}(t)}{x_{k+d-1}(t)}\right)'\right| \leq \frac{x_{k+j-1}(t)}{x_{k+d-1}(t)}\,\frac{u'_{k+d-1}(t)}{u_{k+d-1}(t)}\,w(k+d-1) \leq$$
$$w(k+d-1)\,a^{-\phi_- w(k)^\alpha + \gamma\phi_+^\delta w(k)^\delta}.$$

Hence, the functions $x_{k+j-1}/x_{k+d-1}, j < d$, and their derivatives tend to zero on I faster than any power of $w(k)$ as k increases. We can assume $m_d \neq 0$, as otherwise the problem is reduced to $(d-1)$-dimensional equidistribution. Since the sign of $g(\mathbf{m}, k, l)$ is irrelevant, we assume $m_d > 0$. The functions $h_k(\mathbf{m})$ tend to m_d on the set I:

$$\sup_{t\in I}\left|h_k(\mathbf{m}, t) - m_d\right| \leq \kappa_1(\mathbf{m})\,a^{-\phi_- w(k)^\alpha}.$$

Similarly,

$$\sup_{t\in I}\left|h'_k(\mathbf{m}, t)\right| \leq \kappa_2(\mathbf{m})\,w(k+d-1)\,a^{-\phi_- w(k)^\alpha + \gamma\phi_+^\delta w(k)^\delta}.$$

The functions $g(\mathbf{m}, k, l)$ seem to behave like the functions $g_0(k, l, t) = m_d[x_{k+d-1}(t) - x_{l+d-1}(t)]$. From

$$g'_0(k, l, t) =$$
$$m_d\left[\frac{u'_{k+d-1}(t)}{u_{k+d-1}(t)}\,w(k+d-1)\,x_{k+d-1}(t) - \frac{u'_{l+d-1}(t)}{u_{l+d-1}(t)}\,w(l+d-1)\,x_{l+d-1}(t)\right]$$

it follows

$$g'_0(k, l, t) \geq c_0\,m_d\left[w(k+d-1) - w(l+d-1)\right]x_{k+d-1}(t) \geq c_0\,m_d\,\phi_-\,(k-l)^\beta,$$

as well as

$$g'_0(k, l, t) \geq m_d\,x'_{k+d-1}(t)\left[1 - \frac{x_{l+d-1}(t)}{x_{k+d-1}(t)}\right] \geq m_d c_0\,x_{k+d-1}(t)\left[1 - \frac{x_{l+d-1}(t)}{x_{k+d-1}(t)}\right].$$

Therefore, for each $1/2 > \varepsilon > 0$ there exist $\kappa(\mathbf{m}, \varepsilon) > 0, n(\varepsilon) \in \mathbb{N}$, such that the value of square brackets is larger than $1/2$ when $k > k - l > n(\varepsilon)$.

$$\left|g'(\mathbf{m}, k, l, t) - g'_0(k, l, t)\right| \leq \kappa(\mathbf{m}, \varepsilon)\,g'_0(k, l, t)\,w(k+d-1)\,a^{-\phi_- w(k)^\alpha + \gamma\phi_+^\delta w(k)^\delta}.$$

Thus the sequence of functions $g(\mathbf{m}, k, l)$ has properties P1), P2) of *Corollary 2* for sufficiently large $k - l$. □

4. Discussion

Theorem 2 generalizes results about the functions x_k of (3) by replacing powers t^k with $t^{w(k)}$ where w increases faster than the linear function of k. A similar result exists in which w increases slower than the linear function such as $w(k) = k^\rho, \rho < 1$. Classes of completeley equidistributed numbers are important in description of mathematical samples of infinite length of the uniformly distributed random variable. Therefore, Koksma's numbers as well as classes of *Theorem 2* provide us with non-trivial examples of such samples and Monte Carlo simulated realizations.

Properties P1), P2) of *Corollary 2* seem to be completely analogous to Properties of Satz 3. in Koksma's article [Ko]. Differences are rather technical.

References

[Ca] J. W. S. Cassels, *An Introduction to Diphantine Approximation*, Cambridge University Press, 1965

[Do] J. Doob, *Stochastic Processes*, John Wiley, 1953

[F1] J. N. Franklin, *On the equidistribution of pseudo-random numbers*, Quart.Appl.Math. **16**(1958), 183–191.

[F2] J. N. Franklin, *Deterministic simulation of random processes*, Math. Comp. **17**(1963), 28–59.

[Ha] E. J. Hannan, *Multiple Time Series*, John Wiley, New York, 1970

[Ko] J. F. Koksma, *Ein mengentheoretischer Satz Uber die Gleichvertailung modulo Eins*, Compositio Math. **2**(1935), 250–258.

[We] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77**(1916), 313–352.