On the theorem of N. Singh and K. M. Sharma

ŽIVORAD TOMOVSKI*

Abstract. *A new short proof of the Theorem of N. Singh and K. M. Sharma (see [7]) is given.*

Key words: *quasi-convex sequence, Moore's class, L^1-convergence of Fourier series*

AMS subject classifications: 26D15, 42A20

Received November 2, 2000 Accepted October 15, 2002

1. Introduction and preliminaries

The problem of L^1-convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

is a Fourier series of some $f \in L^1(0,\pi)$ and

$$\|S_n - f\| = o(1), \quad n \to \infty \quad \text{if and only if} \quad a_n \log n = o(1)\,, \quad n \to \infty\,. \tag{2}$$

Here, S_n denotes the n-th partial sum of the series (1) and $\|\ \|$ is the L^1-norm.

Several authors have studied the question of L^1-convergence of the series (1).

The sequence $\{a_n\}$ that satisfies the condition $\sum\limits_{n=1}^{\infty}(n+1)\,|\Delta^2 a_n| < \infty$, where

$$\Delta^2 a_n = \Delta(\Delta a_n) = \Delta a_n - \Delta a_{n+1} = a_n - 2a_{n+1} + a_{n+2}\,, \text{ for all } n\,,$$

is called quasi-convex.

A classical result concerning the integrability and L^1-convergence of a series (1) is the following well-known theorem of Kolmogorov (see [5]).

Theorem 1 [see [4]]. *If $\{a_n\}$ is a quasi-convex null-sequence, then the series* (1) *is the Fourier series of some $f \in L^1(0,\pi)$ and* (2) *holds.*

The following class S of L^1-convergence, was defined by Telyakovskii [9]. A null-sequence $\{a_n\}$ belongs to the class S if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum\limits_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$, for all n.

*Faculty of Mathematical and Natural Sciences, Department of Mathematics, P.O. BOX 162, 1 000 Skopje, Macedonia, e-mail: `tomovski@iunona.pmf.ukim.edu.mk`

Theorem 2 [see [8]]. *Let* $\{a_n\} \in S$. *Then the series* (1) *is the Fourier series of some* $f \in L^1(0, \pi)$ *and* (2) *holds.*

The difference of noninteger order $k \geq 0$ of the sequence $\{a_n\}_{n=0}^{\infty}$ is defined as follows:

$$\Delta^k a_n = \sum_{m=0}^{\infty} \binom{m-k-1}{m} a_{n+m} \quad (n = 0, 1, 2, \dots) \tag{3}$$

where

$$\binom{m+\alpha}{m} = \frac{(1+\alpha)\cdots(m+\alpha)}{m!}\,.$$

It is obvious that if $a_n \to 0$ as $n \to \infty$, then series (3) is convergent and $\lim\limits_{n\to\infty} \Delta^k a_n = 0$.

C. N. Moore in [6] generalized quasi-convexity of null-sequences $\{a_n\}$ in the following way

$$\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty\,, \quad \text{for some} \quad k > 0\,. \tag{M}$$

It is well-known [3] that if $\{a_n\}$ is a null-sequence satisfying the condition (M), then

$$\sum_{n=1}^{\infty} n^r |\Delta^{r+1} a_n| < \infty\,, \quad \text{for} \quad 0 \leq r < k\,. \tag{4}$$

More recently, N. Singh and K. M. Sharma [7] proved the following generalized theorem of Kolmogorov.

Theorem 3 [see [7]]. *Let* k *be a real number such that* $k > 0$. *If*

(i) $\lim\limits_{n\to\infty} a_n = 0$,

(ii) $\sum\limits_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty$,

then the series (1) *is the Fourier series of some* $f \in L^1(0, \pi)$ *and* (2) *holds.*

2. Proof of Theorem 3

Applying *Theorem 2*, it suffices to show that the conditions (i) and (ii) of *Theorem 3* imply condition S. Firstly, we suppose that for some k, $0 < k \leq 1$, the series in (M) converges.

For $0 < k \leq 1$, we construct the sequence

$$A_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} |\Delta^{k+1} a_i|\,.$$

Then, we need the following properties for binomial coefficients $\binom{\alpha+n}{\alpha}$ (see [2], page 885 and [5], page 68):

a) $\alpha > -1 \Rightarrow \begin{pmatrix} \alpha+n \\ \alpha \end{pmatrix} = \dfrac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} > 0$,

b) $\begin{pmatrix} \alpha+n \\ \alpha \end{pmatrix} = \dfrac{n^\alpha}{\Gamma(\alpha+1)} + O(1)$, $0 < \alpha \le 1$,

c) $\sum\limits_{i=0}^{n} \begin{pmatrix} i+\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} n+\alpha+1 \\ n \end{pmatrix}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$.

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n &= \sum_{n=0}^{\infty}\sum_{i=n}^{\infty} \begin{pmatrix} i-n+k-1 \\ i-n \end{pmatrix} |\Delta^{k+1}a_i| \\
&= \sum_{i=0}^{\infty} |\Delta^{k+1}a_i| \sum_{n=0}^{i} \begin{pmatrix} i-n+k-1 \\ i-n \end{pmatrix} \\
&= \sum_{i=0}^{\infty} |\Delta^{k+1}a_i| \sum_{n=0}^{i} \begin{pmatrix} n+k-1 \\ n \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} i+k \\ k \end{pmatrix} |\Delta^{k+1}a_i| \\
&= \frac{1}{\Gamma(k+1)} \sum_{i=0}^{\infty} i^k |\Delta^{k+1}a_i| + O\left(\sum_{i=0}^{\infty} |\Delta^{k+1}a_i| \right).
\end{aligned}$$

Since series (3) is convergent, by condition (M), we obtain

$$\begin{aligned}
\sum_{i=0}^{\infty} |\Delta^{k+1}a_i| &= |\Delta^{k+1}a_0| + \sum_{i=1}^{\infty} |\Delta^{k+1}a_i| \\
&\le \sum_{m=0}^{\infty} \begin{pmatrix} m-k-2 \\ m \end{pmatrix} a_m + \sum_{i=1}^{\infty} i^k |\Delta^{k+1}a_i| < \infty\,.
\end{aligned}$$

Thus, $\sum\limits_{n=0}^{\infty} A_n < \infty$ and $A_n \downarrow 0$.

Then (see [1], Lemma 1)

$$\Delta a_n = \sum_{i=n}^{\infty} \begin{pmatrix} i-n+k-1 \\ i-n \end{pmatrix} \Delta^{k+1} a_i\,,$$

and hence

$$|\Delta a_n| \le \sum_{i=n}^{\infty} \begin{pmatrix} i-n+k-1 \\ i-n \end{pmatrix} |\Delta^{k+1}a_i| = A_n\,, \quad \text{for all } n\,.$$

If $k > 1$, by Bosanquet result (4), we obtain $\sum\limits_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$, i.e. $\{a_n\} \in S$.

Finally, $\{a_n\} \in S$, for all $k > 0$.

References

[1] A. F. ANDERSEN, *Comparison theorems in the theory of Cesàro summability*, Proc. London Math. Soc. **27**(1927), 39–71.

[2] N. K. BARI, *Trigonometric series*, Fizmatgiz, Moscow, 1961 (in Russian).

[3] L. S. BOSANQUET, *Note on convergence and summability factors* (III), Proc. London Math. Soc., 1949, 482–496.

[4] A. N. KOLMOGOROV, *Sur l'ordre de grandeur des coefficients de la serie de Fourier - Lebesque*, Bull. Acad. Polon. Ser A, Sci. Math., 1923, 83–86.

[5] D. S. MITRINOVIĆ, J. D. KEČKIĆ, *Metode Izračunavanja Konačnih Zbirova*, Beograd, 1990.

[6] C. N. MOORE, *On the use of Cesaro means indetermining criteria for Fourier constants*, Soc. Amer. Assoc. Advan. Sc. **21**(1933).

[7] N. SINGH, K. M. SHARMA, *L^1-convergence of modified cosine sums with generalized quasi-convex coefficients*, Journal of Math. Analysis and Appl. **136**(1988), 189-200.

[8] S. A. TELYAKOVSKII, *On a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki **14**(1973), 742–748 (in Russian).

[9] Ž. TOMOVSKI, *Convergence and integrability on some classes of trigonometric series*, PH.D thesis, University of Skopje (2000); RGMIA Monographs, Victoria University, 2000
URL: `http: //rgmia.vu.edu.au/monographs/tomovski_thesis.htm`
to appear in Dissertationes Mathematicae (Warszawa).