Zagreb International Review of Economics & Business, Vol. 16, No. 1, pp. 65-78, 2013

ISSN 1331-5609; UDC: 33+65

Portfolio Selection with Higher Moments and Application on Zagreb Stock Exchange

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Abstract: The Modern Portfolio Theory (MPT) has started a revolution in academic and investors' circles since 1950s. In spite of the popularity of Markowitz's portfolio selection, many critiques have been emerging throughout the years. One of them is the non normality of empirical return distributions. Accordingly, models have been developed in order to incorporate the aforementioned non normality. This paper focuses on the role of these models and optimizes a model with incorporated portfolio higher moments on Zagreb Stock Exchange. The results indicate that incorporating higher moments into the analysis changes the results sustainably when compared to the initial model.

Keywords: portfolio selection, optimization, Zagreb Stock Exchange, stocks, polynomial goal programming.

JEL Classification: G11

Introduction

The Modern Portfolio Theory (MPT) has given many answers to academic and practical researchers regarding questions about investing in the stock market. Many results derived from the MPT have enabled the quantification of risk and return, the selection of the most efficient portfolio on the market, etc. However, many researchers have pointed out many flaws of the MPT. The assumption of normal distributions of stock returns is maybe the most commonly criticized issue. Over time, models have been developed in order to eliminate some of the pitfalls of the MPT. A rather new approach to optimizing portfolios by taking into account the non normality of stock return distributions has been in place over the past two decades. It consists of modeling not only the expected portfolio return and risk (variance), but also the portfolio skewness and kurtosis. Empirical research has pointed out that the empirical stock return distributions are not normal as the MPT is assuming. Theoretical papers

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have proven that individual investors prefer portfolios with excess skewness and with kurtosis smallest possible. In that way, models have been developed in order to take the matters aforesaid in account.

The aim of this paper is to analyze the mentioned models and to optimize a portfolio on Zagreb Stock Exchange by using these models. Domestic research until now has been focused only on the Markowitz model. This paper is a first attempt of modeling higher moments on Croatian stock market and we hope to contribute to the existing sum of works and research on that subject by focusing on a more realistic approach to portfolio optimization. The paper is structured as following. Section 2 briefly reviews the Markowitz portfolio and its critiques. Third Section focuses on previous research dealing with higher portfolio moments. In Section 4 measures of skewness, co-skewness, kurtosis and co-kurtosis are represented, and a model of portfolio selection with higher moments is being derived. Empirical research is provided in Section 5, and the final Section concludes the paper.

Markowitz's Model

The Model

This section will briefly review the Markowitz (1952) model built in his seminal paper. Afterwards, a brief review of the criticisms is given at the end of the Section. Let us assume the investor is dealing with data on prices of N stocks (i=1,…,N) for the last T periods, $P_i(t)$, t=1,…,T. Return on stock i in period t is calculated as it follows (Aljinović, Marasović, Šego 2011):

$$R_i(t) = ln\left(\frac{P_i(t)}{P_i(t-1)}\right), \tag{1}$$

where $R_i(t)$ denotes return and ln natural logarithm. The expected return on stock i, $E(R_i)$ is calculated as following:

$$E\left(R_i\right) = \frac{1}{T}\sum_{t=1}^{T} R_i(t). \tag{2}$$

Variance σ_i^2 as a measure of risk and covariance measures $cov(R_i, R_j)$ are given as it follows:

$$\sigma_i^2 = E\left[R_i(t) - E\left(R_i\right)\right]^2 = \frac{1}{T}\sum_{t=1}^{T}\left[R_i(t) - E\left(R_i\right)\right]^2, \tag{3}$$

and

$$cov\left(R_i,R_j\right)=E\left[\left(R_i(t)-E\left(R_i\right)\right)\left(R_j(t)-E\left(R_j\right)\right)\right]= \\ =\frac{1}{T}\sum_{t=1}^{T}\left(R_i(t)-E\left(R_i\right)\right)\left(R_j(t)-E\left(R_j\right)\right). \tag{4}$$

Now we can define the expected portfolio return $E(R_P)$ and the portfolio variance $\sigma_P^{\ 2}$:

$$E\left(R_P\right)=\sum_{i=1}^{I} w_i E\left(R_i\right) \tag{5}$$

and

$$\sigma_P^2=\sum_{i=1}^{I} w_i^2\sigma_i^2+2\sum_{\substack{i,j=1\\ i\neq j}}^{I} w_i\, w_j cov\left(R_i,R_j\right), \tag{6}$$

where w_i denotes the relative weight on asset i.

Markowitz stated that investors will choose either the portfolio with the highest expected return on a given level of risk, or vice versa. A formal expression of this selection process is given in the following model:

$$\begin{gathered} \max_{w_i}\ \ E\left(R_P\right)=\sum_{i=1}^{N} w_i E\left(R_i\right) \\ \textit{subject to} \\ \sigma_P^2=\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\le c \\ \sum_{i=1}^{N} w_i=1 \\ w_i\ge 0,\quad i=1,2,\ldots,N. \end{gathered} \tag{7}$$

The investor chooses to maximize the expected portfolio return subject to some constraints. The first constraint means that the investor has chosen an arbitrary level of risk c which he does not want to exceed. Only stocks i=1,…,N can consist the portfolio, which is represented by the second constraint, and the last N constraints refer to the absence of short selling. By changing the tolerated risk level c and optimizing the goal function, each time the result is a different efficient portfolio on a different level of risk.

Criticism

There are many criticisms addressed to Markowitz's model, more precisely to its assumptions. Some of the most criticized assumptions are the homogenous expectations of investors, the assumption of efficiency of stock markets, the absence of transaction costs, and the use of the standard deviation as a measure of risk because of its equal valuation of positive and negative deviations from the expected return, etc. Maybe the most frequently mentioned assumption is the one regarding the distribution of returns. Markowitz assumed that stock returns are normally distributed. This means that it is sufficient to take the first and second moment of return distributions into account when optimizing the portfolio. However, empirical research has shown that empirical stock return distributions are not normal; they are either leptokurtic or platykurtic, and positively or negatively skewed. In that way, the first moment (expected return) and the second one (variance) are not representative anymore. Mandelbrot (1963), Fama (1965) and others are one of the first authors to recognize that the stock returns are not normal. Since 1960s, many other papers have proven that empirical stock returns distributions are in most cases leptokurtic and skewed. Thus, the approach of this paper is to incorporate higher moments of distributions into the analysis of portfolio selection.

Research Dealing with Higher Moments

Previous Theoretical and Empirical Research

Markowitz (1952) has caused a revolution by deriving the portfolio selection model. Still, in the following years many authors started to develop interest for stock return and higher moments of portfolio distributions. We can divide these papers into two groups: the first group provides empirical evidence on non normality of aforementioned distributions, and the other group gives theoretical justification of incorporating higher moments into the analysis. The empirical papers were first to emerge. Some authors have been studying the statistical properties of return distributions, and Kendall and Hill (1953) were probably first to notice that the stock returns are not normally distributed. Their research was followed by Fama (1965) and Alderfer and Bierman (1970) who came up with similar conclusions by examining empirical return distributions. Afterwards, more and more papers emerged showing the same problems when modeling stock returns.

The second group of papers consists of those which theoretically model investors' preferences. In 1967 Arditti (1967) showed that investors prefer positive portfolio skewness as a result of decreasing absolute risk aversion. Samuelson (1970) was the first author to examine the investors expected utility by incorporating the third moment of portfolio, and three years later Rubinstein (1973) extended the Capital Asset Pricing Model by the same moment. By 1980s many authors agreed that higher moments have to be incorporated into investor's utility functions. Müller and Machina (1987:351) have derived a theorem: "An expected utility maximizer with continuous von Neumann-Morgenstern utility function *U*(.) will rank probability distributions on the basis of their first *m* absolute moments if and only if *U*(.) is a polynomial of at most degree *m*." It is evident that in order to evaluate higher moments when choosing a portfolio, investors must include them into their utility functions. Tending to maximize the skewness of a portfolio means that the investor seeks to maximize the probability of occurrence of above average returns.

The coefficient of kurtosis was introduced into the analysis in 1988, in Graddy and Homaifar (1988). As it can be seen, it took nearly 20 years to include the fourth moment into the analysis because it is harder to interpret and calculate it. By seeking to minimize the kurtosis of a portfolio investor is aiming at minimizing the probability of occurrence of extreme negative returns. These are the reasons why empirical modeling of kurtosis started only a few years ago. If we look at papers dealing with optimization of the portfolio by taking into account higher moments, they started to emerge in late 1990s and in 2000s and today there are only few of them. First one to empirically evaluate the selection of efficient portfolio was Lai (1991) when he introduced the polynomial goal programming approach to solving the portfolio problem. In the paper he optimized a portfolio by taking into account the first three moments, and took only a couple of stocks into the analysis. Afterwards, some authors have been evaluating international portfolios by including higher moments into optimization: Prakash et. al. (2003), Sun and Yan (2003), Yang and Hung (2010) are some of them. They use the same approach as Lai (1991) – the polynomial goal programming.

If we examine domestic papers dealing with portfolio optimization, we can notice that Croatian authors mostly deal with the implementation of Markowitz's model. Miljan (2002a, 2002b) implemented the model on the Zagreb Stock Exchange and on EMU markets. Aljinović, Marasović and Tomić-Plazibat (2005) have incorporated financial statements into the portfolio modeling. Marasović and Šego (2006) and Jakšić (2007) gave a good overview of the formulae of the model and demonstrated it on a sample of stocks on Croatian stock market. We can also mention papers of Briš, Kristek and Mijoč (2008), Jerončić and Aljinović (2011), etc. Examining this previous research, there has not been any regarding portfolio's higher moments optimization. Investigation of European stock markets in this context is also very rare. In that way this paper is the first paper in this region dealing with these issues, and we hope to contribute to the existing literature by showing the importance of incorporating higher moments into the analysis.

Optimization Model with Higher Moments

According as presented by now, this Section is going to deal with basic concepts needed for the analysis. The individual coefficient of skewness s_{iii}, coefficient regarding two stocks s_{iij} and three s_{ijk} - the coefficients of co-skewness, are defined as:

$$s_{iii} = E\left[\left(R_i - E\left(R_i\right)\right)^3\right], \tag{8}$$

$$s_{iij} = E\left[\left(R_i - E\left(R_i\right)\right)^2\left(R_j - E\left(R_j\right)\right)\right] \tag{9}$$

and

$$s_{ijk} = E\left[\left(R_i - E\left(R_i\right)\right)\left(R_j - E\left(R_j\right)\right)\left(R_k - E\left(R_k\right)\right)\right]. \tag{10}$$

Analogously, the coefficients of kurtosis and co-kurtosis k_{iiii}, k_{iijj}, k_{ijjj} and k_{ijkl} are defined as:

$$k_{iiii} = E\left[\left(R_i - E\left(R_i\right)\right)^4\right], \tag{11}$$

$$k_{iijj} = E\left[\left(R_i - E\left(R_i\right)\right)^2\left(R_j - E\left(R_j\right)\right)^2\right], \tag{12}$$

$$k_{ijjj} = E\left[\left(R_i - E\left(R_i\right)\right)\left(R_j - E\left(R_j\right)\right)^3\right] \tag{13}$$

and

$$k_{ijkl} = E\left[\left(R_i - E\left(R_i\right)\right)\left(R_j - E\left(R_j\right)\right)\left(R_k - E\left(R_k\right)\right)\left(R_l - E\left(R_l\right)\right)\right]. \tag{14}$$

It can be noticed that the calculation of the higher moments is more complex when compared to the first two moments. However, due to the symmetry of the coefficients, if we are dealing with N stocks, we need to calculate only $\binom{N+2}{3}$ coefficients of skewness and co-skewness, and $\binom{N+3}{4}$ coefficients of kurtosis and co-kurtosis. For example, if the investor has data on 100 stocks, he will need to calculate "only" 171.700 coefficients of skewness and co-skewness instead of one million, and "only" 4.421.275 coefficients of kurtosis and co-kurtosis instead of one hundred million. We hope now that this small example showed the complexity of dealing with higher moments.

The portfolio skewness is defined as following:

$$S_P = E\left[R_P - E\left(R_P\right)\right]^3 = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k s_{ijk}, \tag{15}$$

whereby all of the notation have been already defined. As it can be seen, by focusing on the third moment around portfolio's expected return, the portfolio skewness is a weighted sum of individual coefficients of skewness, and the co-skewness between stock returns. Portfolio kurtosis is defined as

$$K_P = E\left[R_P - E\left(R_P\right)\right]^4 = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} w_i w_j w_k w_k k_{ijkl}. \tag{9}$$

As already mentioned, all of the notations have been explained earlier. The portfolio kurtosis is defined as a weighted sum of individual coefficients of kurtosis and the co-kurtosis. Usually, portfolio skewness and kurtosis are standardized by using the portfolio's standard deviation:

$$S'_P = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} w_i w_j w_k w_k k_{ijkl}}{\sigma_P^4} \tag{10}$$

and

$$K'_P = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k s_{ijk}}{\sigma_P^3}. \tag{11}$$

Thus, these coefficients can be comparable when analyzing different portfolios. Since all of the moments have been defined, the optimization model this paper is going to implement is the following one:

$$\begin{gathered}
\min_{w_i} \; K'_P = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} w_i w_j w_k w_k k_{ijkl}}{\sigma_P^4} \\
\max_{w_i} \; E\left(R_P\right) = \sum_{i=1}^{N} w_i E\left(R_i\right) \\
\min_{w_i} \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} \\
\max_{w_i} S'_P = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k s_{ijk}}{\sigma_P^3} \\
\textit{subject to} \quad \sum_{i=1}^{N} w_i = 1 \\
w_i \geq 0, \quad i = 1, 2, \ldots, N.
\end{gathered} \tag{12}$$

As it can be seen, there are four goal functions that investors seek to optimize. They want to maximize the expected return and skewness on one hand, and to minimize the portfolio risk (variance) and kurtosis on the other. In order to solve these conflicted goals, this paper adopted the framework from Lai (1991) and Lai et. al. (2006). In order to solve the problem given in (12), we follow two stages. In the first stage four individual problems are formed and solved. The first problem *P*(1) refers to maximizing the expected return; the second *P*(2) deals with minimizing the portfolio variance, etc., in the following way:

$$P(1)\begin{cases} \max\limits_{w_i} E^*\left(R_P\right)=\sum\limits_{i=1}^{N} w_i E\left(R_i\right) \\ \text{subject to} \\ \qquad \sum\limits_{i=1}^{N} w_i=1 \\ \qquad w_i \geq 0, \quad i=1,2,\ldots,N. \end{cases}$$

$$P(2)\begin{cases} \min\limits_{w_i} \sigma_P^{2*}=\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} w_i w_j \sigma_{ij} \\ \text{subject to} \\ \qquad \sum\limits_{i=1}^{N} w_i=1 \\ \qquad w_i \geq 0, \quad i=1,2,\ldots,N. \end{cases}$$

$$P(3)\begin{cases} \max\limits_{w_i} S'^{*}{}_P=\dfrac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N} w_i w_j w_k s_{ijk}}{\sigma_P^3} \\ \text{subject to} \\ \qquad \sum\limits_{i=1}^{N} w_i=1 \\ \qquad w_i \geq 0, \quad i=1,2,\ldots,N. \end{cases}$$

and

$$P(4)\begin{cases} \min\limits_{w_i} \quad K'^{*}{}_P=\dfrac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N}\sum\limits_{l=1}^{N} w_i w_j w_k w_k k_{ijkl}}{\sigma_P^4} \\ \text{subject to} \\ \qquad \sum\limits_{i=1}^{N} w_i=1 \\ \qquad w_i \geq 0, \quad i=1,2,\ldots,N. \end{cases}$$

By solving individual problems 1-4, we obtain ideal solutions in each of the problem:

$$\left(E^*\left(R_P\right), \sigma_P^{2*}, S'^*{}_P, K'^*{}_P\right). \tag{13}$$

The second stage is defined as a new optimization problem in which we minimize the deviations from the ideal point given in (13). In order to solve a new problem, new variables have to be defined. d_1, d_2, d_3 and d_4 represent nonnegative variables which account for the deviations of expected return, variance, skewness and kurtosis from the ideal solutions. A new goal function is a specific form of the general Minkovski distance, which is defined as (Kemalbay, Özkut and Franko 2011):

$$M = \left(\sum_{i=1}^{N}\left|\frac{d_i}{M_i}\right|^p\right)^{\frac{1}{p}}, \tag{14}$$

where M_i is used as a basis for normalizing the i-th variable. Specifically, the second stage optimization problem is as follows:

$$min\, M = \left|\frac{d_1}{E^*\left(R_P\right)}\right| + \left|\frac{d_2}{\sigma_P^{2*}}\right| + \left|\frac{d_3}{S'^*{}_P}\right| + \left|\frac{d_4}{K'^*{}_P}\right|$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{N} w_i E\left(R_i\right) + d_1 = E^*\left(R_P\right)\\
&\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - d_2 = \sigma_P^{2*}\\
&\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k s_{ijk}}{\sigma_P^3} + d_3 = S'^*{}_P\\
&\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} w_i w_j w_k w_k k_{ijkl}}{\sigma_P^4} - d_4 = K'^*{}_P\\
&\sum_{i=1}^{N} w_i = 1\\
&d_m \geq 0, \quad m = 1,\ldots,4\\
&w_i \geq 0, \quad i = 1,2,\ldots,N.
\end{aligned} \tag{15}$$

First of all, the problem given in (15) is, as it is mentioned, minimizing the deviations from the aspired levels of return, variance, skewness and kurtosis. There are *N*+5 constraints and the first four refer to the nonnegative variables, which have to be incorporated into the portfolio selection. The model in (15) will be optimized by using selected stocks on Zagreb Stock Exchange.

Empirical Research

This Section implements the model described in the previous section on Zagreb Stock Exchange. A sample of 10 stocks was taken from the web page of ZSE (2012). It refers to daily data on prices ranging from January 31st to December 4th 2012 for the following stocks: ATGR-R-A, DDJH-R-A, ERNT-R-A, HUPZ-R-A, INA-R-A, KNZM-R-A, KOEI-R-A, KORF-R-A, LEDO-R-A and LKRI-R-A. A random sample was chosen: stocks which have been listed on the stock exchange for at least 200 working days in a row. The whole analysis was performed in Excel. Based on the data on daily prices, returns were calculated using the formula given in (1). This resulted with total of 212 daily return observations on each stock. Then, 10 expected returns, 55 coefficients of variance and covariance, 220 coefficient of skewness and co-skewness and 715 coefficients of kurtosis and co-kurtosis were calculated. First of all, the Markowitz model was optimized on several risk levels, the results are given in table 1. It is evident that the relative portfolio weights of each stock vary in each portfolio, depending on the level of risk chosen for the optimization.

Table 1. Results of portfolio optimization, Markowitz model

Variance	Expected return	Portfolio weights									
		ATGR	DDJH	ERNT	HUPZ	INA	KNZM	KOEI	KORF	LEDO	LKRI
0,007	0,0011	0,1788	0,1486	0,0153	0,0448	0,0578	0,1957	0,0319	0,2054	0,1980	0,1025
0,008	0,0013	0,1365	0,0100	0,0028	0,0256	0,1565	0	0,2588	0,2724	0,1373	0
0,009	0,0014	0,0931	0	0	0	0,0939	0	0,3055	0,3413	0,1661	0
0,01	0,0015	0,0488	0	0	0	0,0229	0	0,3399	0,4035	0,1849	0

Source: author's calculation

By moving on to the modeling of higher moments portfolio problem, it is important to solve four individual problems, the maximization of expected portfolio return, minimization of portfolio variance, maximization of relative skewness and minimization of relative kurtosis. Optimal solutions to each of the problems are given in table 2. These are the ideal solutions, which we can never be obtained, due to the conflict between the goal functions.

Table 2. Results of the individual optimization problems

$P(1)$	$E^*(R_P)=0,0019$
$P(2)$	$\sigma_P^{2*}=5,16\cdot 10^{-5}$
$P(3)$	$S^{'*}{}_P=1,37$
$P(4)$	$K^{'*}{}_P=0,26$

Source: author's calculation

When the ideal solutions were obtained, the problem given in (15) was solved and the results are the following: d_1^*=0,0014, d_2^*=0,00023, d_3^*=1,39421 and d_4^*=0,04895. d_1^*, d_2^*, d_3^* and d_4^* are the optimal values of the nonnegative variables accounting for the deviations from the optimal values given in table 2. The portfolio moments resulting in this second stage of optimization are: $E(R_P)$=0,00048, σ_P^2=2,8×10^{-4}, S'_P= –0,028 and K'_P=0,309. These values are closest an investor can get to the ideal solution. A structure of the resulting portfolio is given in table 3. It can be seen that the relative weights differ sustainably when comparing to the results of the Markowitz model in table 1. This is not surprising due to the inclusion of higher moments in investor's preferences.

Table 3. Portfolio weights, optimization of higher moments portfolio

ATGR	DDJH	ERNT	HUPZ	INA	KNZM	KOEI	KORF	LEDO	LKRI
0	0	0,155	0	0,443	0,241	0,061	0	0	0,1

Source: author's calculation

In order to compare the results given in table 3, a new Markowitz model was optimized, with the same level or risk as the risk given in higher moments portfolio, σ_P^2=2,8×10^{-4}. Therefore, a more detailed insight of the differences between the initial model and the one when incorporating higher moments is given. The relative portfolio weights in each model are given in table 4. It is obvious how the structures of the two optimal portfolios differ to a great extent. The only thing in common is the relative weight of KOEI stock; in the Markowitz model it consists 7% of the portfolio, and in the higher moments portfolio 6,1%. The KORF stock, which consists 93% percent of the initial portfolio is not included in the higher moments portfolio. This is a huge difference, as well as the weights of ERNT, INA, KNZM and LKRI, which are not included in the initial portfolio, but they represent 93,9% of the higher moments portfolio. There is a great significance of these results; by including the higher moments into the analysis, the structure of the optimal portfolio changes sustainably.

Table 4. Comparison of portfolio weights

Model	ATGR	DDJH	ERNT	HUPZ	INA	KNZM	KOEI	KORF	LEDO	LKRI
Markowitz	0	0	0	0	0	0	0,07	0,93	0	0
Higher moments	0	0	0,155	0	0,443	0,241	0,061	0	0	0,1

Source: author's calculation

Finally, let us compare the portfolio moments of each model by looking into table 5. The levels of risk are the only equal moments. Although the expected return and relative skewness are higher in the Markowitz model, the relative kurtosis is also much higher, which means that the probability of occurrence of extreme negative returns is very high. On the other hand, the higher moments portfolio has a lower expected return and relative skewness, but it has a much lower relative kurtosis coefficient, which compensates for the lower return and skewness.

Table 5. Comparison of portfolio moments

Model	Expected return	Variance	Relative skewness	Relative kurtosis
Markowitz	0,00186	0,00028	1,49	12,63
Higher moments	0,00048	0,00028	-0,028	0,309

Source: author's calculation

Conclusion

Many useful concepts important to investors and academics have derived from the Modern Portfolio Theory. However, the criticism has been addressed since its beginning, and it was primarily focused on the assumptions of the Markowitz's portfolio. The normality of stock return distributions is probably one of the most questioned issues. Hence, in 1970s authors started to incorporate higher moments into the portfolio optimization. The inclusion of portfolio skewness meant that investors want to maximize the probability of occurrence of above average returns, and the inclusion of portfolio kurtosis meant that investors try to minimize the probability of occurrence of extreme negative events (negative returns).

This paper focused on the theoretical background and previous empirical research in order to optimize a higher moments portfolio on Zagreb Stock Exchange. The results showed that the inclusion of higher moments into the optimization changes the efficient portfolio substantially. This is mostly emphasized in the portfolio structure, which changes to a great extent when including higher moments. If an investor includes higher moments into his preferences, it is important to incorporate them into the optimization process as well. This paper showed for a sample of 10 stocks on Croatian stock market the differences arising from optimizing a simple Markowitz portfolio in which investors do not take into account higher moments, and a model which includes them. The analysis done in this paper has included only 10 stocks; however this is an initial empirical investigation of this type on Croatian stock market. There is much more work to be done. The research in the future should extend the analysis on a broader sample of stocks and investigate the consequences of changing preferences towards portfolio's higher moments. This refers to the goal function given in the second stage of polynomial goal programming (relation 15), which can be changed, so that each of the summands can have their individual weight, depending on investors preferences. Finally, we can conclude that the main purpose of the paper has been fulfilled by showing the importance of including higher moments into portfolio selection process.

REFERENCES

Alderfer, C. P, Bierman, H., (1970), Choices with Risk: Beyond the Mean and Variance, Journal of Business, Vol. 43, Issue 3: 341-353.

Aljinović, Z., Marasović, B. Tomić-Plazibat, N., (2005), Multi-criterion approach versus Markowitz in selection of the optimal portfolio, Proceedings of the 8th International Symposium on Operational Research, Ljubljana: 261-266.

Aljinović, Z., Marasović, B., Šego, B., (2011), Financijsko modeliranje, 2. izmijenjeno i dopunjeno izdanje, Ekonomski fakultet Sveučilišta u Splitu.

Arditti, F., (1967), Risk and the Required Return on Equity, Journal of Finance 22: 19–36.

Briš, M., Kristek, I., Mijoč, I., (2008), Selection of optimal portfolio by use of risk diverzificaton method, Interdisciplinary management research IV, Faculty of Economics in Osijek, Osijek: 329-343.

Fama, E. F., (1965), The Behaviour of Stock Market Prices, Journal of Business 38: 34-105.

Fama, E., (1965), The Behaviour of Stock-Market Prices, Journal of Business 38: 34–105.

Graddy, D., Homaifar, G., (1988), Equity Yields in Models Considering Higher Moments of the Return Distribution, Applied Economics 20: 325–334.

Jakšić, S., (2007), Primjena Markowitzeve teorije na tržište dionica Zagrebačke burze, Zbornik Ekonomskog fakulteta u Zagrebu, Vol.5 No.1 Prosinac 2007: 331-344.

Jerončić, M., Aljinović, Z., (2011), Formiranje optimalnog portfelja pomoću Markowitzevog modela uz sektorsku podjelu kompanija, Ekonomski pregled, 62 (9-10): 583-606.

Kemalbay, G., Özkut, C. M., and Franko, C., (2011), Portfolio Selection with Higher Moments: a Oolynomial Goal Programming Approach to ISE-30 Index, Ekonometri ve İstatistik Sayı:13 (12. Uluslararası Ekonometri, Yöneylem Araştırması, İstatistik Sempozyumu Özel Sayısı) 2011: 41–61.

Kendall, M., Hill, B., (1953), The Analysis of Economic Time-series – Part I: Prices, Journal of the Royal Statistical Society Series A, 116 (1):11–34.

Lai K.K, Yu L, and Wang S., (2006), Mean-variance-skewness-kurtosis-based portfolio optimization, Proceedings of the First International Multi-Symposiums on Computer and Computational Sciences - Volume 2 (IMSCCS'06): 292-297.

Lai, T., (1991), Portfolio with Skewness: A Multiple-objective Approach, Review of Quantitative Finance and Accounting 1: 293–305.

Mandelbrot, B., (1963), The Variation of Certain Speculative Prices, Journal of Business 36(4): 394-419.

Marasović, B., Šego, B., (2006), „Markowitzev model optimizacije portfelja“, Računovodstvo i financije, LII (2006): 57-61.

Markowitz, H., (1952), Portfolio Selection, The Journal of Finance, Vol. 7, No. 1: 77-91.

Miljan, I., (2002a), Ocjena efikasnosti ulaganja u indeks Zagrebačke burze prema MPT modelu, Ekonomska istraživanja, Vol. 15, No. 3: 33-41.

Miljan, I., (2002b), International portfolio management – The case of Croatian and EMU stock markets, International Conference on Globalization and Entrepreneurship: Fears, Challenges and Opportunities / Krbec, Denisa (ur.). - Zagreb: FET, 2002.: 372-378.

Müller, S. M., Machina, M. J., (1987), Moment preferences and polynomial utility, Economics Letters 23: 349-353.

Prakash, A. J., Chang, C-H., Pactwa, T. E., (2003), Selecting a portfolio with skewness: Recent evidence form US, European and Latin American equity markets, Journal of Banking & Finance 27: 1375-1390.

Rubinstein, M., (1973), The Fundamental Theorem of Parameter-preference Security Valuation, Journal of Financial and Quantitative Analysis 8: 61–69.

Sun, Q., Yan, Y., (2003), Skewness persistence with optimal portfolio selection, Journal of Banking & Finance 27: 1111-1121.

Yang, C. W., Hung, K., (2010), A Generalized Markowitz Portfolio Selection Model with Higher Moments, International Review of Accounting, Banking and Finance, Vol. 2, No. 1: 1-7.

Zagreb Stock Exchange ZSE, (2012), web site www.zse.hr [accessed December 5th 2012].