Scaling laws for fractional Volterra equations with chi-square random data*

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Abstract. *This paper presents Gaussian and non-Gaussian scenarios for the renormalized solutions of fractional integro-differential equations of Volterra type. The solutions are obtained under random initial conditions which are subordinated to chi-square random fields with weak or strong dependence.*

Key words: *fractional integro-differential equation, fractional diffusion-wave equation, chi-square subordination, Mittag-Leffler function, spectral representation, long-range dependence*

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1. Introduction

We are interested in fractional-in-time diffusion-wave equations with random initial conditions as models of random fields which describe the singular and fractal behaviour of data arising in many applied fields such as hydrology, ecology, geophysics, turbulence, economics and finance (see Friedman [13], Prüss [22], Anh and Leonenko [3], [5], [4], [6] and the references therein). A typical example is the following fractional integro-differential equation of the Volterra type:

$$u(t,x) = u_0(x) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \Delta u(\tau,x)\, d\tau, \quad t>0,\ x\in\mathbb{R}^n,\ 0<\beta\le 1, \quad (1)$$

under some initial condition

$$u(0,x) = u_0(x), \quad (2)$$

where Δ is the n-dimensional Laplacian. Equation (1) was introduced by Friedman [13], whose other variants may be found in Schneider and Wyss [24], Prüss [22],

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Engler [12] and Anh and Leonenko [3], [5], [4], [6]. Equation (1) is equivalent to the fractional-in-time diffusion equation

$$\frac{\partial^\beta u}{\partial t^\beta} = \Delta u, \quad 0 < \beta \leq 1 \tag{3}$$

subject to the initial condition (2), where the fractional derivative-in-time is interpreted in the Caputo-Djrbashian sense (see Schneider and Wyss [24], Kochubei [16], [17], Anh and Leonenko [1], [2], [3], [5], [4], [6]), that is,

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \frac{\partial u}{\partial t}(t,x) & \text{if } \beta = 1, \\ \left(\mathcal{D}_t^\beta u\right)(t,x) & \text{if } \beta \in (0,1], \end{cases}$$

where

$$\left(\mathcal{D}_t^\beta u\right)(t,x) = \frac{1}{\Gamma(1-\beta)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\beta} u(\tau,x)\,d\tau - \frac{u(0,x)}{t^\beta}\right].$$

Using some results of Schneider and Wyss [24], Schneider [23], and Anh and Leonenko [3], [4], the solution of the Cauchy problem (1) and (2) (or (3) and (2)) may be written as

$$u(t,x) = \int_{\mathbb{R}^n} G_\beta(t,x-y)\,u_0(y)\,dy, \tag{4}$$

where the Green function $G_\beta(t,x)$ is radial in x and satisfies

$$\int_{\mathbb{R}^n} e^{i\langle\lambda,x\rangle} G_\beta(t,x)\,dx = E_{\beta,1}\left(-t^\beta|\lambda|^2\right), \quad \lambda \in \mathbb{R}^n \tag{5}$$

with $\int_{\mathbb{R}^n} G_\beta(t,x)\,dx = 1, 0 < \beta < 2$. Here, $E_{\beta,1}$ is a special case of the generalized Mittag-Leffler function

$$E_{a,b}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(ak+b)}, \quad a,b > 0,\ z \in \mathbb{C} \tag{6}$$

(see Djrbashian [9], Mainardi and Gorenflo [21]). Other special cases of $E_{a,b}(z)$ are

$$E_{1,1}(-x) = e^{-x},\ E_{1/2,1}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-y^2-2xy}dy,$$

$$E_{1/2,3/2}(-x) = \frac{2}{\sqrt{\pi}x}\int_0^\infty e^{-y^2}\left(e^{-2yx}-1\right)dy,\ E_{1,2}(-x) = \frac{1-e^{-x}}{x}, \quad x \geq 0.$$

Note that for $n = 1$, by definition

$$G_0(t,x) = \frac{1}{2}e^{-|x|},\ G_1(t,x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},\ G_2(t,x) = \frac{\delta(t-x)+\delta(t+x)}{2}.$$

Along the same line, we may introduce the fractional equation

$$u(t,x) = u_0(x) + tu_1(x) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\Delta u(\tau,x)\,d\tau, 1 < \beta \leq 2, \tag{7}$$

which reduces to the integrated wave equation when $\beta = 2$. The solution of (7) can be expressed in terms of the initial conditions

$$u(t,x)|_{t=0} = u_0(x)\,; \; \frac{\partial}{\partial t} u(t,x)\bigg|_{t=0} = u_1(x) \tag{8}$$

as

$$u(t,x) = \int_{\mathbb{R}^n} G_\beta(t, x-y)\, u_0(y)\, dy + \int_{\mathbb{R}^n} G_\beta^{(1)}(t, x-y)\, u_1(y)\, dy, 1 < \beta \leq 2, \tag{9}$$

where G_β is defined in (5) and

$$G_\beta^{(1)}(t,x) = \int_0^t G_\beta(\tau, x)\, d\tau. \tag{10}$$

Note that for $n = 1$ and $u_1(x) \equiv 0$, the Green function G_β, $0 < \beta \leq 2$, can be expressed in terms of Wright's function (see Mainardi [20], Anh and Leonenko [2]). In a stochastic situation, this type of equations has been studied by Anh and Leonenko [4]. They obtained Gaussian and non-Gaussian scenarios as limits of the rescaled solution of (4) or (7) with random initial condition (2) or (8). These scaling laws are mostly concerned with random initial conditions which are subordinated to Gaussian random fields with weak or strong dependence.

In this paper, we generalize the above results and obtain new Gaussian and non-Gaussian scenarios for random initial conditions (2) or (8) which are subordinated to chi-square random fields. In a sense, our results are analogous to the Gaussian and non-Gaussian central limit theorems for local functionals of random fields with weak or strong dependence (see Taqqu [26], Dobrushin and Major [10], Breuer and Major [8]), but the normalizing factors and types of non-Gaussian limiting fields obtained in this paper are new.

2. Spectral representation of mean-square solutions

We shall use extensively the spectral theory of random fields (see Yadrenko [29] or Leonenko [18] and the references therein). Let $\eta_j(x) = \eta_j(\omega, x)$, $x \in \mathbb{R}^n, \omega \in \Omega$, $j \in \{0,1\}$ be two real, uncorrelated, mean-square continuous homogeneous (in the wide sense) random fields on the complete probability space $(\Omega, \mathcal{F}, P)$ with means

$$E\eta_j(x) = m_j, \quad j \in \{0,1\}$$

and covariance functions

$$B_j(x) = cov(\eta_j(0), \eta_j(x)) = \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} F_j(d\lambda), \tag{11}$$

where F_j, $j \in \{0,1\}$ are the spectral measures. In view of Karhunen's Theorem (see Gihman and Skorokhod [14]), there exist a complex-valued orthogonally scattered random measures Z_j, $j \in \{0,1\}$, such that the random fields have the spectral representations

$$\eta_j(x) = m_j + \int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle} Z_j(d\lambda), \quad j \in \{0,1\}, \tag{12}$$

where

$$E\left|Z_j\left(A\right)\right|^2 = F_j\left(A\right), \quad A \in \mathcal{B}\left(\mathbb{R}^n\right), j \in \{0,1\}.$$

For $0 < \beta \leq 1$ we define the mean-square solution of the initial-value problem (1) and (2) with random initial condition

$$u_0\left(x\right) = \eta_0\left(x\right), \quad x \in \mathbb{R}^n, \tag{13}$$

as the stochastic integral in $L_2\left(\Omega\right)$-sense:

$$u\left(t,x\right) = \int_{\mathbb{R}^n} G_\beta\left(t,x-y\right)\eta_0\left(y\right)dy = m_0 + \int_{\mathbb{R}^n} e^{i\langle\lambda,x\rangle} E_{\beta,1}\left(-t^\beta\left|\lambda\right|^2\right)Z_0\left(d\lambda\right) \tag{14}$$

with covariance structure

$$cov\left(u\left(t,x\right),u\left(s,y\right)\right) = \int_{\mathbb{R}^n} e^{i\langle\lambda,x-y\rangle} E_{\beta,1}\left(-t^\beta\left|\lambda\right|^2\right)E_{\beta,1}\left(-s^\beta\left|\lambda\right|^2\right)F_0\left(d\lambda\right), \tag{15}$$

$E_{\beta,1}$ being the Mittag-Leffler function defined in (5) and (6).

For $1 < \beta \leq 2$, we define the mean-square solution of the initial-value problem (7) and (8) with random initial conditions

$$u_0\left(x\right) = \eta_0\left(x\right), u_1\left(x\right) = \eta_1\left(x\right), \quad x \in \mathbb{R}^n, \tag{16}$$

as the stochastic integral in $L_2\left(\Omega\right)$-sense:

$$\begin{aligned} u\left(t,x\right) &= m_0 + m_1 t + \int_{\mathbb{R}^n} e^{i\langle\lambda,x\rangle} E_{\beta,1}\left(-t^\beta\left|\lambda\right|^2\right)Z_0\left(d\lambda\right) \\ &\quad + t\int_{\mathbb{R}^n} e^{i\langle\lambda,x\rangle} E_{\beta,2}\left(-t^\beta\left|\lambda\right|^2\right)Z_1\left(d\lambda\right) \end{aligned} \tag{17}$$

with covariance structure

$$\begin{aligned} cov\left(u\left(t,x\right),u\left(s,y\right)\right) &= \int_{\mathbb{R}^n} e^{i\langle\lambda,x-y\rangle} E_{\beta,1}\left(-t^\beta\left|\lambda\right|^2\right)E_{\beta,1}\left(-s^\beta\left|\lambda\right|^2\right)F_0\left(d\lambda\right) \\ &\quad + ts\int_{\mathbb{R}^n} e^{i\langle\lambda,x-y\rangle} E_{\beta,2}\left(-t^\beta\left|\lambda\right|^2\right)E_{\beta,2}\left(-s^\beta\left|\lambda\right|^2\right)F_1\left(d\lambda\right), \end{aligned} \tag{18}$$

where $E_{\beta,1}$ and $E_{\beta,2}$ are Mittag-Leffler functions (6). Note that (17) follows from (5), (9), (10) and the following formula (see Djrbashian [9], p. 1):

$$\int_0^t E_{\beta,1}\left(-t^\beta\left|\lambda\right|^2\right)d\tau = tE_{\beta,2}\left(-t^\beta\left|\lambda\right|^2\right). \tag{19}$$

3. Chi-square random fields

We consider a class of chi-square random fields

$$\chi_d\left(x\right) = \frac{1}{2}\sum_{k=1}^{d}\xi_k^2\left(x\right), \quad x \in \mathbb{R}^n, \tag{20}$$

where $\xi_1(x), ..., \xi_d(x)$ are independent copies of a homogeneous isotropic Gaussian random field $\xi(x)$, $x \in \mathbb{R}^n$, with $E\xi(x) = 0$ and $E\xi(x)\xi(y) = B_\xi(x-y)$, $x, y \in \mathbb{R}^n$. Note that the random field (20) has a marginal density of the form

$$p(u) = e^{-u}u^{\frac{d}{2}-1}/\Gamma(d/2), \quad u > 0. \tag{21}$$

It is known that a complete orthogonal system in the Hilbert space $L_2((0,\infty), p(u)\,du)$ has the form

$$e_k(u) = L_k^{\left(\frac{d}{2}-1\right)}(u)\left\{k!\Gamma\left(\frac{d}{2}\right)/\Gamma\left(\frac{d}{2}+k\right)\right\}^{1/2},$$

where $L_k^{(c)}$ are generalized Laguerre polynomials of index c for $k \geq 0$ (see, for instance, Srivastava and Manocka [25], p. 74). The two-dimensional density of the random field (20) has the form

$$p(u,w,r) = p(u)p(w)\left[1 + \sum_{k=1}^{\infty} r^k e_k(u) e_k(w)\right]$$

$$= \left(\frac{uw}{r}\right)^{\left(\frac{d}{2}-1\right)/2} \exp\left\{-\frac{u+w}{1-r}\right\} I_{\frac{d}{2}-1}\left(2\frac{\sqrt{uwr}}{1-r}\right)\frac{1}{(1-r)\Gamma(d/2)}, \tag{22}$$

$u, w > 0, 0 < r \leq 1$, $I_\nu(z) = \sum_{m=\nu}^{\infty}\left(\frac{z}{2}\right)^{2m+\nu}/[m!\Gamma(m+\nu+1)]$, $z > 0$ being the modified Bessel function of the first kind of order ν, and

$$\begin{aligned} r &= R_{\chi_d}(x-y) = cov(\chi_d(x), \chi_d(y))/var\chi_d(0) \\ &= B_\xi^2(x-y), \quad x, y \in \mathbb{R}^n \end{aligned} \tag{23}$$

(see Anh and Leonenko [1] for further details). We note that the formula (22) is known as the Hille-Hardy formula.

From (20) - (23), we obtain the following moment properties:

$$E\chi_d(x) = \frac{d}{2}, var\chi_d(x) = \frac{d}{2},$$

$$Ee_k(\chi_d(x)) = 0, Ee_k(\chi_d(x))e_m(\chi_d(y)) = \delta_m(k)R_{\chi_d}^m(x-y), \tag{24}$$

where $\delta_m(k)$ is the Kronecker delta function.

In the next section, we will consider the Cauchy problem (1) and (2) with random initial condition (13) for $\beta \in (0,1]$ and the Cauchy problem (7) and (8) with random initial condition (16) for $\beta \in (1,2)$. For these initial conditions, we introduce the following conditions:

A. The initial conditions (13) and (16) specify independent random fields which are subordinated to chi-square random fields, that is,

$$u_j(x) = h_j\left(\chi_{d_j}(x)\right), \quad x \in \mathbb{R}^n, j \in \{0,1\}, \tag{25}$$

where

$$\chi_{d_0}(x) = \frac{1}{2}\sum_{k=1}^{d_0}\left[\xi_k^{(0)}(x)\right]^2, \chi_{d_1}(x) = \frac{1}{2}\sum_{k=1}^{d_1}\left[\xi_k^{(1)}(x)\right]^2, x \in \mathbb{R}^n,$$

$\xi_1^{(j)},...,\xi_{d_j}^{(j)}, j \in \{0,1\}$, are independent copies of homogeneous isotropic Gaussian random fields $\xi^{(j)}(x), j \in \{0,1\}$, with $E\xi^{(j)}(x) = 0, E\xi^{(j)}(x)\xi^{(j)}(y) = B_{\xi^{(j)}}(x-y)$, $x, y \in \mathbb{R}^n$, $j \in \{0,1\}$, and h_j, $j \in \{0,1\}$, are two real non-random Borel functions such that

$$Eh_j^2\left(\chi_{d_j}(x)\right) < \infty, \quad j \in \{0,1\}. \tag{26}$$

Under the condition A, we have the following expansion in the Hilbert space $L_2((0,\infty), p(u)\,du)$:

$$h_j(u) = \sum_{k=0}^{\infty} C_k^{(j)} e_k(u), C_k^{(j)} = \int_0^{\infty} h_j(u) e_k(u) p(u)\,du,\ k = 0,1,2,...,j \in \{0,1\}. \tag{27}$$

Additionally, we assume that the functions h_j, $j \in \{0,1\}$ satisfy the following condition:

B. Condition A holds and there exist integers $m_j \geq 1$, $j \in \{0,1\}$ such that

$$C_1^{(j)} = ... = C_{m_j-1}^{(j)} = 0, C_{m_j}^{(j)} \neq 0,\ j \in \{0,1\}. \tag{28}$$

The integers $m_0 \geq 1$ and $m_1 \geq 1$ are the Laguerre ranks of the functions h_0 and h_1, respectively.

For random fields with long-range dependence (LRD), we introduce the following condition:

C. Condition A holds and

$$R_{\chi_{d_j}}(x) = \left(1 + |x|^2\right)^{-\varkappa_j}, 0 < \varkappa_j < n/2,\ j \in \{0,1\}. \tag{29}$$

Note that (29) means that

$$B_{\xi^{(j)}}(x) = \left(1 + |x|^2\right)^{-\varkappa_j/2}, 0 < \varkappa_j < n,\ j \in \{0,1\}. \tag{30}$$

By the Bochner-Khintchine Theorem, the correlation functions (30) have the spectral representations

$$B_{\xi^{(j)}}(x) = \int_{\mathbb{R}^n} \cos\langle\lambda, x\rangle f_j(\lambda)\,d\lambda, \quad j \in \{0,1\}, \tag{31}$$

where the isotropic spectral densities $f_j(\lambda)$, $\lambda \in \mathbb{R}^n$, $j \in \{0,1\}$, have the following explicit form (see Donoghue [11], p. 293):

$$\begin{aligned} f_j(\lambda) &= \left[\pi^{n/2} 2^{((\varkappa_j - n)/2)} \Gamma(\varkappa_j/2)\right]^{-1} K_{(\varkappa_j-n)/2}(|\lambda|)\,|\lambda|^{(\varkappa_j-n)/2} \\ &= c(n,\varkappa_j)\,|\lambda|^{\varkappa_j - n}\left(1 - \theta_j(|\lambda|)\right), \quad j \in \{0,1\}, \end{aligned} \tag{32}$$

with

$$c(n,\varkappa) = \Gamma\left(\frac{n-\varkappa}{2}\right) / \left[2^{\varkappa}\pi^{n/2}\Gamma\left(\frac{\varkappa}{2}\right)\right], \varkappa \in (0,n), \tag{33}$$

and $\theta_j(|\lambda|) \to 0$ as $|\lambda| \to 0, j \in \{0,1\}$; moreover $\theta_j(|\lambda|) = O\left(|\lambda|^{n-\varkappa_j}\right), j \in \{0,1\}$. Note that we have used the following expansion of the modified Bessel function of the third kind or McDonald's function (see Watson [27]):

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}\left(I_{-\nu}(z) - I_\nu(z)\right) \sim \Gamma(\nu)\, 2^{\nu-1} z^{-\nu} \tag{34}$$

as $z \downarrow 0, \nu > 0$. As $z \to \infty$, the following expansion holds (see Watson [27]):

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z}\left(1 + \frac{4\nu^2 - 1}{8z} + ...\right). \tag{35}$$

Relation (32), which can be found in Donoghue [11], p. 295, is a special case of Tauberian theorem, and constant (33) is known as the Tauberian constant (see Leonenko [18], pp. 64-66).

Observe that under condition C

$$\int_{\mathbb{R}^n} R_{\chi_{d_j}}(x)\, dx = \infty, \quad j \in \{0,1\}, \tag{36}$$

and $f_j(0) = \infty$, $j \in \{0,1\}$. Thus, the random fields with correlation functions (29) display LRD.

4. Gaussian and non-Gaussian scenarios

We now give our main results which present the Gaussian and non-Gaussian scenarios for renormalized random fields (14) or (17) with random initial conditions (25) with weak or strong dependence. The results yield Gaussian and non-Gaussian central limit theorems for fractional Volterra equations. These results are analogous to the results by Anh and Leonenko [2], [3], [5], [4], [6] for Gaussian random fields and their subordinated ones, but the normalizing factors and the type of non-Gaussian limiting fields are new.

Theorem 1. *Let $n = 1, 2$ or 3. Consider the random field $u(t,x)$, $t > 0$, $x \in \mathbb{R}^n$, defined by (14), in which $\eta_0(x)$ is of the form (25) with $j = 0$, where $\chi_{d_0}(x)$ is a chi-square random field satisfying conditions A and B with $j = 0$, that is, the non-random function h_0 has Laguerre rank 1 and*

$$\int_{\mathbb{R}^n} \left|R_{\chi_{d_0}}(x)\right| dx < \infty, \sigma_0^2 = \sum_{k=1}^{\infty}\left[C_k^{(0)}\right]^2 \int_{\mathbb{R}^n}\left[R_{\chi_{d_0}}(x)\right]^k dx > 0. \tag{37}$$

Then the finite-dimensional distributions of random fields

$$U_\varepsilon(t,x) = \frac{1}{\varepsilon^{n\beta/4}}\left[u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}}\right) - C_0^{(0)}\right], t > 0, x \in \mathbb{R}^n, 0 < \beta \le 1, \varepsilon > 0, \tag{38}$$

converge weakly as $\varepsilon \to 0$ to the finite-dimensional distributions of the homogeneous (in space) Gaussian random fields $U(t,x)$, $t > 0$, $x \in \mathbb{R}^n$, with $EU(t,x) = 0$ and covariance function

$$EU(t,x)\,U(s,y) = \frac{\sigma_0^2}{(2\pi)^n}\int_{\mathbb{R}^n} \cos\langle\lambda, x-y\rangle\, E_{\beta,1}\left(-t^\beta|\lambda|^2\right) E_{\beta,1}\left(-s^\beta|\lambda|^2\right) d\lambda, \tag{39}$$

where σ_0^2 is defined in (37) and $E_{\beta,1}$ is defined in (6).

We will give the proof in *Section 5*. Note that for $\beta < 2, \beta \neq 1$, there is an asymptotic expansion:

$$E_{a,b}(z) = -\sum_{k=1}^{N} \frac{z^{-k}}{\Gamma(b-ak)} + O\left(|z|^{-N-1}\right) \tag{40}$$

as $z \to \infty$, which is valid in a sector about the negative real axis (see, for example, Djrbashian [9], p. 5). Thus, if $\beta < 2$, $\beta \neq 1$, and $n = 1, 2$ or 3

$$T(\lambda) = E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-s^{\beta}|\lambda|^2\right) \in L_1(\mathbb{R}^n) \tag{41}$$

since by (40) $T(\lambda) = O\left(|\lambda|^{-4}\right)$.

The non-Gaussian scenarios for renormalized random field (14) with a strongly dependent initial condition (26) (with $j = 0$), which satisfies condition C (with $j = 0$) are obtained in Anh and Leonenko [3], [4] in terms of Wiener-Itô multiple stochastic integrals. These non-Gaussian limiting random fields may possess LRD and/or intermittency.

Theorem 2. *Let $n = 1, 2$ or 3. Consider the random field $u(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, defined by (17) with random initial conditions (26) satisfying conditions A and B with $m_j = 1$, $j = 1, 2$ and*

$$\int_{\mathbb{R}^n} \left|R_{\chi_{d_j}}(x)\right| dx < \infty, \sigma_0^2 = \sum_{k=1}^{\infty} \left[C_k^{(j)}\right]^2 \int_{\mathbb{R}^n} \left[R_{\chi_{d_j}}(x)\right]^k dx > 0,\ j \in \{0, 1\}. \tag{42}$$

Then the finite-dimensional distributions of random fields

$$\begin{aligned} V_\varepsilon(t, x) &= \varepsilon^{\frac{(4-n\beta)}{4}} \left[u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}}\right) - C_0^{(0)} - C_0^{(1)}\frac{t}{\varepsilon}\right], \\ t &> 0, x \in \mathbb{R}^n, \varepsilon > 0, 1 < \beta < 2, \end{aligned} \tag{43}$$

converge weakly as $\varepsilon \to \infty$ to the finite-dimensional distributions of the Gaussian random field $V(t, x)$, $t > 0, x \in \mathbb{R}^n$, with $EV(t, x) = 0$ and covariance function

$$EV(t, x) V(s, y) = \frac{\sigma_1^2}{(2\pi)^n} ts \int_{\mathbb{R}^n} \cos\langle\lambda, x - y\rangle E_{\beta,2}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,2}\left(-s^{\beta}|\lambda|^2\right) d\lambda, \tag{44}$$

where σ_1^2 is defined in (42) and $E_{\beta,2}$ is defined in (6).

Let us now consider the case when the random fields $u_0(x)$, $x \in \mathbb{R}^n$, and $u_1(x)$, $x \in \mathbb{R}^n$, have LRD (see condition C and (36)). Anh and Leonenko [3], [4] developed the theory of renormalization for the case $\beta \in (0, 1]$ and the Laguerre rank $m_0 = 1$ or $m_0 = 2$. Now, we consider the case $\beta \in (1, 2)$. For simplicity, we include in the analysis the Laguerre ranks $m_0 = 1$ and $m_1 = 1$, but similar results (with necessary modifications) can be obtained for the Laguerre ranks $m_0 \geq 1$ and $m_1 \geq 1$.

Theorem 3. *Let $n = 1, 2$ or 3 and $\beta \in (1, 2)$. Consider the random fields $u(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, in which the fields $u_j(x)$, $x \in \mathbb{R}^n$, $j \in \{0, 1\}$ satisfy the conditions A, B, C with $m_j = 1$, $\varkappa_j \in (0, n/2)$, $j \in \{0, 1\}$. Then*

1) if $2\varkappa_0\beta < 2\varkappa_1\beta - 4$, *the finite-dimensional distributions of the random fields*

$$T'_\varepsilon(t,x) = \frac{1}{\varepsilon^{\varkappa_0\beta/2}}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right) - C_0^{(0)} - C_0^{(1)}\frac{t}{\varepsilon}\right], t>0, x\in\mathbb{R}^n, \tag{45}$$

converge weakly, as $\varepsilon \to 0$, *to the finite-dimensional distributions of the random field*

$$T(t,x) = -\frac{C_1^{(0)}c(n,\varkappa_0)}{\sqrt{2d_0}}\sum_{k=1}^{d_0}\int'_{\mathbb{R}^{2n}} e^{i\langle\lambda_1+\lambda_2,x\rangle}$$

$$\times E_{\beta,1}\left(-t^\beta|\lambda_1+\lambda_2|^2\right)\frac{W_k^{(0)}(d\lambda_1)W_k^{(0)}(d\lambda_2)}{|\lambda_1\lambda_2|^{\frac{n-\varkappa_0}{2}}}, t>0, x\in\mathbb{R}^n, \tag{46}$$

where W_k *are independent copies of Gaussian measure* $W^{(0)}$ *such that* $E\left|W^{(0)}(d\lambda)\right|^2 = d\lambda$, $\int'_{\mathbb{R}^{2n}}\dots$ *is a multiple Wiener-Itô stochastic integral and* $c(n,\varkappa_0)$ *is defined in (33);*

2) If $2\varkappa_1\beta - 4 < 2\varkappa_0\beta$, *the finite-dimensional distributions of random fields*

$$T''_\varepsilon(t,\varepsilon) = \frac{1}{\varepsilon^{(\varkappa_1\beta-2)/2}}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right) - C_0^{(0)} - C_0^{(1)}\frac{t}{\varepsilon}\right], t>0, x\in\mathbb{R}^n, \tag{47}$$

converge weakly, as $\varepsilon \to 0$, *to the finite-dimensional distributions of the random field*

$$T(t,\varepsilon) = -\frac{C_1^{(1)}c(n,\varkappa_1)}{\sqrt{2d_1}}\sum_{k=1}^{d_1}\int'_{\mathbb{R}^{2n}} e^{i\langle\lambda_1+\lambda_2,x\rangle}$$

$$\times tE_{\beta,2}\left(-t^\beta|\lambda_1+\lambda_2|^2\right)\frac{W_k^{(1)}(d\lambda_1)W_k^{(1)}(d\lambda_2)}{|\lambda_1\lambda_2|^{\frac{n-\varkappa_1}{2}}}, t>0, x\in\mathbb{R}^n, \tag{48}$$

where $W_k^{(1)}$ *are independent copies of Gaussian measure* $W^{(1)}$, *which is independent of* $W^{(0)}$;

3) If $2\varkappa_0\beta = 2\varkappa_1\beta - 4$, *the finite-dimensional distributions of random fields*

$$T'''_\varepsilon(t,x) = \frac{1}{\varepsilon^{\varkappa_0\beta/2}}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right) - C_0^{(0)} - C_0^{(1)}\frac{t}{\varepsilon}\right], t>0, x\in\mathbb{R}^n, \tag{49}$$

converge weakly, as $\varepsilon \to 0$, *to the finite-dimensional distributions of the random field*

$$T'''(t,x) = T'(t,x) + T''(t,x), t>0, x\in\mathbb{R}^n, \tag{50}$$

where the two independent random fields T' *and* T'' *are defined in (46) and (48), respectively.*

Remark 1. *It is not difficult to modify the results of* Theorems 1 - 3 *for the case when a)* $u_0(x)$ *has weak dependence in the sense of condition (37) but* $u_1(x)$ *has LRD in the sense of condition C with* $j=1$, *or b)* $u_0(x)$ *has LRD in the sense of condition C with* $j=0$ *but* $u_1(x)$ *has weak dependence in the sense of condition*

(42) with $j=1$. Thus, with necessary modifications of the conditions of Theorems 1 - 3 *we have the following results for the rescaled solutions*

$$T_{\varepsilon}(t,x)=\frac{1}{A_{\varepsilon}}\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right)-C_0^{(0)}-C_1^{(0)}\frac{t}{\varepsilon}\right]$$

of Eq. (7) as $\varepsilon\to 0$:

a) $T_{\varepsilon}(t,x)\xrightarrow{d}T''(t,x)$ *with* $A_{\varepsilon}=\varepsilon^{(\varkappa_1\beta-2)/2}$;

b) $T_{\varepsilon}(t,x)\xrightarrow{d}T(t,x)$ *with* $A_{\varepsilon}=\varepsilon^{\varkappa_0\beta/2}$ *if* $2\varkappa_0\beta<n\beta-4$;

$T_{\varepsilon}(t,x)\xrightarrow{d}V(t,x)$ *with* $A_{\varepsilon}=\frac{n\beta-4}{4}$ *if* $n\beta-4<2\varkappa_0\beta$;

$T_{\varepsilon}(t,x)\xrightarrow{d}T(t,x)+V(t,x)$ *with* $A_{\varepsilon}=\varepsilon^{\varkappa_0\beta/2}$ *if* $2\varkappa_0\beta=n\beta-4$, *where* $\xrightarrow{d}$ *stands for convergence of finite-dimensional distributions of the random fields defined in* Theorems 2 - 3.

Remark 2. *For the wave equations ($\nu=2$), similar results may be obtained for d'Alembert random field* $u(t,x)=m_0+\int_{\mathbb{R}^1}e^{i\lambda x}\cos(\lambda t)\,Z_0(d\lambda)$ *using exact hyperbolic renormalization (see Woyczynski [28]):*

$$\left[u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)-Eu\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\right]/A_{\varepsilon}$$

as $\varepsilon\to 0$ *and* $A_{\varepsilon}=\varepsilon^{-\varkappa_0}$ *(see Anh and Leonenko [6]).*

5. Proofs

We will use the standard notations $\xrightarrow{p}$ and $\xrightarrow{d}$ for convergence of random variables in probability and distribution, respectively, and $\stackrel{d}{=}$ for equality of random variables in distribution.

Proof of Theorem 1

From (5), we obtain

$$\int_{\mathbb{R}^n}G_{\beta}(t,u)\,G_{\beta}(t',x'-x+z+u)\,du$$

$$\begin{aligned}&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\delta(\lambda-\lambda')\,e^{-i\langle\widetilde{\lambda},x'-x+z\rangle}E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right)E_{\beta,1}\left(-t'^{\beta}|\lambda|^2\right)d\lambda d\lambda'\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-i\langle\lambda,z-x+x'\rangle}E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right)E_{\beta,1}\left(-t'^{\beta}|\lambda|^2\right)d\lambda,\end{aligned}$$

where

$$\delta\left(\lambda-\widetilde{\lambda}\right)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i\langle x,\lambda-\widetilde{\lambda}\rangle}dx \tag{51}$$

is the Dirac delta function. From (4), (14), (25) and (51), we obtain the covariance function of the random field $u(t,x)$ as

$$\begin{aligned} cov\left(u(t,x), u(t',x')\right) &= \int_{\mathbb{R}^n} \cos\langle\lambda, x-x'\rangle E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-s^{\beta}|\lambda|^2\right) F_0(d\lambda) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} B(y-y') G(t',x'-y') G(t,x-y)\, dy dy' \\ &= \int_{\mathbb{R}^n} B(z) \int_{\mathbb{R}^n} G(t,u) G(t',x'-x+z+u)\, du, \end{aligned} \tag{52}$$

where F_0 and B are the spectral measure and the covariance function of the random field (25) with $j=0$, which satisfies condition (26) with $j=0$ and (37). From (24), (25) and (26), we can write

$$cov\left(h_0\left(\chi_{d_0}(x)\right), h_0\left(\chi_{d_0}(0)\right)\right) = \sum_{k=m_0}^{\infty}\left[C_k^{(0)}\right]^2 R^k_{\chi_{d_0}}(x). \tag{53}$$

Thus, from (4), (51) to (53), we obtain

$$cov\left(u(t,x), u(t',x')\right) = \sum_{k=m_0}^{\infty}\left[C_k^{(0)}\right]^2$$
$$\times \int_{\mathbb{R}^n} R^k_{\chi_{d_0}}(z)\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\langle\lambda, z-x+x'\rangle} E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-t'^{\beta}|\lambda|^2\right) d\lambda dz\right]. \tag{54}$$

From (37) and (54), we can write

$$\begin{aligned} &\lim_{\varepsilon\to 0} cov\left(U_\varepsilon(t,x), U_\varepsilon(t',x')\right) = \lim_{\varepsilon\to 0}\varepsilon^{-n\beta/2}\cos\left(u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right) - u\left(\frac{t'}{\varepsilon},\frac{x'}{\varepsilon^{\beta/2}}\right)\right) \\ &= \lim_{\varepsilon\to 0}\varepsilon^{-n\beta/2}\sum_{k=m_0}^{\infty}\left[C_k^{(0)}\right]^2\int_{\mathbb{R}^n} R^k_{\chi_{d_0}}(z) \\ &\quad\times\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\left\langle\lambda, z-\frac{x-x'}{\varepsilon^{\beta/2}}\right\rangle} E_{\beta,1}\left(-\left(\frac{t}{\varepsilon}\right)^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-\left(\frac{t'}{\varepsilon}\right)^{\beta}|\lambda|^2\right) d\lambda\right] dz \\ &= \lim_{\varepsilon\to 0}\sum_{k=m_0}^{\infty}\left[C_k^{(0)}\right]^2\int_{\mathbb{R}^n} R^k_{\chi_{d_0}}(z) \\ &\quad\times\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\cos\left\langle\lambda, z\varepsilon^{1/\beta}-(x-x')\right\rangle E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-t'^{\beta}|\lambda|^2\right) d\lambda\right] dz \\ &= \frac{\sigma_0^2}{(2\pi)^n}\int_{\mathbb{R}^n}\cos\langle\lambda, x-x'\rangle E_{\beta,1}\left(-t^{\beta}|\lambda|^2\right) E_{\beta,1}\left(-t'^{\beta}|\lambda|^2\right) d\lambda \end{aligned} \tag{55}$$

by the dominated convergence theorem (see (41)).

Our proof is based on the Markov method of moments (see for example Breuer and Major [8] or Ivanov and Leonenko [15], p. 72), which consists of showing that for any integer $p \geq 2$

$$\lim_{\varepsilon\to 0} E\left(\zeta_\varepsilon\right)^p = E\zeta^p, \tag{56}$$

where for $u_j \in \mathbb{R}^1$, $j = 1, ..., k$

$$\zeta_\varepsilon = \sum_{j=1}^{k} u_j U_\varepsilon(t_j, x_j), \ \zeta = \sum_{j=1}^{k} u_j U(t_j, x_j),$$

U_ε and U being defined in *Theorem 1.* It is well-known that for a Gaussian random field $U(t, s)$

$$E\zeta^p = \begin{cases} (p-1)!! \left[\sum_{j=1}^{k} \sum_{j'=1}^{k} u_j u_{j'} EU(t_j, x_j) U(t_{j'}, x_{j'})\right]^{p/2}, \ p = 2\nu, \\ 0, \quad p = 2\nu + 1. \end{cases} \tag{57}$$

To prove (56), we use the properties of multidimensional Hermite polynomials

$$\Pi_\nu(u) = \prod_{j=1}^{d_0} H_{k_j}(u_j), \ u = (u_1, ..., u_{d_0}) \in \mathbb{R}^{d_0}, \nu = (k_1, ..., k_{d_0}), \ k_j \geq 0, j = 1, ..., d_0,$$

where

$$H_k(u) = (-1)^m e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \quad m = 0, 1, 2, ...$$

are Hermite polynomials. The polynomials $\{e_\nu(u)\}_\nu$ form a complete orthogonal system in the Hilbert space

$$L_2\left(\mathbb{R}^{d_0}, \phi(|u|)\, du\right) = \left\{h : \int_{\mathbb{R}^{d_0}} h^2(u)\, \phi(|u|)\, du < \infty\right\},$$

where

$$\phi(|u|) = \prod_{j=1}^{d_0} \phi(u_j), \ \phi(u_j) = \frac{1}{\sqrt{2\pi}} e^{-u_j^2/2}.$$

Now, the function $h_0(u_1, ..., u_{d_0}) = \frac{1}{2}\sum_{j=1}^{d_0} u_j^2 \in L_2\left(\mathbb{R}^{d_0}, \phi(|u|)\, du\right)$ and admits the expansion

$$h_0(u_1, ..., u_{d_0}) = \sum_{k=0}^{\infty} \sum_{\nu \in S_k} \frac{\widetilde{C}_\nu \Pi_\nu(u)}{\nu!},$$

where

$$S_k = \left\{\nu = (k_1, ..., k_{d_0}) : \sum_{j=1}^{d_0} k_j = k, \ k_j \geq 0\right\},$$

$\nu! = k_1!...k_{d_0}!$ and

$$\widetilde{C}_\nu = \int_{\mathbb{R}^{d_0}} \left(\frac{1}{2}\sum_{j=1}^{d_0} u_j^2\right) \Pi_\nu(u)\, \phi(|u|)\, du.$$

We then have the following representation of the nonlinear functional of the Gaussian vector field $\xi(x) = (\xi_1(x), ..., \xi_{d_0}(x))$:

$$\begin{aligned} h_0\left(\frac{1}{2}\sum_{j=1}^{d_0}\xi_j^2(x)\right) &= h_0(\xi_1(x), ..., \xi_d(x)) \\ &= \widetilde{C}_0 + \sum_{k=1}^{\infty}\sum_{\nu\in S_k}\left(\frac{\widetilde{C}_\nu}{\nu!}\right)\Pi_\nu(\xi(x)). \end{aligned}$$

It is clear that $\widetilde{C}_0 = C_0^{(0)}$, where $C_0^{(0)}$ is the zero Laguerre coefficient in the Laguerre expansion of the function

$$h_0\left(\frac{1}{2}\sum_{j=1}^{d_0}\xi_j^2(x)\right) = \sum_{j=1}^{d_0} C_j^{(0)} e_k\left(\frac{1}{2}\sum_{j=1}^{d_0}\xi_j^2(x)\right).$$

Thus, the random field

$$\begin{aligned} U_\varepsilon(t,x) &= \frac{1}{\varepsilon^{n\beta/4}}\left[\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}} - y\right) h_0(\chi_{d_0}(y))\, dy - C_0^{(0)}\right] \\ &= \sum_{k=1}^{\infty}\sum_{\nu\in S_k}\frac{1}{\varepsilon^{n\beta/4}}\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/4}} - y\right) \\ &= \sum_{k=1}^{\infty}\sum_{\nu\in S_k}\frac{\widetilde{C}_\nu}{\nu!}\frac{1}{\varepsilon^{n\beta/4}}\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/4}} - y\right)\Pi_\nu(\xi(y))\, dy. \qquad (58) \end{aligned}$$

The diagram formulae are valid for our situation (see Arcones [7], Leonenko and Deriev [19]). Thus, the main statement (55) can be proved by using (57), (58) and the diagram method for multidimensional Hermite polynomials. The details are given in Anh and Leonenko ([5]) for one-dimensional Hermite expansion. The multidimensional generalization is straightforward since our multidimensional Hermite polynomials are the products of one-dimensional Hermite polynomials. A detailed exposition of multidimensional Hermite polynomial expansions can be found in Arcones [7], Leonenko and Deriev [19].

Proof of Theorem 2

Similar to the proof of *Theorem 1*, we may represent the random fields

$$\begin{aligned} V_\varepsilon(t,x) &= \varepsilon^{\frac{4-n\beta}{4}}\left[\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}} - y\right) h_0(\chi_{d_0}(y))\, dy\right. \\ &\quad \left. + \int_{\mathbb{R}^n} G_\beta^{(1)}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}} - y\right) h_1(\chi_{d_1}(y))\, dy - C_0^{(0)} - C_0^{(1)}\frac{t}{\varepsilon}\right. \\ &= R_{1\varepsilon} + R_{2\varepsilon}, \end{aligned}$$

where

$$R_{1\varepsilon} = \sum_{k=1}^{\infty}\sum_{\nu\in S_k}\frac{\widetilde{C}_\nu^{(0)}}{\nu!}\varepsilon^{\frac{4-n\beta}{4}}\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\beta/2}} - y\right)\Pi_\nu\left(\xi^{(0)}(y)\right) dy,$$

$$R_{2\varepsilon}=\sum_{k=1}^{\infty}\sum_{\nu\in S_k}\frac{\widetilde{C}_{\nu}^{(1)}}{\nu!}\varepsilon^{\frac{4-n\beta}{4}}\int_{\mathbb{R}^n}G_{\beta}^{(1)}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right)\Pi_{\nu}\left(\xi^{(1)}(y)\right)dy,$$

$\xi^{(0)}(x)=\left(\xi_1^{(0)}(x),...,\xi_{d_0}^{(0)}(x)\right)$, $\xi^{(1)}(x)=\left(\xi_1^{(1)}(x),...,\xi_{d_1}^{(1)}(x)\right)$ being two independent Gaussian vector fields, and

$$\widetilde{C}_{\nu}^{(j)}=\int_{\mathbb{R}^{d_j}}\left(\frac{1}{2}\sum_{k=1}^{d_j}\left[u_k^{(j)}\right]^2\right)\Pi_{\nu}\left(u^{(j)}\right)du,\quad j\in\{0,1\}$$

being Hermite coefficients. From an asymptotic variance analysis, we obtain that

$$varR_{1\varepsilon}\to 0,\ \varepsilon\to 0$$

and by Slutsky's argument, the limiting distribution of this functional is the same as the limiting distribution of $R_{2\varepsilon}$, which can be analyzed in a similar fashion to the proof of *Theorem 1* by making use of the diagram formula (see Anh and Leonenko [5]).

Proof of Theorem 3

We use the Laguerre polynomial expansion (see Anh and Leonenko [1]), from which we obtain the following expansion:

$$u\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}\right)-C_0^{(0)}-C_0^{(1)}\frac{t}{\varepsilon}=Q_{1\varepsilon}+Q_{2\varepsilon},$$

where

$$\begin{aligned}Q_{1\varepsilon}&=\sum_{k=1}^{\infty}C_k^{(0)}\int_{\mathbb{R}^n}G_{\beta}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right)e_k\left(\chi_{d_0}(y)\right)dy\\&=C_1^{(0)}\int_{\mathbb{R}^n}G_{\beta}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right)e_1\left(\chi_{d_0}(y)\right)dy+S_{1\varepsilon},\end{aligned}$$

$$\begin{aligned}Q_{2\varepsilon}&=\sum_{k=1}^{\infty}C_k^{(1)}\int_{\mathbb{R}^n}G_{\beta}^{(1)}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right)e_k\left(\chi_{d_1}(y)\right)dy\\&=C_1^{(1)}\int_{\mathbb{R}^n}G_{\beta}^{(1)}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right)e_1\left(\chi_{d_1}(y)\right)dy+S_{2\varepsilon}.\end{aligned}$$

Note that for case (1)

$$var\left[\varepsilon^{-\varkappa_0\beta/2}S_{1\varepsilon}\right]\to 0,var\left[\varepsilon^{-\varkappa_0\beta/2}Q_{2\varepsilon}\right]\to 0\text{ as }\varepsilon\to 0\ ,$$

and for case (2)

$$var\left[\varepsilon^{(2-\varkappa_1\beta)/2}Q_{1\varepsilon}\right]\to 0,var\left[\varepsilon^{(2-\varkappa_1\beta)/2}S_{2\varepsilon}\right]\to 0\text{ as }\varepsilon\to 0.$$

Thus, in case (1), the asymptotic distribution of $T'_{\varepsilon}(t,\varepsilon)$ is the same as the asymptotic distribution of the functional

$$\frac{C_1^{(0)}}{\varepsilon^{\beta\varkappa_0/2}}\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right) e_1\left(\chi_{d_0}(y)\right) dy, \tag{59}$$

while for case (2), the asymptotic distribution of $T''_{\varepsilon}(t,\varepsilon)$ is the same as the asymptotic distribution of the functional

$$\varepsilon^{-(\varkappa_1\beta-2)/2}\int_{\mathbb{R}^n} G_\beta^{(1)}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right) e_1\left(\chi_{d_1}(y)\right) dy. \tag{60}$$

In case (3), the asymptotic distribution is the same as the asymptotic distribution of the sum (59) and (60). To obtain the asymptotic distribution of (59), we note that the first Laguerre polynomial $e_1(u)=\left(\frac{d_0}{2}-u\right)/\sqrt{\frac{d_0}{2}}$, and as a result

$$e_1\left(\frac{1}{2}\sum_{k=1}^{d_0}\left(\xi_k^{(0)}(x)\right)^2\right)=-\frac{1}{\sqrt{2d_0}}\sum_{k=1}^{d_0}H_2\left(\xi_k^{(0)}(x)\right),$$

where $H_2(u)=u^2-1$ is the second Hermite polynomial. The asymptotic distribution of the functional

$$\frac{1}{\varepsilon^{\varkappa_0\beta/2}}C_1^{(0)}\int_{\mathbb{R}^n} G_\beta\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon^{\beta/2}}-y\right) H_2\left(\xi_k^{(0)}(y)\right) dy$$

is obtained in Anh and Leonenko [5]. Thus, our limiting distribution is the sum (46) of d_0 copies of such asymptotic distributions.

In the same manner, we may derive the asymptotic distributions for cases (2) and (3).

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