Network cost allocation games based on threshold discounting

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Abstract. *Consider networks in which each pair of nodes needs to communicate. The communication flow between any pair of nodes can be delivered through a direct link or via some connecting path in the network. By discounting the cost of flow through links for which the high flow volume is anticipated, network designers exploit economies of scale. This approach encourages the concentration of flows and use of relatively small number of links. This led to the design of well known hub networks and more recently hub-like networks. Applications include telecommunications, airline traffic flow, and mail delivery networks. The cost of services delivered through such networks is distributed among its users who may be individuals or organizations with possibly conflicting interests. The cooperation of these users is essential for the exploitation of economies of scale. Consequently, there is a need to find a fair distribution of the cost of providing the service among network users. In this paper, we present a survey of some recent results in the development of cooperative game theory based mechanisms to efficiently characterize cost allocation solutions for hub and hub-like networks. Specifically, we formulate the associated hub and hub-like network cost allocation games. Then, while paying special attention to users' contribution to economies of scale, we demonstrate that some attractive cost allocation solutions, which provide users with the incentive to cooperate, can be efficiently computed.*

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1. Introduction

The objective for designers of communication networks is often to minimize the cost, while satisfying various service constraints like reliability, congestion, speed,

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capacity and coverage. Economies of scale play an important role in their considerations. Namely, the creation of high capacity links and concentration of flows reduces the number of needed links and the unit flow cost. Applications of this approach are seen, for example, in high capacity lines of backbone networks in telecommunications and in high volume traffic between major airports in air transportation networks.

Most of the related work in the literature was performed in the context of the hub networks. In those networks, a certain subset of focal nodes (i.e. hubs) is fully interconnected, while other nodes are connected to those hubs and the economies of scale are achieved by discounting the cost of traffic among hubs. The hub networks were extensively studied over the last couple of decades (see for example, [1],[2],[6],[7],[8],[9],[19] and [20]). Numerous computational studies showed that hub networks are quite attractive and practical. Nevertheless, the restrictions imposed with the hub network model are sometimes too prohibitive. For example, in some cases a high traffic between a non-hub and a hub node is not discounted and/or the traffic between two hubs is not big enough to warrant any discounts. For an extensive discussion on these issues see [10].

Recently, Podnar et al. [10] introduced the network model in which each pair of nodes can communicate via any path, and the cost of sending flow through each link is discounted if and only if the amount of flow exceeds certain threshold. This approach also gives incentive to concentrate flows. It seems however, that the above threshold based discounting model is even more 'efficient' than hub networks in its use of a relatively small number of links and in the exploitation of economies of scale. We will refer to the threshold based discounting network model as to the hub-like network (HLN) model. Podnar et al. in [10] provided combinatorial formulations and efficient heuristic for finding the minimum cost HLN.

The cost of services delivered through hub and hub-like networks is distributed among users who may be individuals or organizations with possibly conflicting interests. The cooperation of these users is essential for the exploitation of economies of scale. Consequently, there is a need to find a fair distribution of the cost of providing the service among network users. Failing to do so may cause some users to secede and seek services from some other competing network. Such secession would inevitably result in a higher cost per unit of delivered service.

A cooperative game theory was used to analyze several classes of network cost allocation problems in the literature. Some examples include: spanning tree games ([5]), Steiner tree games ([14]), network flow games ([3],[4]), cost allocation arising from routing in networks ([11]), capacitated network design games ([13],[18]). For a survey and numerous references on cost allocation models in networks see [12]. A common approach to above papers is the formulation of the associated cost allocation problem as a cooperative game in the characteristic function form, followed by the evaluation of various solution concepts such as core, nucleolus, kernel, the least ϵ-core, Shapley value, etc. It is well known that these game theoretic solution concepts are computationally prohibitive even for relatively small problems. Moreover, there are no general practical algorithms for the computation of these solutions. Consequently, researchers have concentrated on individual classes of games to demonstrate that computation of cost allocation solution concepts is sometimes feasible in the context of a particular problem.

We take a similar approach in the study of the cost allocation problem in hub and hub-like networks. Namely, we will show that attractive cost allocation solutions are possible in the context of some practical problems. In order to analyze the cost allocation problem we will define hub and hub-like games. In defining those games special attention will be paid to users' contribution to economies of scale. Then, we will demonstrate that those network games are decomposable, thus enabling us to simplify the cost allocation problem. Finally, we will efficiently characterize some attractive cost allocation solutions, which encourage cooperation of users.

The plan of the paper follows. In *Section 2*, we present combinatorial formulations of hub and hub-like optimization problems. Specifically, we formulate the minimum cost hub network and the minimum cost hub like network problems. In addition, we define some basic game theoretic concepts. In *Section 3*, we outline some cost optimization results on hub and hub-like networks. In *Section 4*, we discuss various cost allocation games associated with these problems. In particular, we formulate hub network and hub-like network cost allocation games based on threshold discounting. *Section 5* provides the analysis of the core of the above cooperative games. In *Section 6*, we summarize our findings and present concluding remarks.

2. Definitions and preliminaries

The linear mixed integer programming formulation of the single allocation hub network problem (HNP) by Skorin-Kapov et al. [20] can be summarized as follows. Let N be the set of users as well as the set of potential hub locations. For $i, j, k, m \in N$, let us define the following variables: x_{ijkm} =the fraction of flow from location (origin) i to location (destination) j, routed via hubs at locations k and m in that order; $z_{ik} = 1$ if non-hub location i is allocated to hub k, and 0 otherwise. The input data are given as: p =the required number of hubs to be open; f_{ij} = the flow from location i to location j; c_{ij} =the cost per unit of flow from location i to location j (it is proportional to the distance and it is assumed that $c_{ij} \geq 0$); $0 \leq \alpha \leq 1$ is the discount on the unit cost of flow between hubs. Observe that the cost per unit of flow between origin i and destination j, routed via hubs k and m in that order, is given by $c_{ik} + \alpha c_{km} + c_{mj}$. We assume that $c_{ii} = 0$, $i = 1,, ..., n$, so the above formula remains valid when i and/or j is a hub. The HNP is then:

$$min \sum_{i\in N} \sum_{j\in N} \sum_{k\in N} \sum_{m\in N} f_{ij} \, (c_{ik} + \alpha c_{km} + c_{mj}) x_{ijkm} \tag{1}$$

s.t.

$$\sum_{k\in N} z_{kk} = p \tag{2}$$

$$\sum_{k\in N} z_{ik} = 1, \quad i \in N \tag{3}$$

$$z_{ik} \leq z_{kk}, \quad i \in N, k \in N \tag{4}$$

$$\sum_{m \in N} x_{ijkm} = z_{ik}, \quad i \in N, j \in N, k \in N \tag{5}$$

$$\sum_{k \in N} x_{ijkm} = z_{jm}, \quad i \in N, j \in N, m \in N \tag{6}$$

$$z_{ik} \in \{0,1\}, \quad i \in N, k \in N \tag{7}$$

$$x_{ijkm} \geq 0, \quad i \in N, j \in N, k \in N, m \in N \tag{8}$$

The objective is to minimize the overall transportation cost (1) subject to the following constraints: there should be exactly p hubs (2), each node should be allocated to exactly one hub (3), a non-hub node i can be allocated to node k only if a hub is established at k (4), the entire flow from origin i to destination j will be routed via link (i,k) iff i is allocated to hub k independently of destination (5), the entire flow from origin i to destination j will be routed via (m,j) if and only if j is assigned to hub m independently of origin (6), hub assignment indicators are restricted to 0 or 1 (7), and flow variables x are non negative (8).

Next,we state the 4-dimensional formulation of the HLN problem from [10]. Let N be the set of nodes. The cost of sending the unit of flow is assigned to every link by cost matrix $D=(d_{km})$. Input matrix $F=(f_{ij})$ contains the required amounts of flow associated with every origin-destination pair (i,j). The decision variables are given as follows. Variable $x1^{ij}_{km}$ captures the fraction of flow that goes from node i to node j via link (k,m) which is not discounted. Variable $x2^{ij}_{km}$ is the fraction of flow from i to j (via (k,m)) that is discounted. Parameter α, $0 < \alpha < 1$ is the discount factor. Binary variable y_{km} is 1 if link (k,m) is discounted and 0 otherwise. The HLN problem is then:

$$\min \sum_{i,j,k,m,\, i \neq j, k \neq m, m \neq i, k \neq j} d_{km}(x1^{ij}_{km} + \alpha x2^{ij}_{km}) \tag{9}$$

s.t.

$$\sum_{i,j,\, i \neq j, i \neq m, j \neq k} x2^{ij}_{km} \geq y_{km} M, \qquad \text{for all } k,m : k \neq m \tag{10}$$

$$x2^{ij}_{km} \leq y_{km} f_{ij}, \qquad \text{for all } i,j,k,m : i \neq j, k \neq m, m \neq i, k \neq j \tag{11}$$

$$\sum_{m : m \neq i} (x1^{ij}_{im} + x2^{ij}_{im}) = f_{ij}, \qquad \text{for all } i,j : i \neq j \tag{12}$$

$$\sum_{k:k\neq l,k\neq j} (x1^{ij}_{kl} + x2^{ij}_{kl}) = \sum_{m:m\neq l,m\neq i} (x1^{ij}_{ml} + x2^{ij}_{ml}), \quad \text{for all } l,i,j\text{: } l \neq i,j,\ i \neq j \tag{13}$$

$$x1, x2 \geq 0,\ y - \text{binary}. \tag{14}$$

If the flow through a link (k, m) is discounted, then it must be greater than the threshold M (constraints (10)). In the case when the link (k, m) is not discounted, the variables $x2^{ij}_{km}$ are set to zero (constraint (11)). The complete flow f_{ij} must leave the origin i (constraint (12)). Conservation of flow follows from constraint (13). Non-negativity and integrality is assured by constraints (14). The complexity of this model is $(\mathcal{O}(n^4))$. The computational studies in [10] showed that the LP relaxation of this model is also very tight.

In order to analyze the cost allocation problem associated with the HLN problem, we need to introduce the following game theoretic definitions and notation. Let $P = \{1, 2, ..., n\}$ be a finite set of players, and let $c : 2^P \to \mathbb{R}$, with $c(\emptyset) = 0$, be a characteristic function defined over subsets of P referred to as coalitions. If c(P) designates a cost that has to be shared by all the players, then the pair $(P$;c) is called a (cost) cooperative game, or simply a game. For $x \in \mathbb{R}^{|P|}$ and $S \subseteq P$, let $x(S) \equiv \sum_{j\in S} x_j$. We can interpret $x(S)$ as the part of the total cost paid by the coalition S. A cost allocation vector x in a game $(P; c)$ satisfies $x(P) = c(P)$, and the solution theory of cooperative games is concerned with the selection of a reasonable subset of cost allocation vectors.

Central to the solution theory of cooperative games is the concept of solution referred to as the core of a game. The core of a game (P, c) consists of all vectors $x \in \mathbb{R}^{|P|}$ such that $x(S) \leq c(S)$ for all $S \subseteq P$, and $x(P) = c(P)$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede. In general, the core of a game may be empty.

3. Optimization results

Since optimal or best known solutions to the HNP and the HLN problems serve as an input to our cost allocation considerations, we point to several important related optimization results from the literature.

Note that the single allocation version of the HNP, as defined in $(1) - (8)$, was preceded by O'Kelly's [8] formulation as a quadratic integer programming problem. It received considerable attention by the research community during the last decade. Since modeling of the HNP leads to NP-hard problems, the researchers have naturally resorted to the development of heuristic solutions. It is important to mention the well known Civil Aeronautics Board (CAB) benchmark data-set (data collected on airline traffic between major US cities in early seventies) on which much of the computational testing from the literature was performed, thus enabling some comparison.

A number of heuristic algorithms to solve the HNP location problem have been proposed, including: complete evaluation of all locational patterns with respect to allocations based on distances, such as nearest hub allocation and allocation to one of the two nearest hubs [8]; exchange and clustering heuristics [6]; tabu search and GRASP strategies with distance based allocations [7]; tabu search method with allocations of non-hub nodes based jointly on distances as well as on flows between the nodes [19]. For an extensive survey of these methods, see [2].

For the CAB data set, the best known solutions for the single allocation HNP were obtained by the tabu search heuristic (TABUHUB) developed by Skorin-Kapov and Skorin-Kapov [19]. The quality of the above mentioned tabu search heuristic was further confirmed by obtaining good lower bounds for cases when distances satisfy the triangle inequality (O'Kelly et al., [9]).

Campbell [1] formulated the single and multiple allocation versions of the HNP as mixed 0/1 linear programs. However, integrality restrictions imposed on a subset of variables, coupled with a large size of formulations (for a network of size n, the number of variables is $O(n^4)$) restrict the suitability of those formulations to small instances. Since LP relaxations of Campbell's models resulted with highly fractional solutions, tighter LP relaxations were needed. Skorin-Kapov et al. [20] have proposed new mixed 0/1 linear formulations (presented herein $(1)-(8)$) whose linear programming relaxations often provide integral solutions. For the CAB data set, the LP relaxations proposed in their study resulted in almost all cases with integral solutions. Where this was not the case, the LP objective function value for the multiple (respectively, single) allocation case was less than 0.1% (respectively, 1%) below the optimal integer objective function value. Specifically, all considered instances of CAB data set were solved to optimality. Note that these problems are already large (the above LP relaxations for the case with 25 nodes and 4 hubs have 391,250 variables and 31,901 constraints). However, the results of these studies suggest that TABUHUB algorithm could be used with a reasonable confidence on larger problems, since for all considered cases of CAB data it achieved optimal solutions. Those results should be used as an input to our hub network cost allocation games considered in *Section 4*.

As mentioned in the introduction the HNP design has many advantages but also some deficiencies. In particular, the prescribed discount for the cost of traffic between hubs might be applied to relatively low traffic volume and some high volume between non-hub node and hub might not get deserved discount. Podnar et al. in [10] addressed these issues by introducing networks with threshold based discounting. Therein the discount is warranted to any high volume flow, regardless of the link through which it goes. There are no hubs in those networks. We call them hub-like networks, since in those networks the amalgamation of flows tends to resemble those in hub networks.

In [10], the authors presented several heuristic algorithms to solve the HLN problem. Moreover, they performed extensive computational experiments on the CAB (Civil Aeronautics Board) benchmark data set. The efficiency of their formulations and heuristic was demonstrated by obtaining the gaps between the upper and the lower bound within few percent. Those solutions will be used as the input to hub-like network cost allocation games presented in *Section 5*.

4. Cost allocation games

In this section, we will use a game theoretic approach to describe the cost allocation problem in hub and hub-like networks. Specifically, we will formulate corresponding cooperative cost allocation network games that capture the user's contribution to economies of scale discounting. We will refer to those games as hub and hub-like network games.

4.1. Hub network games

The total cost of delivering the service through hub network is obtained from the optimal (or best known) objective function value to the HNP ((1) − (8)). The objective is to fairly allocate this cost among hub network users. This problem was first formulated as the cooperative game by D. Skorin-Kapov in [15].

There are several concerns in deriving the definition of the cost allocation game. The first dilemma is how to define the set of players for the hub game. One natural choice is to identify the set of players P with the set of nodes (users) N, namely, $P = N$. Since we need to satisfy the flow requirements for all pairs of users (nodes), the other natural choice for the set of players is the set of all node pairs, namely $P = N * N$.

In case when $P = N$, the characteristic function is $c : 2^P \to \mathbb{R}$ and the value of $c(P)$ is the entire cost of delivering service through the hub network, which can be obtained from the best known objective function value of the HNP. The value of the characteristic function for each subset of players $S, S \subseteq P$, $c(S)$ should describe the cost associated with the delivery of service to S. This could, for example, represent the cost of communication between users in S only, or $c(S)$ could represent the cost of all the communication generated by users in S (*i.e.* between users in S and users in N).

On the other hand, if players are pairs of users, namely $P = N * N$ and $T \subseteq P$, $c(T)$ should be the cost of sending traffic between pairs of users in T. The assumption here is that the cost allocated to each pair would be later equally divided between nodes in the pair.

Moreover, in both cases (i.e. $P = N$ and $P = N * N$) there is a question how should the above cost $c(S)$, for $S \subset P$, be determined. It could represent the cost of delivering service to S in the globally optimal network associated with the optimal solution to the HNP, or from some network that subset of players S could potentially construct in order to optimally provide the service only to players in S.

There are advantages, as well as drawbacks, of these approaches. If we set players to be nodes (users) (i.e. $P = N$), and the value of $c(s)$ for the coalition S is representing the cost of traffic between nodes in S only, we ignore the demand for communication of players in S with players out of S. It is easy to see that such characteristic function would be too restrictive. For example, core constraints for (N, c) would imply that all single player coalitions should not pay anything (for each $i \in N, 0 = c(\{i\}) \geq x(\{i\})$.

If we change the approach and for $S \subset N$, assume that the value of the characteristic function $c(S)$ represents the cost of the entire communication initiated by players in S, we would face a different problem. Namely, the characteristic function would attribute the entire cost of traffic from users in S to users out of S, to coalition S. This may lead to unfairness if the flow matrix is not symmetric.

The above problems arising from the fact that each coalition communicates with users who do not belong to that coalition can be taken care of by a different choice of players. Namely, we could define players as pairs of users, i.e. $P = N * N$.

Finally, for $T \subset P = N * N$, we are concerned with the interpretation of the value of the characteristic function $c(T)$. It should be the cost of providing service to pairs in T. We might assume that the coalition T, even when acting on its own, would use the globally optimal network, or we might assume that $c(T)$ could represent the cost of service obtained if coalition T re-optimize and create their own subnetwork.

If we assume that the coalition T, if acting on its own, would construct another network which would serve T optimally, the computation of cost allocation solution concepts would become very prohibitive. In particular, in order to compute the characteristic function value for each coalition, we would have to solve an NP-hard problem.

In view of these considerations we choose players to be the pairs of users, i.e. $P = N * N$, and we assume that for $T \subset P$, $c(T)$ represent the cost of providing the service to pairs in T in the globally optimal network. Note however that in HNP the number of hubs is predetermined and we expect heavy traffic between hubs. However, if coalition T acts on its own, the generated traffic might be too small to justify the discount applied to inter hub communication. Consequently, the above characteristic functions of the hub game might be too generous to certain coalitions.

We are now ready to define the hub game that takes into account all the above considerations, as well as the coalition's contribution to economies of scale. Let x_{ijkm} be an optimal solution to $(1) - (8)$. Then, for each $T \subseteq P = N * N$, $f_{km}^{T} = \sum_{i \in T} \sum_{j \in N} f_{ij} x_{ijkm}$ is the total amount of flow generated by users in T, which is routed via hubs k, m. We assume that the cost of traffic between two hubs can be discounted by α if its amount exceeds certain threshold M. The indicator y_{km}^{T} whether the amount of traffic between hubs k and m generated by subset T is above M can be defined as follows: $y_{km}^{T} = 1$ if $f_{km}^{T} \geq M$ and zero otherwise. Next we define the characteristic function of the hub game $(N * N$,c) that allows the discounts only in case of heavy corresponding inter hub traffic. For each $T \subseteq N * N$,

$c(T) = \sum_{(i,j) \in T} \sum_{k \in N} \sum_{m \in N} f_{ij}(c_{ik} + \alpha y_{km}^{T} c_{km} + (1 - y_{km}^{T}) c_{km} + c_{mj}) x_{ijkm}$,
where x_{ijkm} is an optimal solution to $(1) - (8)$.

4.2. Hub-like network games

The total cost of delivering the service through a hub-like network is obtained from the best known objective function value to the HLN problem ((9) − 14)). The objective is to allocate this cost among network users in a 'fair' manner. In order to define the hub-like network game, we need to define the players and the characteristic function on the set of all coalitions (all subsets of players).

When choosing a set of players, we face similar considerations as in the case of hub games. Namely, if nodes are players, i.e. $P = N$, for each subset of players

S, the value of the characteristic function $c'(S)$ could represent the cost of providing service to S. If $c'(S)$ captures only the cost of traffic between nodes in S and ignores the demand for communication of players in S with players out of S, the definition would be too restrictive and similarly to the case in hub games it would lead to an empty core. On the other hand, if we attempt to capture with $c'(S)$ the cost of all traffic involving users in S, then the characteristic function would attribute the entire cost of traffic between users in S and users in N to coalition S. This would again lead to unfairness if the flow matrix is not symmetric.

Since we need to satisfy the flow requirements for all pairs of users (nodes), we choose for a set of players the set of all node pairs, i.e. $P = N * N$. Consider now the definition of the characteristic function $c' : 2^P \to \mathbb{R}$. If $P = N * N$, the value of $c'(P)$ is simply the entire cost of delivering service through the hub-like network and it can be obtained from the best known objective function value of the HLN problem. The dilemma is how to define the value of the characteristic function for each subset of players $Q \subseteq P = N * N$. (The detailed discussion of different versions of characteristic functions for this problem is provided in [16].) One approach could be to define characteristic function c', such that $c'(Q)$ reflects the cost of providing service to Q in the globally best known HLN. In this case, it is easy to compute the characteristic function value for each coalition Q. The drawback of this approach is that a coalition Q might benefit by getting the discount on the cost of flow through a certain link (k, m), even if Q by itself does not generate sufficient amount of traffic to warrant the discount on (k, m). This could make the characteristic function c' too generous to certain coalitions.

Finally, we informally introduce the characteristic function c', which assigns the value to each coalition that is based on the globally optimal solution and which takes into account the coalition's contribution to economies of scale. For $\emptyset \neq Q \subseteq N * N$, $c'(Q)$ is the cost of satisfying the communication requirements for all pairs of users in Q. The value $c'(Q)$ is essentially the sum of the cost of flow generated by Q through links in the optimal (best known) network. Note however, that special attention is paid to economies of scale. Specifically, the important notion here is that the cost of flow generated by Q through a particular link is discounted, if and only if, the amount of this flow exceeds the threshold M. It appears that the characteristic function c' captures the economic side of the problem by taking into account the coalition's contribution to economies of scale. Moreover, the characteristic function c' is computationally promising. Consequently, we will define the hub-like game as a pair $(N * N, c')$.

Next we will work on the formal definition of the hub-like network game. Recall that our objective is to allocate the cost of service provided by hub-like network in a computationally practical manner. To accomplish this goal we will employ a divide and conquer strategy. Specifically, in the next section, we will demonstrate that the cost allocation problem of the entire network can be decomposed into cost allocation problems associated with specific links. For that purpose, for each link (k, m), we will first define the so-called link game $(N * N, c_{km})$, which will deal with the fair allocation of the cost of the flow that goes through the link (k, m). The hub-like game will then be defined as the composition of link games.

We now define the characteristic functions c_{km} and c'. For an empty set, $c_{km}(\emptyset) =$

0 and $c'(\emptyset) = 0$. For $\emptyset \neq Q \subseteq N * N$, let $c_{km}(Q)$ be the cost of flow that players in coalition Q send through link (k, m) in the optimal hub like network. We assume here that the coalition Q uses the same links as in an optimal solution to the HLN problem. The difference is that the cost of the above flow through a particular link (k, m) is discounted if and only if Q itself generates enough traffic for the discount (i.e. the amount of flow through link (k, m) generated by the coalition Q exceeds the threshold M).

We define c_{km} as follows. For $\emptyset \neq Q \subseteq N * N$,

$$c_{km}(Q) = d_{km}(x1(Q)^{ij}_{km} + \alpha x2(Q)^{ij}_{km}),$$

where

$$x1(Q)^{ij}_{km} = \begin{cases} \sum_{(i,j)\in Q} x1^{ij}_{km} + x2^{ij}_{km}, & \text{if } \sum_{(i,j)\in Q} x2^{ij}_{km} < M \\ & \text{and } x1^{ij}_{km}, x2^{ij}_{km} \text{ is optimal to (9)-(14)} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$x2(Q)^{ij}_{km} = \begin{cases} \sum_{(i,j)\in Q} x2^{ij}_{km}, & \text{if } \sum_{(i,j)\in Q} x2^{ij}_{km} \geq M \\ & \text{and } x2^{ij}_{km} \text{ is optimal to (9)-(14)} \\ 0, & \text{otherwise.} \end{cases}$$

The pair $(N * N, c_{km})$ is a link game. Further, for each $\emptyset \neq Q \subseteq N * N$, we define $c'(Q) = \sum_{(k,m)} c_{km}(Q)$. The hub-like network game is a pair $(N * N, c')$.

5. Cost allocation solutions

Next we will analyze the core of hub network and hub-like network games. Recall that the core consists of all cost allocation vectors that provide no incentive for any coalition to secede. Specifically, each cost allocation vector in the core allocates to each coalition of players at most the cost needed to provide service to that coalition. Namely, there is no cross-subsidization. In this Section, we will show that the core of the hub and hub-like network games can be efficiently characterized.

5.1. Core of the hub network game

The core of the hub network game $(N * N, c)$ was first analyzed by Skorin-Kapov [15]. Therein, it was proven that the core of $(N * N, c)$ is not empty. Moreover, it was demonstrated that most of core constraints are redundant, and that the core of the hub game $(N * N, c)$ can be characterized with the polynomial number of constraints. Herein, we summarize some results from [15].

Lemma 1. *For any two hubs k and m, let $T_{km} \subseteq N*N$ consist of all node pairs (i,j) whose traffic is routed via hubs k,m. Then for each x in the core of the game (N∗N,c), we have $c(T_{km}) = x(T_{km})$.*

Corollary 1. *The game (N∗N,c) is decomposable and its core is a Cartesian product of all the cores of games (T_{km}, c), where k,m are pairs of hubs. Namely, in* *order to characterize the core of the game* $(N{*}N{,}c)$ *it is sufficient to characterize, for all pairs of hubs k,m, the cores of games* (T_{km},c).

In view of *Corollary 1* it is sufficient to consider the characterization of the core of the game $(N * N, c)$ for the 2-hub location problems. Further, define the excess for a coalition T relative to the cost allocation x as the quantity $e(x, T) = c(T) - x(T)$. The excess could be interpreted as the level of satisfaction of a coalition T with the cost allocation x. The following Lemma shows that the excess is monotonically decreasing for coalitions that produce enough traffic to enable the discount on inter hub traffic.

Lemma 2. *Let* $T{\subseteq}N{*}N$, *be a coalition which generates enough traffic to get the discount* α *on the inter hub traffic, and let x be the cost allocation which satisfies the core constraints associated with coalitions* $N{*}N\backslash(i,j)$,*for all* $(i,j) \in N{*}N\backslash T$. *Then, for each* $(i,j) \in N{*}N\backslash T$:

$$c(T) - x(T) \geq c(T \cup \{(i, j)\}) - x(T \cup \{(i, j)\}).$$

Now, let the collection $\mathcal{S}_1 = \{\{(i, j)\} \,|\, (i, j) \in N * N \text{ and } f_{km}^{\{(i,j)\}} < M\}$ consist of all single player coalitions which themselves do not generate sufficient amount of traffic between hubs k and m to warrant the discount α, and let $\mathcal{S}_2 = \{N * N\backslash(i, j) \,|\, (i, j) \in N * N\}$ be the collection of all coalitions that are missing only a single player. Then the following theorem holds.

Theorem 1. *The core constraints associated with coalitions in collections* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *completely determine the core of the game* $(N{*}N,\ c)$.

Theorem 1 implies that most of the core constraints for the hub game are redundant and the core of the hub network game can be efficiently characterized. Indeed, assume that we use the optimal or the best known solution to the HNP as the input to our cost allocation problem. Then, in order to characterize the core of 2-hub game, we only need to generate $2n^2$ linear constraints from *Theorem 1.* Then, *Corollary 1* implies that in order to characterize the core of a p-hub network game (for a fixed number of hubs p), we need to generate only $\mathcal{O}(n^2)$ constraints.

5.2. Core of the hub-like network game

The core of the hub-like network game was first analyzed by Skorin-Kapov in [16] and [17]. It was shown therein that the core of that game is not empty and that some points of its points could be found efficiently. Next, we summarize some of the results presented in [16].

We first demonstrate that the hub-like network game can be decomposed into link games. Namely, in order to analyze the core of the hub-like network game it is sufficient to consider all corresponding link games.

Theorem 2. *For each link* (k, m), *let* x_{km} *be a point in the core* $\mathcal{C}(N * N, c_{km})$ *of the link game* $(N * N, c_{km})$. *Then,* $x = \sum_{(k,m)} x_{km}$ *is in the core* $\mathcal{C}(N * N, c')$ *of the hub-like game* $(N * N, c')$.

Let x be a cost allocation in the hub-like game $(N * N, c')$. *Define allocations for the related link games,* $(N * N, c_{km})$, *as follows. For each link* (k, m) *and a player* $(i, j) \in Q \subseteq N * N$, *let the portion of the cost covered by the player* (i, j) *be*

$x_{km}(i,j) = x(i,j)\frac{c_{km}(i,j)}{c'(i,j)}$. *If x is in the core $\mathcal{C}(N*N, c')$ of the hub-like game then for each link (k,m), x_{km} is in the core $\mathcal{C}(N{*}N, c_{km})$ of the related link game.*

Corollary 2. *In order to characterize the core $\mathcal{C}(N{*}N, c')$ of the hub $-$ like game $(N{*}N, c')$, it is sufficient to characterize the cores $\mathcal{C}(N{*}N, c_{km})$ of link games $(N{*}N, c_{km})$, where (k,m) are links in the optimal hub-like network.*

Next we provide an efficient characterization of the core of the hub-like game. Recall that the main computational difficulty is that the core is determined with an exponential number of constraints. However, we will show that most of the core constraints for the core of each link game are redundant.

Lemma 3. *Consider a link game $(N{*}N, c_{km})$. Let $Q \subseteq N{*}N$, be a coalition which generates enough traffic to warrant the discount α on the traffic through link (k,m), and let x_{km} be the cost allocation which satisfies the core constraints associated with coalitions $N*N\backslash\{i,j\}$, for all $(i,j) \in N*N\backslash Q$. Then, for each $(i,j) \in N*N\backslash Q$:*

$$c_{km}(Q) - x_{km}(Q) \geq c_{km}(Q \cup \{(i,j)\}) - x_{km}(Q \cup \{(i,j)\}).$$

Let $f_{km}^{\{(i,j)\}}$ be the amount of flow that the user (i,j) generates through a link (k,m) in the optimal solution to the HLN problem. For each link (k,m), let the collection $\mathcal{S}_1^{km} = \{\{(i,j)\} \mid (i,j) \in N*N \text{ and } f_{km}^{\{(i,j)\}} < M\}$ consist of all single player coalitions which themselves do not generate sufficient amount of traffic through the link (k,m) to warrant the discount α, and let $\mathcal{S}_2 = \{N*N\backslash\{(i,j)\} \mid (i,j) \in N*N\}$ be the collection of all coalitions that are missing only a single player. The core can now be characterized as follows.

Theorem 3. *The core constraints associated with coalitions in collections $\mathcal{S}_1^{km}$ and $\mathcal{S}_2$ completely determine the core of the link game $(N{*}N, c_{km})$.*

Consequently, *Corollary 1* and *Theorem 3* imply that the core of the hub-like network game is completely determined with the constraints associated with collections $\mathcal{S}_1 = \{\mathcal{S}_1^{km} \mid (k,m) \text{ is a link in an optimal hub-like network}\}$, and the collection $\mathcal{S}_2$.

6. Conclusions

In this paper, we surveyed some recent results on the cost allocation problem associated with the hub and hub-like network design problem. The objective of these results was to allocate the network cost in a fair manner using cooperative game theory approach. Here, by fair we mean that each subset of users should be charged a share of the network cost which provide them no incentive to seek services from another network. There are no definite answers to issue of fairness, but theory of cooperative games proposes several solution concepts. To that end we presented cooperative games referred to as the hub and hub-like network game, respectively. Our models emphasize the users' contribution to economies of scale and utilize the so-called threshold based discounts.

We considered various choices for the set of players. We opted to identify the players with the set of pairs of users. In defining the costs for each coalition (characteristic function values), the important issue was whether the cost of traffic for a particular coalition should be taken from the globally optimal network, or from some potential network that is optimal from the point of view of that particular coalition. Games in which each coalition (subset of users) assumes the potential use of another network (optimal from their point of view) seem to be computationally prohibitive for hub and hub-like networks. Note however, that the network optimal from the point of view of a particular coalition still may involve other users out of that coalition who would not necessarily support such solution. We conclude that the most promising practical approach is in using pairs of nodes as players, and determining the characteristic function values by the use of globally optimal (best known) optimization solution.

It was confirmed in the literature that the core points associated with the hub and hub-like network games exist. Moreover, we also showed that they can be efficiently computed. Specifically, we demonstrated that the HNP network game can be decomposed into 2-hub games and that the hub-like game can be decomposed into link games. Using the above decomposition it can be shown that the core of the hub network game for a fixed number of hubs can be characterized with $\mathcal{O}(n^2)$ constraints while the core of the hub-like game can be characterized with $\mathcal{O}(n^4)$ constraints. Moreover, the above constraints are associated only with single member coalitions and grand coalitions that are missing only one player. Due to this special property, those constraints are very easy to generate.

In summary, we developed a framework for the efficient computation of some hub and hub-like network cost allocation solutions (the core) in which users are charged a 'fair' share with respect to their contribution to economies of scale. Fairness is in the sense that there is no cross-subsidization. Namely, each coalition of users is not paying more than they would pay to provide their own service. The most useful part for the cost allocation practice is our scheme to efficiently generate 'attractive' cost allocations which give the users incentive to cooperate and exploit economies of scale via participation in hub and hub-like networks.

References

[1] J. F. Campbell, *Integer programming formulations of discrete hub location problems*, European Journal of Operational Research **72**(1994), 387–405.

[2] J. F. Campbell, *A survey of network hub location*, Studies in locational analysis **6**(1994), 31–49.

[3] D. Granot, F. Granot, *On some network flow games*, Mathematics of Operations Research **17**(1992), 792–841.

[4] D. Granot, M. Hojati, *On the cost allocation in telecommunication networks*, Networks **20**(1990), 209–229.

[5] D. Granot, G. Huberman, *Minimum cost spanning tree games*, Math. Programming **21**(1981), 1–18.

[6] J. G. Klincewicz, *Heuristics for the p-hub location problem*, European Journal of Operational Research, **53**(11991),25–37.

[7] J. G. Klincewicz, *Avoiding local optima in the p-hub location problem using tabu search and GRASP*, Annals of Operations Research **40**(1992), 283–302.

[8] M. E. O'Kelly, *A quadratic integer program for the location of interacting Hub Facilities*, European Journal of Operational Research **32**(1987), 393–404.

[9] M. E. O'Kelly, D. Skorin-Kapov, J. Skorin-Kapov, *Lower bounds for the hub location problem*, Management Science, **41**(1995), 713–728.

[10] H. Podnar, J. Skorin-Kapov, D. Skorin-Kapov, *Network cost optimization using threshold based discounting*, European Journal of Operations Research **137**(2002), 371–386.

[11] J. A. M. Potters, I. J Curiel, S. H. Tijs, *Traveling salesman games*, Mathematical Programming **53**(1987), 199–211.

[12] W. W. Sharkey, *Networks Models in Economics*, in: *Handbooks in OR&MS*, Vol. 8, Chapter 9, (M. O. Ball et al., Eds.), Elsevier, 1995.

[13] D. Skorin-Kapov, *On a cost allocation problem arising from a capacitated concentrator covering problem*, Operations Research Letters **13**(1993), 315–323.

[14] D. Skorin-Kapov, *On the core of the minimum Steiner tree game in networks*, Annals of Operations Research **57**(1995), 233–249.

[15] D. Skorin-Kapov, *Hub network games*, Networks **31**(1998), 293–302.

[16] D. Skorin-Kapov, *On cost allocation in hub-like networks*, Annals of Operations Research **106**(2001), 63–78.

[17] D. Skorin-Kapov, *On cost allocation in networks with threshold based discounting*, Journal of Computing and Information Technology **8**(2000) 311–317.

[18] D. Skorin-Kapov, H. F. Beltran, *An efficient characterization of cost allocation solutions associated with capacitated network design problems*, Telecommunication Systems **3**(1994), 91–107.

[19] D. Skorin-Kapov, J. Skorin-Kapov, *On tabu search for the location of interacting hub facilities*, European Journal of Operational Research **73**(1994),502–509.

[20] D. Skorin-Kapov, J. Skorin-Kapov, M. O'Kelly, *Tight linear programming relaxations of uncapacitated p-hub median problems*, European Journal of Operational Research **94**(1996), 582–593.