On some subspaces of an FK-space

İ. Dağadur*

Abstract. *In this paper we study the subspaces C_1S, C_1W, C_1F and C_1B for a locally convex FK-space X containing ϕ, the space of finite sequences.*

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1. Introduction and notation

Let w denote the space of all complex-valued sequences. An FK-space is a locally convex vector subspace of w which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK-space is a normed FK-space. The basic properties of FK-spaces may be found in [7], [8] and [10]. We now define the Cesáro summability matrix which is used throughout this paper: The Cesáro mean is given by the matrix C_1 whose $nkth$ entry is

$$C_1[n,k] = \begin{cases} \frac{1}{n+1}, \text{ if } 0 \leq k \leq n \\ 0, \text{ if } k > n. \end{cases}$$

The sequence spaces

$$\sigma_0 = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{j=1}^{n} x_j = 0 \right\},$$

$$\sigma b = \left\{ x \in w : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

*Gaziosmanpaşa University, Faculty of Arts and Science, Department of Mathematics, 60100, Tokat, Turkey, e-mail: ilhandagadur@yahoo.com

and

$$\sigma s=\left\{x\in w:\lim_n\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j \text{ exists}\right\}$$

are BK-spaces with the norm

$$\|x\|_{\sigma_0}=\sup_n\frac{1}{n}\left|\sum_{k=1}^{n}x_k\right|$$

and

$$\|x\|_{\sigma s}=\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|$$

respectively ([1] , [2] and [9]).

Throughout the paper δ^j, $(j=1,2,...)$, the sequence $(0,0,...,0,1,0,...)$ with the one in the j-th position; ϕ the linear span of the δ^j 's. The topological dual of X is denoted by X'. A sequence x in a locally convex sequence space X is said to have the property AK (respectively σK) if $x^{(n)}\to x$ (respectively $\frac{1}{n}\sum\limits_{k=1}^{n}x^{(k)}\to x$) in X where $x^{(n)}=(x_1,x_2,...,x_n,0,...)=\sum\limits_{k=1}^{n}x_k\delta^k$. It is known that if an FK-space $\phi\subset X$ is said to have σB if $\left\{\frac{1}{n}\sum\limits_{k=1}^{n}x^{(k)}\right\}$ is a bounded set in X for each $x\in X$. Also, an FK- space X is said to have $F\sigma K$ (functional σK) if $X\subset C_1F^+$ i.e., $X=C_1F$ ([1] , [2] and [4]).

We recall (see [3] and [4]) that the $f,\sigma-$ and $\sigma b-$ duals of a subset X of w are defined to be

$$X^f=\left\{\left\{f\left(\delta^k\right)\right\}:f\in X'\right\},$$

$$\begin{aligned} X^{\sigma} &= \left\{x\in w:\lim_n\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_jy_j \text{ exists for all } y\in X\right\}\\ &= \{x\in w: x.y\in\sigma s \text{ for all } y\in X\}, \end{aligned}$$

$$\begin{aligned} X^{\sigma b} &= \left\{x\in w:\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|<\infty \text{ for all } y\in X\right\}\\ &= \{x\in w: x.y\in\sigma b \text{ for all } y\in X\}, \end{aligned}$$

respectively, where $x.y=(x_ny_n)$.

2. Some subspaces of X

Following [4] we recall some important subspaces of a locally convex FK-space X containing ϕ.

Definition1. *Let X be an FK-space$\supset \phi$. Then*

$$W := W(X) = \left\{x \in X : x^{(k)} \to x(\textit{weakly}) \textit{ in } X\right\}$$

$$\begin{aligned}
C_1W := \quad C_1W(X) \quad &= \left\{x \in X : \tfrac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x(\textit{weakly}) \textit{ in } X\right\} \\
&= \left\{x \in X : f(x) = \lim_n \tfrac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f\left(\delta^j\right) \textit{ for all } f \in X'\right\} \\
&= \left\{x \in X : x \textit{ has } S\sigma K \textit{ in } X\right\}, \\
C_1S := \quad C_1S(X) \quad &= \left\{x \in X : \tfrac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x\right\} \\
&= \left\{x \in X : x \textit{ has } \sigma K \textit{ in } X\right\} \\
&= \left\{x \in X : x = \lim_n \tfrac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j \delta^j\right\}, \\
C_1F^+ := \quad C_1F^+(X) \quad &= \left\{x \in w : \left(\tfrac{1}{n}\sum_{k=1}^{n} x^{(k)}\right) \text{ is } \textit{weakly Cauchy in } X\right\} \\
&= \left\{x \in w : \left(x_n f\left(\delta^n\right)\right) \in \sigma s \textit{ for all } f \in X'\right\}, \\
C_1B^+ := \quad C_1B^+(X) \quad &= \left\{x \in w : \left(\tfrac{1}{n}\sum_{k=1}^{n} x^{(k)}\right) \text{ is } \textit{bounded in } X\right\} \\
&= \left\{x \in w : \left(x_n f\left(\delta^n\right)\right) \in \sigma b \textit{ for all } f \in X'\right\},
\end{aligned}$$

also

$$C_1F := \quad C_1F^+ \cap X \qquad \textit{and} \quad C_1B := C_1B^+ \cap X.$$

We note that subspaces W and C_1W are closely related to conullity and Cesáro conullity of the $FK-$space X (see [5] and [6]).

We now study some inclusions which are analogous to those given in [8; *Chapter* 10].

Theorem 2. *Let X be an FK-space$\supset \phi$. Then*

$$\phi \subset C_1S \subset C_1W \subset C_1F \subset C_1B \subset X \textit{ and } \phi \subset C_1S \subset C_1W \subset \overline{\phi}.$$

Proof. The only non-trivial part is $C_1W \subset \overline{\phi}$. Let $f \in X'$ and $f = 0$ on ϕ. The definition of C_1W shows that $f = 0$ on C_1W. Hence, the Hahn-Banach theorem gives the result. □

Theorem 3. *The subspaces $E = C_1S$, C_1W, C_1F, C_1F^+, C_1B, and C_1B^+ of X FK-space are monotone i.e., if $X \subset Y$ then $E(X) \subset E(Y)$.*

Proof. The inclusion map $i : X \to Y$ is continuous by Corollary 4.2.4 of [8], so $\frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x$ in X implies the same in Y. This proves the assertion for C_1S. For C_1W it follows from the fact that i is weakly continuous by (4.0.11) of [8]. Now

$z \in C_1F^+, C_1B^+$ if and only if $(z_n f(\delta^n)) \in \sigma s, \sigma b$ respectively for all $f \in X'$, hence for all $g \in Y'$ since $g|X \in X'$ by Corollary 4.2.4 of [8]. The result follows for C_1F^+, C_1B^+ and so for C_1F, C_1B. □

Since σ_0 is an $AK-$space, we immediately get the following

Theorem 4. *Let X be an FK-space$\supset \sigma_0$. Then $\sigma_0 \subset C_1S \subset C_1W$.*

Theorem 5. *Let X be an FK-space$\supset \phi$. Then $C_1B^+ = X^{f\sigma b}$.*

Proof. By *Definition 1*, $z \in C_1B^+$ if and only if $z.u \in \sigma b$ for each $u \in X^f$. This is precisely the assertion. □

Theorem 6. *Let X be an FK-space$\supset \phi$. Then C_1B^+ is the same for all FK-spaces Y between $\overline{\phi}$ and X; i.e., $\overline{\phi} \subset Y \subset X$ implies $C_1B^+(Y) = C_1B^+(X)$. Here the closure of ϕ is calculated in X.*

Proof. By *Theorem 3* we have $C_1B^+(\overline{\phi}) \subset C_1B^+(Y) \subset C_1B^+(X)$. By *Theorem 5* and by (7.2.4) of [8] the first and the last are equal. □

Theorem 7. *Let X be an FK-space such that $C_1B \supset \overline{\phi}$. Then $\overline{\phi}$ has σK and $C_1S = C_1W = \overline{\phi}$.*

Proof. Suppose first that X has σB. Define $f_n : X \to X$ by

$$f_n(x) = x - \frac{1}{n}\sum_{k=1}^{n} x^{(k)}.$$

Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by (7.0.2) of [8]. Since $f_n \to 0$ on ϕ then also $f_n \to 0$ on $\overline{\phi}$ by (7.0.3) of [8]. This is the desired conclusion. □

Theorem 8. *Let X be an FK-space$\supset \phi$. Then $C_1F^+ = X^{f\sigma}$.*

Proof. This may be proved as in *Theorem 5*, with σs instead of σb. □

Theorem 9. *Let X be an FK-space$\supset \phi$. Then C_1F^+ is the same for all FK-spaces Y between $\overline{\phi}$ and X; i.e., $\overline{\phi} \subset Y \subset X$ implies $C_1F^+(Y) = C_1F^+(X)$. (The closure of ϕ is calculated in X).*
The proof is similar to that of *Theorem 6*.

Lemma 10. *Let X be an FK-space in which $\overline{\phi}$* has *σK. Then $C_1F^+ = (\overline{\phi})^{\sigma\sigma}$.*

Proof. Observe that $C_1F^+ = X^{f\sigma}$ by *Theorem 8*. Since $X^f = (\overline{\phi})^f$ by Theorem 7.2.4 of [8], we have $X^{f\sigma} = (\overline{\phi})^{f\sigma}$. Hence, by Theorem 1.9 of [4] the result follows. □

Theorem 11. *Let X be an FK-space$\supset \phi$. Then X has $F\sigma K$ if and only if $\overline{\phi}$ has σK and $X \subset (\overline{\phi})^{\sigma\sigma}$.*

Proof. *Necessity.* X has σB since $C_1F \subset C_1B$ so $\overline{\phi}$ has σK by *Theorem 7*. The remainder of the proof follows from *Lemma 10*. Sufficiency is given by *Lemma 10*. □

Theorem 12. *Let X be an FK-space$\supset \phi$. The following are equivalent:*
(i) X has $F\sigma K$,

(*ii*) $X \subset C_1 S^{\sigma\sigma}$,
(*iii*) $X \subset C_1 W^{\sigma\sigma}$,
(*iv*) $X \subset C_1 F^{\sigma\sigma}$,
(*v*) $X^{\sigma} = C_1 S^{\sigma}$,
(*vi*) $X^{\sigma} = C_1 F^{\sigma}$.

Proof. Observe that (*ii*) implies (*iii*) and (*iii*) implies (*iv*) and that they are trivial since

$$C_1 S \subset C_1 W \subset C_1 F.$$

If (*iv*) is true, then $X^f \subset C_1 F^{\sigma} = X^{f\sigma\sigma} \subset X^{\sigma}$ so (*i*) is true by Theorem 1.9 of [4]. If (*i*) holds, then *Theorem 11* implies that $\overline{\phi} = C_1 S$ and that (*ii*) holds. The equivalence of (*v*), (*vi*) with the others is clear. □

Theorem 13. *Let X be an FK-space* $\supset \phi$. *The following are equivalent:*
(*i*) *X has* $S\sigma K$,
(*ii*) *X has* σK,
(*iii*) $X^{\sigma} = X'$.

Proof. Clearly (*ii*) implies (*i*). Conversely if X has $S\sigma K$ it must have AD for $C_1 W \subset \overline{\phi}$ by *Theorem 2*. It also has σB since $C_1 W \subset C_1 B$. Thus X has σK by *Theorem 7*, this proves that (*i*) and (*ii*) are equivalent. Assume that (*iii*) holds. Let $f \in X'$, then there exists $u \in X^{\sigma}$ such that

$$f(x) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Since $f\left(\delta^j\right) = u_j$, it follows that each $x \in C_1 W$ which shows that (*iii*) implies (*i*). That (*ii*) implies (*iii*) is known (see [2], page 97). □

Theorem 14. *Let X be an FK-space* $\supset \phi$. *The following are equivalent:*
(*i*) $C_1 W$ *is closed in X*,
(*ii*) $\overline{\phi} \subset C_1 B$,
(*iii*) $\overline{\phi} \subset C_1 F$,
(*iv*) $\overline{\phi} = C_1 W$,
(*v*) $\overline{\phi} = C_1 S$,
(*vi*) $C_1 S$ *is closed in X*.

Proof. (*ii*) implies (*v*): By *Theorem 7*, $\overline{\phi}$ has σK, *i.e.* $\overline{\phi} \subset C_1 S$. The opposite inclusion is *Theorem 2*. Note that (*v*) implies (*iv*), (*iv*) implies (*iii*) and (*iii*) implies (*ii*) because

$$C_1 S \subset C_1 W \subset \overline{\phi},\ C_1 W \subset C_1 F \subset C_1 B;$$

(*i*) implies (*iv*) and (*vi*) implies (*v*) since $\phi \subset C_1 S \subset C_1 W \subset \overline{\phi}$. Finally (*iv*) implies (*i*) and (*v*) implies (*vi*). □

References

[1] M. BUNTINAS, *Convergent and bounded Cesáro sections in FK-spaces*, Math. Z. **121**(1971), 191–200.

[2] G. GOES, S. GOES, *Sequences of bounded variation and sequences of Fourier coefficients, I*, Math. Z. **118**(1970), 93–102.

[3] G. GOES, *Sequences of bounded variation and sequences of Fourier coefficients, II*, J. Math. Anal. Appl. **39**(1972), 477–494.

[4] G. GOES, *Summan von FK-Räumen funktionale Abschnıttskonvergenz und Umkehrsatz*, Tôhoku. Math. Journ. **26**(1974), 487–504.

[5] H. G. İNCE, *Cesáro conull FK-spaces*, Demonstratio Math. **33**(2000), 109–121.

[6] C. ORHAN, *Conull absolute summability domains*, Indian J. Pure & Applied Math. **24**(1993), 539–542.

[7] A. WILANSKY, *Functional Analysis*, Blaisdell Press, 1964.

[8] A. WILANSKY, *Summability Through Functional Analysis*, North Holland, 1984.

[9] K. ZELLER, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53**(1951), 463–487.

[10] K. ZELLER, *Theorie der Limitierungsverfahren*, Berlin-Heidelberg- New York, 1958.