The butterfly theorem for conics

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Abstract. *The butterfly theorem holds for any diameter of any conic.*

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In a series of papers (cf. [1]-[11]) the well-known butterfly theorem for circles was proved and generalized. In [12] a generalization for conics was developed. Here we shall show the following theorem.

Theorem 1. *Let A, B, C, D be four points on a conic $\mathcal{K}$ and $\mathcal{M}$ any straight line in the same plane. Let $\mathcal{N}$ be the diameter of $\mathcal{K}$ which is conjugate to the line $\mathcal{M}$ and let $M = \mathcal{M} \cap \mathcal{N}$. If M is the midpoint of two points $E = \mathcal{M} \cap AB$ and $F = \mathcal{M} \cap CD$, then M is the midpoint of the points $G = \mathcal{M} \cap AC$ and $H = \mathcal{M} \cap BD$ and the midpoint of the points $K = \mathcal{M} \cap AD$ and $L = \mathcal{M} \cap BC$.*

If $\mathcal{N}$ is an axis of $\mathcal{K}$, then we have the theorem in [12].

Before proving the theorem we recall some facts from the analytic geometry of conics. It is well-known that any conic can be represented by an equation of the form

$$y^2 = px - qx^2 \tag{1}$$

where $p > 0$ and $q > 0$ for an ellipse, $q = 0$ for a parabola and $q < 0$ for a hyperbola. In the cases of an ellipse or a hyperbola we substitute x and y by $\frac{p}{|q|}x$ and $\frac{p}{\sqrt{|q|}}y$ respectively and multiply the obtained equation by $\frac{|q|}{p^2}$ and with $\omega = \frac{|q|}{q}$ we get the equation

$$y^2 = x - \omega x^2, \tag{2}$$

where $\omega = 1$ for an ellipse and $\omega = -1$ for a hyperbola. In the case of a parabola it suffices in (1) to substitute x by $\frac{x}{p}$ and we obtain equation (2) again, but now with $\omega = 0$. Applying an affine transformation we conclude that any conic section has an equation of the form (2), where $\omega \in \{1, 0, -1\}$.

If we put $x = 0$ in (2), we obtain the equation $y^2 = 0$ with two solutions $y = 0$, i.e. our conic has the y-axis $\mathcal{Y}$ for a tangent in the origin. Let us prove that the

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x-axis $\mathcal{X}$ is a diameter of the conic. In the cases of an ellipse or a hyperbola the centre of the conic (2) is the point $S = (\frac{\omega}{2}, 0)$. Indeed, this point is the midpoint of the points $T = (x, y)$ and $T' = (\omega - x, -y)$. If the point T is on the conic (2), then we have (because of $\omega^2 = 1$)

$$(\omega - x) - \omega(\omega - x)^2 = x - \omega x^2 = y^2 = (-y)^2$$

and the point T' is on this conic too. Any straight line through the origin has an equation of the form $y = tx$ and intersects the conic (2) at two points, whose abscissas are the solutions of the equation $t^2x^2 = x - \omega x^2$. The first solution $x = 0$ gives the origin and the second one is

$$x = \frac{1}{t^2 + \omega}. \tag{3}$$

For the intersection with the abscissa (3) the ordinate is

$$y = \frac{t}{t^2 + \omega}. \tag{4}$$

In the case of a parabola we cannot have the value $t = 0$. But, then we put $y = 0$ directly in (2) and obtain the unique solution $x = 0$, i.e. the axis $\mathcal{X}$ does not have a second intersection with the parabola. Therefore, $\mathcal{X}$ is a diameter of this parabola. Our conic (2) has $\mathcal{X}$ as a diameter and the conjugate diameter (in the case of an ellipse or a hyperbola) is parallel with the axis $\mathcal{Y}$ and has the equation $x = \frac{\omega}{2}$. If we put $x = \frac{\omega}{2}$ in (2), we obtain the equation $y^2 = \frac{\omega}{4}$ with a real solution only for an ellipse, i.e. in the case of a hyperbola this diameter does not have the intersections with the hyperbola. By the way, we have the parametric representation (3) and (4) of the conic under consideration. The point $T = (x, y)$ given by (3) and (4), where $t \in \mathbb{R}$, will be denoted by $T = (t)$.

Proof of theorem. 1° If the diameter $\mathcal{N}$ intersects the conic $\mathcal{K}$ let $\mathcal{N}$ be the axis $\mathcal{X}$ of an affine coordinate system, where the conic $\mathcal{K}$ has the equation (2), i.e. the parametric representation (3) and (4). The straight line $\mathcal{M}$ has the equation $x = m$, where $M = (m, 0)$. Let $A = (a), B = (b), C = (c), D = (d)$ and let the points E, F, G, H, K, L have the ordinates e, f, g, h, k, l, respectively. The condition for the collinearity of the points $A = (a), B = (b)$ and $E = (m, e)$ has (after the multiplication of two rows by $a^2 + \omega$ and $b^2 + \omega$, respectively) the form

$$\begin{vmatrix} 1 & m & e \\ a^2 + \omega & 1 & a \\ b^2 + \omega & 1 & b \end{vmatrix} = 0,$$

i.e.

$$e(a^2 - b^2) - m\left[a^2b - ab^2 - \omega(a - b)\right] - (a - b) = 0$$

or

$$(a + b)e = (ab - \omega)m + 1. \tag{5}$$

The analogous condition for the points B, C and F is

$$(c+d)f = (cd-\omega)m + 1. \tag{6}$$

The point M is the midpoint of two points E and F iff $e+f=0$. Because of (5) and (6) this equality is equivalent to the equality

$$[(ab-\omega)m+1]\,(c+d) + [(cd-\omega)m+1]\,(a+b) = 0,$$

i.e.

$$(abc+abd+acd+bcd)m + (1-m\omega)(a+b+c+d) = 0.$$

This equality is symmetrical with respect to the parameters a, b, c, d. Therefore, we obtain the same condition for $g+h=0$ and for $k+l=0$.

2° If the diameter $\mathcal{N}$ does not intersect the conic $\mathcal{K}$, then $\mathcal{K}$ is a hyperbola and let the conjugate diameter of $\mathcal{N}$ be the axis $\mathcal{X}$, let $\mathcal{K}$ have the equations (3) and (4) again and now $\omega = -1$. The diameter $\mathcal{N}$ has the equation $x = -\frac{1}{2}$ and if $M = (-\frac{1}{2}, m)$ then the straight line $\mathcal{M}$ has the equation $y = m$. Let $A = (a), B = (b), C = (c), D = (d)$ and let the points E, F, G, H, K, L have the abscissas e, f, g, h, k, l, respectively. The collinearity conditions for the points A, B, E and C, D, F can be obtained from (5) and (6) by the substitutions $m \leftrightarrow e$ resp. $m \leftrightarrow f$ and because of $\omega = -1$ have the forms

$$\begin{aligned}(ab+1)e &= (a+b)m - 1,\\ (cd+1)f &= (c+d)m - 1.\end{aligned} \tag{7}$$

The point $M = (-\frac{1}{2}, m)$ is the midpoint of two points E and F iff $e+f=-1$. Because of (7) this condition is equivalent to the equality

$$-(ab+1)(cd+1) = [(a+b)(cd+1) + (c+d)(ab+1)]\,m - (cd+1+ab+1),$$

i.e.

$$(abc+abd+acd+bcd+a+b+c+d)m + abcd - 1 = 0.$$

Again, we have the symmetry of this equality with respect to a, b, c, d.

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