A correction for a result on convergence of Ishikawa iteration for strongly pseudocontractive maps

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Abstract. *We give a correction to the main result from [13].*

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1. Introduction

Let X be a real Banach space. Let B be a nonempty, convex subset of X. Let $T: B \to B$ be a map. Let $x_1 \in B$. We consider the following iteration, see [4]:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n = 1, 2, \dots . \end{aligned} \tag{I}$$

We suppose that $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$, and the sequence $(\alpha_n)_n$ satisfies

$$0 < w \leq \alpha_n \leq 1. \tag{1}$$

For $\beta_n = 0, \forall n \in N$ we get Mann iteration, see [5]. Ishikawa iteration with condition (1) is studied in [13]. In [7] it was proven that two assumptions of the main theorem from [13] are contradictory. In this note we will prove that renouncing to one assumption from [13] and supposing true an assumption à la [3], the above theorem from [13] is true.

The map $J : X \to 2^{X^*}$ given by

$$Jx := \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X,$$

is called *the normalized duality mapping*. The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$. It is easy to see that we have

$$\langle j(x), y \rangle \leq \|x\| \|y\|, \forall x, y \in X, \forall j(x) \in J(x). \tag{2}$$

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Definition 1. *Let X be a real Banach space, let B be a nonempty subset. A map $T : B \to B$ is called strongly pseudocontractive if for all $x, y \in B$, there exists $j(x-y) \in J(x-y)$ such that*

$$\exists \gamma \in (0,1) : \langle Tx - Ty, j(x-y) \rangle \leq \gamma \left\| x - y \right\|^2 . \tag{3}$$

The following Lemma could be found in [6], [12], with different proofs. A particular form of this lemma is in [11].

Lemma 1. *[6], [11], [12] If X is a real Banach space, then the following relation is true*

$$\left\| x + y \right\|^2 \leq \left\| x \right\|^2 + 2 \langle y, j(x+y) \rangle , \quad \forall x, y \in X, \forall j(x+y) \in J(x+y). \tag{4}$$

The following result is from [9]. Three other proofs could be found in [10].

Proposition 1. *[9], [10].Let $(a_n)_n$ be a nonnegative sequence which satisfies*

$$a_{n+1} \leq (1-w)a_n + \sigma_n S, \tag{5}$$

where $w \in (0,1)$, $S > 0$ are fixed numbers, $\sigma_n \geq 0, \forall n \in N$, $\lim_{n\to\infty} \sigma_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main result

We are able now to give the following result.

Theorem 1. *Let X be a real Banach space, and let $T : X \to X$ be a continuous, strongly pseudocontractive with bounded range map. If $\lim_{n\to\infty} \left\| Ty_n - Tx_{n+1} \right\| = 0$, and $\gamma \in (0, 1/2)$, then the iteration (I) strongly converges to the unique fixed point of T.*

Proof. . The existence follows from [2] and the uniqueness from strongly pseudocontractivity. Let $x^* = Tx^*$. Using (4) for the first inequality, (2) and (3) for the third one, we can see:

$$\begin{aligned}
&\left\| x_{n+1} - x^* \right\|^2 = \left\| (1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*) \right\|^2 \\
&\leq (1-\alpha_n)^2 \left\| x_n - x^* \right\|^2 + 2\alpha_n \langle Ty_n - x^*, j(x_{n+1} - x^*) \rangle \\
&\leq (1-\alpha_n)^2 \left\| x_n - x^* \right\|^2 + 2\alpha_n \langle Ty_n - Tx_{n+1}, j(x_{n+1} - x^*) \rangle \\
&+ 2\alpha_n \langle Tx_{n+1} - x^*, J(x_{n+1} - x^*) \rangle \\
&\leq (1-\alpha_n)^2 \left\| x_n - x^* \right\|^2 + 2\alpha_n \left\| Ty_n - Tx_{n+1} \right\| \left\| x_{n+1} - x^* \right\| \\
&+ 2\alpha_n \gamma \left\| x_{n+1} - x^* \right\|^2 , \quad \forall j(x_{n+1} - x^\star) \in J(x_{n+1} - x^\star).
\end{aligned}$$

There results

$$(1 - 2\alpha_n\gamma) \left\| x_{n+1} - x^* \right\|^2 \leq (1-\alpha_n)^2 \left\| x_n - x^* \right\|^2 + 2\alpha_n \left\| Ty_n - Tx_{n+1} \right\| \left\| x_{n+1} - x^* \right\| ,$$

$$\left\| x_{n+1} - x^* \right\|^2 \leq \frac{(1-\alpha_n)^2}{(1-2\alpha_n\gamma)} \left\| x_n - x^* \right\|^2 + \frac{2\alpha_n}{(1-2\alpha_n\gamma)} \left\| Ty_n - Tx_{n+1} \right\| \left\| x_{n+1} - x^* \right\| .$$

Because $\gamma \in (0, 1/2)$, $\alpha_n \in (0, 1) \Rightarrow \frac{2(1-\gamma)-\alpha_n}{1-2\alpha_n\gamma} \geq 1$ *i.e.* $-\left(\frac{2(1-\gamma)-\alpha_n}{1-2\alpha_n\gamma}\right) \leq -1$, we have

$$\frac{(1-\alpha_n)^2}{(1-2\alpha_n\gamma)} = \frac{1-2\alpha_n+\alpha_n{}^2}{(1-2\alpha_n\gamma)} = \frac{\left((1-2\alpha_n\gamma)+2\alpha_n\gamma-2\alpha_n+\alpha_n^2\right)}{(1-2\alpha_n\gamma)} =$$
$$= 1-\left(\frac{2(1-\gamma)-\alpha_n}{1-2\alpha_n\gamma}\right)\alpha_n \leq 1-\alpha_n. \tag{6}$$

Also, the sequence (x_n) is bounded. We will prove that by induction. Let us denote by $d := \sup\{\|Tx\| : x \in B\} + \|x^*\|$. Because the range of T is bounded we have $d < \infty$. We denote by $M := d + \|x_0 - x^*\| + 1$. Observe that

$$\begin{aligned}\|x_1 - x^*\| &\leq (1-\alpha_0)\|x_0 - x^*\| + \alpha_0\|Ty_0 - x^*\| \\ &\leq (1-\alpha_0)M + \alpha_0(\|Ty_0\| + \|x^*\|) \leq (1-\alpha_0)M + \alpha_0 M = M.\end{aligned}$$

Supposing $\|x_n - x^*\| \leq M$, we will prove that $\|x_{n+1} - x^*\| \leq M$. Indeed we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|Ty_n - x^*\| \\ &\leq (1-\alpha_n)M + \alpha_n(\|Ty_n\| + \|x^*\|) \leq (1-\alpha_n)M + \alpha_n M = M.\end{aligned}$$

Thus we have

$$\exists M > 0 : \|x_{n+1} - x^*\| \leq M, \forall n \geq 0. \tag{7}$$

Conditions (6), (7) and (8) lead us to

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \|Ty_n - Tx_{n+1}\|\frac{2\alpha_n}{(1-2\alpha_n\gamma)}M.$$

But $(1-\alpha_n) \leq (1-w)$, and $\frac{2\alpha_n}{(1-2\alpha_n\gamma)} \leq \frac{2}{1-2\gamma}$. So, we have

$$\|x_{n+1} - x^*\|^2 \leq (1-w)\|x_n - x^*\|^2 + \|Ty_n - Tx_{n+1}\|\frac{2}{1-2\gamma}M.$$

Let us denote be $a_n := \|x_n - x^*\|^2$, $\sigma_n := \|Ty_n - Tx_{n+1}\|$, and $S := \frac{2}{1-2\gamma}M$. Then we have $\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty} x_n = x^*$. □

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