On cyclic characterizations of regular pentagons and heptagons: Two approaches

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Abstract. *In this paper we present two different proofs of an algebraic characterization of regular pentagons and regular pentagrams in terms of two cyclic (complex) algebraic equations on a five–dimensional torus (Theorem 1 and Theorem 2). The problem arose in functional analysis (as communicated to one of the authors by A. Björner some twenty years ago). No published proof has appeared so far. Apparently a proof was given by L. Lovász (unpublished and not known to the authors). Here we give two different proofs, both somewhat tricky. The first one relies upon discrete Fourier transform and the second one is more direct. Also several generalizations to heptagons are presented including an explicit description of some new irregular heptagrams. Some additional conjectures on general polygons are stated.*

Key words: *pentagon, heptagon, cyclic equations, discrete Fourier transform, bi–unimodular sequences*

AMS subject classifications: 65T50, 51Mxx, 14Mxx

Received May 23, 2002 Accepted July 4, 2002

1. Preliminaries on discrete Fourier transform

Let $\omega = e^{2\pi i/N}$ denote the N–th root of unity. Then the one–dimensional Fourier transform of the sequence of complex numbers $(u_0, u_1, \dots, u_{N-1})$ is defined to be the sequence $(\widehat{u}_0, \widehat{u}_1, \dots, \widehat{u}_{N-1})$, where

$$\widehat{u}_s = \sum_{0 \le t < N} \omega^{st} u_t, \quad 0 \le s < N. \tag{1}$$

Letting $(\widehat{v}_0, \widehat{v}_1, \dots, \widehat{v}_{N-1})$ be defined in the same way as the Fourier transform of $(v_0, v_1, \dots, v_{N-1})$ it is easy to see that $(\widehat{u}_0\widehat{v}_0, \widehat{u}_1\widehat{v}_1, \dots, \widehat{u}_{N-1}\widehat{v}_{N-1})$ is the transform

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of $(w_0, w_1, \ldots, w_{N-1})$ where

$$w_r = \sum_{i+j \equiv r(\mathrm{mod} N)} u_i v_j. \tag{2}$$

To get the inverse Fourier transform $(u_0, u_1, \ldots, u_{N-1})$ from the values of $(\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_{N-1})$, we may note that the "double transform" is

$$\begin{aligned} \widehat{\widehat{u}}_r &= \sum_{0 \le s < N} \omega^{rs} \widehat{u}_s = \sum_{0 \le s,t < N} \omega^{rs} \omega^{st} u_t \\ &= \sum_{0 \le t < N} u_t \left(\sum_{0 \le s < N} \omega^{s(t+r)} \right) = N u_{-r(\mathrm{mod} N)}, \end{aligned} \tag{3}$$

since the geometric series $\sum_{0 \le s < N} \omega^{sj}$ sums to zero unless j is a multiple of N. Therefore the inverse transformation can be computed in the same way as the transform itself except that the final result must be divided by N and shuffled slightly:

$$N u_r = \sum_{0 \le s < N} \overline{\omega}^{rs} \widehat{u}_s \tag{4}$$

2. The problem of five points on a circle

Let z_0, z_1, z_2, z_3, z_4 be five complex numbers lying on the unit circle $\{|z| = 1\}$ and satisfying the symmetric conditions $z_0 + z_1 + z_2 + z_3 + z_4 = 0$ and $z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_0 + z_0 z_2 + z_1 z_3 + z_2 z_4 + z_3 z_0 + z_4 z_1 = 0$. It follows easily, by conjugating these two equations and then using Vietè's formulas, that z_k's are necessarily vertices of a regular pentagon.

In the following theorem we show that the same conclusion follows if we replace the symmetric conditions by cyclic ones, but the proof is not so immediate.

Theorem 1. *Any five complex numbers $z_0, z_1, \ldots, z_4$ lying on the unit circle $\{|z| = 1\}$ and satisfying the conditions*

(A_1) $z_0 + z_1 + z_2 + z_3 + z_4 = 0$,

(A_2) $z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_0 = 0$,

are necessarily vertices of a regular pentagon.

Proof. Let $(c_0, c_1, c_2, c_3, c_4)$ be the Fourier transform of the sequence $(z_0, z_1, z_2, z_3, z_4)$ given by (1). Then, by using (4), the conditions (A_1) and (A_2) can be expressed in terms of c_i's as follows:

$$c_0 = 0 \tag{A_1}$$

$$a c_1 c_4 - (1+a) c_2 c_3 = 0, \tag{A_2}$$

where $a = \omega + \overline{\omega} = 2\cos(2\pi/5)$, $(\omega = cos(2\pi/5) + i\sin(2\pi/5))$. By the inverse Fourier transformation, the conditions $|z_0| = |z_1| = |z_2| = |z_3| = |z_4| = 1$ imply two more equations satisfied by c_i's:

$$c_1\overline{c}_2 + c_2\overline{c}_3 + c_3\overline{c}_4 = 0, \tag{B_1}$$

$$c_1\overline{c}_3 + c_2\overline{c}_4 + c_4\overline{c}_1 = 0, \tag{B_2}$$

which can be proved directly as follows. For $r = 1, 2$ we have:

$$\textstyle\sum_{0\le k<5} c_k\overline{c}_{k+r} = \displaystyle\sum_{0\le k,s,t<5} \omega^{ks}\overline{\omega}^{t(k+r)} z_s\overline{z}_t = \sum_{0\le s,t<5} \overline{\omega}^{rt}\left(\sum_{0\le k<5}\omega^{k(s-t)}\right) z_s\overline{z}_t$$
$$= 5\sum_{0\le t<5}\overline{\omega}^{rt}|z_t|^2 = 5\sum_{0\le t<5}\overline{\omega}^{rt} = 0.$$

Lemma 1. *Conditions (A_1),(A_2), (B_1) and (B_2) imply that exactly one of the c_i's is nonzero.*

Proof. We first eliminate c_1 from (A_2) and (B_1):

$$c_3\left(a|c_4|^2 + (1+a)|c_2|^2\right) = -ac_2c_4\overline{c}_3, \tag{5}$$

and then take absolute values of both sides in (5):

$$|c_3|\left(a|c_4|^2 - a|c_2||c_4| + (1+a)|c_2|^2\right) = 0. \tag{6}$$

Note that the (real) quadratic form $ax^2 - axy + (1+a)y^2$ is positive definite (its discriminant $D = -a(4+3a)$ is negative). Hence (6) implies two cases.

Case 1.: $c_3 = 0 \overset{(B_1)}{\Longrightarrow} c_1 = 0$ *or* $c_2 = 0 \overset{(B_2)}{\Longrightarrow} (c_2 = 0$ or $c_4 = 0)$ or $(c_1 = 0$ or $c_4 = 0)$

Case 2.: $c_2 = c_4 = 0 \overset{(B_2)}{\Longrightarrow} c_1 = 0$ or $c_3 = 0$. □

Finally, (A_1) and *Lemma 1* show that z_k's must be of the form (by (4))

$$z_k = \tfrac{1}{5}\overline{\omega}^{kj}c_j \tag{7}$$

for some $j \in \{1,2,3,4\}$, so z_k $(k = 0,1,\ldots,4)$ represent vertices of a regular pentagon or vertices of a regular pentagram. □

3. A direct proof

Theorem 2. *Let z_1, z_2, z_3, z_4, z_5 be five distinct points (complex numbers) on a unit circle $S^1 = \{z : |z| = 1\}$. Then they are consecutive vertices of either a regular pentagon or a regular pentagram (i.e. $z_k = \lambda\omega^k$, $\omega = exp(2\pi i/5)$, $|\lambda| = 1$) if and only if the following two equations are satisfied:*

i) $z_1 + z_2 + z_3 + z_4 + z_5 = 0,$

ii) $z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_1 = 0.$

Proof. First we note that the equations (i) and (ii) hold for regular pentagons and for regular pentagrams.

Let Z be the set of all solutions of (i) and (ii) belonging to $S^1 \times S^1 \times S^1 \times S^1 \times S^1 = T^5$ (a five-dimensional torus). The set Z is clearly S^1–invariant with respect to the action given by $(\lambda, (z_1, z_2, z_3, z_4, z_5)) \mapsto (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, \lambda z_5)$.

We shall now seek for special representatives $(z_1, z_2, z_3, z_4, z_5)$ of S^1–orbits in Z obeying the additional property

$$z_5 = \overline{z}_1 \quad \text{(equivalently } z_1z_5 = 1). \tag{8}$$

(Take any orbit representative $(z_1, z_2, z_3, z_4, z_5)$ and scale it by $\lambda := \frac{1}{\sqrt{z_1z_5}}$.)

The following fact is a key observation here: If $|z| = 1$ and $|v| = |w|$, $v \neq -w$ then there is a real number μ such that

$$vz + wz^{-1} = \mu(v + w). \tag{9}$$

Hence by using this fact we can write the part $z_1z_2 + z_4z_5$ of the sum in (ii) as follows

$$\begin{aligned} z_1z_2 + z_4z_5 &= z_2z_1 + z_4z_1^{-1} \quad \text{by (9)} \\ &= \mu(z_2 + z_4) \end{aligned} \tag{10}$$

where μ is a real number. Here we only need to comment that $z_2 + z_4$ is nonzero. Otherwise, we would have from (i): $0 = z_1 + z_3 + z_5 = z_1 + \overline{z}_1 + z_3 \Rightarrow z_3$ is a real number, hence $z_3 = \pm 1$ and $z_1 = \mp \frac{1 \pm \sqrt{3}}{2}$ what contradicts (ii): $0 = z_1z_2 + z_4z_5 + z_5z_1 = (z_1 - \overline{z}_1)z_2 + 1 = \pm i\sqrt{3}z_2 + 1 (\neq 0$, because $|z_2| = 1)$.

Now we plug in (10) into (ii):

$$(\mu + z_3)(z_2 + z_4) + 1 = 0. \tag{11}$$

From (i), by using (8) we obtain

$$z_2 + z_4 = -(z_3 + z_1 + \overline{z}_1). \tag{12}$$

Substituting (12) into (11) leads to

$$(\mu + z_3)(z_3 + z_1 + \overline{z}_1) = 1, \tag{13}$$

or equivalently

$$|\mu + z_3|^2(z_3 + z_1 + \overline{z}_1) = \mu + \overline{z}_3. \tag{13'}$$

The imaginary part of (13') gives

$$|\mu + z_3|^2 \Im(z_3) = -\Im(z_3) \Rightarrow \Im(z_3) = 0 \Rightarrow z_3 = \pm 1. \tag{14}$$

Now (12) becomes

$$z_2 + z_4 = \mp 1 - z_1 - \overline{z}_1, \tag{15}$$

implying that $\Im(z_4) = -\Im(z_2)$ and because $|z_4| = |z_2|$ $(= 1)$ we infer

$$z_4 = \overline{z}_2, \tag{16}$$

because $z_2 + z_4 \neq 0$. Now (16) and (8) imply

$$z_4 z_5 = \overline{z_1 z_2} \tag{17}$$

(This implies $z_1 z_2 z_3 z_4 z_5 = \pm 1$.)

Now we substitute $z_3 = \pm 1$ from (14) into (*ii*) and use (17) and (8):

$$\begin{aligned} 0 &= z_1 z_2 + \frac{1}{z_1 z_2} \pm (\mp 1 - z_1 - \overline{z}_1) + 1 \\ &= z_1 z_2 + \frac{1}{z_1 z_2} \mp (z_1 + \frac{1}{z_1}) \\ &= (z_1 \mp \frac{1}{z_1 z_2})(z_2 \mp 1). \end{aligned}$$

This implies that (because z_k's are distinct)

$$z_2 = \pm z_1^{-2}. \tag{18}$$

Now we substitute (18) into (16) and then into (15):

$$\pm z_1^{-2} \pm z_1^2 = \mp 1 - z_1 - z_1^{-1},$$

or equivalently

$$z_1^4 \pm z_1^3 + z_1^2 \pm z_1 + 1 = 0.$$

Hence

$$z_1^5 = \pm 1.$$

Now formulas (18), (14), (16) and (8) can be written as

$$(z_1, z_2, z_3, z_4, z_5) = (z_1, z_1^3, z_1^5, z_1^7, z_1^9) = (z_1, z_1^3, \pm 1, \pm z_1^2, \pm z_1^4)$$

where z_1 is any primitive root of $z^5 = \pm 1$.

This means that for each choice of sign ± 1 we obtain exactly four cyclic arrangements of five points on the unit circle: $(1, 2, 3, 4, 5)$, $(5, 4, 3, 2, 1)$ (regular pentagon), or $(1, 3, 5, 2, 4)$, $(1, 4, 2, 5, 3)$ (regular pentagram). The theorem is proved. □

We state now the general conjecture, mentioned also by A. Björner ([B]):

Conjecture 1. *Let N be any odd integer, $N = 2n + 1, n \geq 1$. Let z_0, z_1, $\cdots$, z_{N-1} be any distinct N points (complex numbers) on a unit circle $S^1 = \{z : |z| = 1\}$ which satisfy the following n equations :*

(A_s) $\sum_{0 \leq k < N} z_k z_{k+1} \cdots z_{k+s-1} = 0, \quad 1 \leq s \leq n. \quad (z_{N+k} := z_k, k \geq 0)$

Then these N numbers represent vertices of a regular N-gon.

Remark 1. *Note that for unit complex numbers the conjugate of the cyclic equation $A_s(1 \le s \le n)$ is just the complementary cyclic equation $A_{2n+1-s}(n+1 \le 2n+1-s \le 2n)$.*

Now we reformulate this conjecture via the Fourier transform as follows:

Conjecture 2. *Let $\omega = e^{2\pi i/N}$ denote the N–th root of unity. Let $N(=2n+1)$ complex numbers $c_0 = 0, c_1, c_2, \dots, c_N$ satisfy the following equations:*

$$(\widehat{A}_s) \quad \sum_{1\le i_1,\cdots,i_s<N; i_1+\cdots+i_s=0(mod N)} \overline{\omega}^{i_2+2i_3+\cdots+(s-1)i_s} c_{i_1}\cdots c_{i_s} = 0, 2 \le s \le n,$$

$(\widehat{B}_s)$ $\sum_{0\le k<N} c_k\overline{c}_{k+s} = 0, (1 \le s \le n)$, *where* $(c_{N+k} = c_k, k \ge 0, c_0 = 0)$.

Then exactly one of c_i's is nonzero. For example, for $N = 7$ the conditions in the *Conjecture 2* read as follows:

$(\widehat{A_2})$ $(\omega+\overline{\omega})c_1c_6 + (\omega^2+\overline{\omega}^2)c_2c_5 + (\omega^3+\overline{\omega}^3)c_3c_4 = 0,$

$(\widehat{A_3})$ $(1+\omega^3+\overline{\omega}^3)(c_1^2c_5 + c_2c_6^2) + (1+\omega^2+\overline{\omega}^2)(c_1c_3^2 + c_4^2c_6) + (1+\omega+\overline{\omega})(c_2^2c_3 + c_4c_5^2) - c_1c_2c_4 - c_3c_5c_6 = 0,$

$(\widehat{B_1})$ $c_1\overline{c}_2 + c_2\overline{c}_3 + c_3\overline{c}_4 + c_4\overline{c}_5 + c_5\overline{c}_6 = 0,$

$(\widehat{B_2})$ $c_1\overline{c}_3 + c_2\overline{c}_4 + c_3\overline{c}_5 + c_4\overline{c}_6 + c_6\overline{c}_1 = 0,$

$(\widehat{B_3})$ $c_1\overline{c}_4 + c_2\overline{c}_5 + c_3\overline{c}_6 + c_5\overline{c}_1 + c_6\overline{c}_2 = 0,$

where $\omega = e^{2\pi i/7}$.

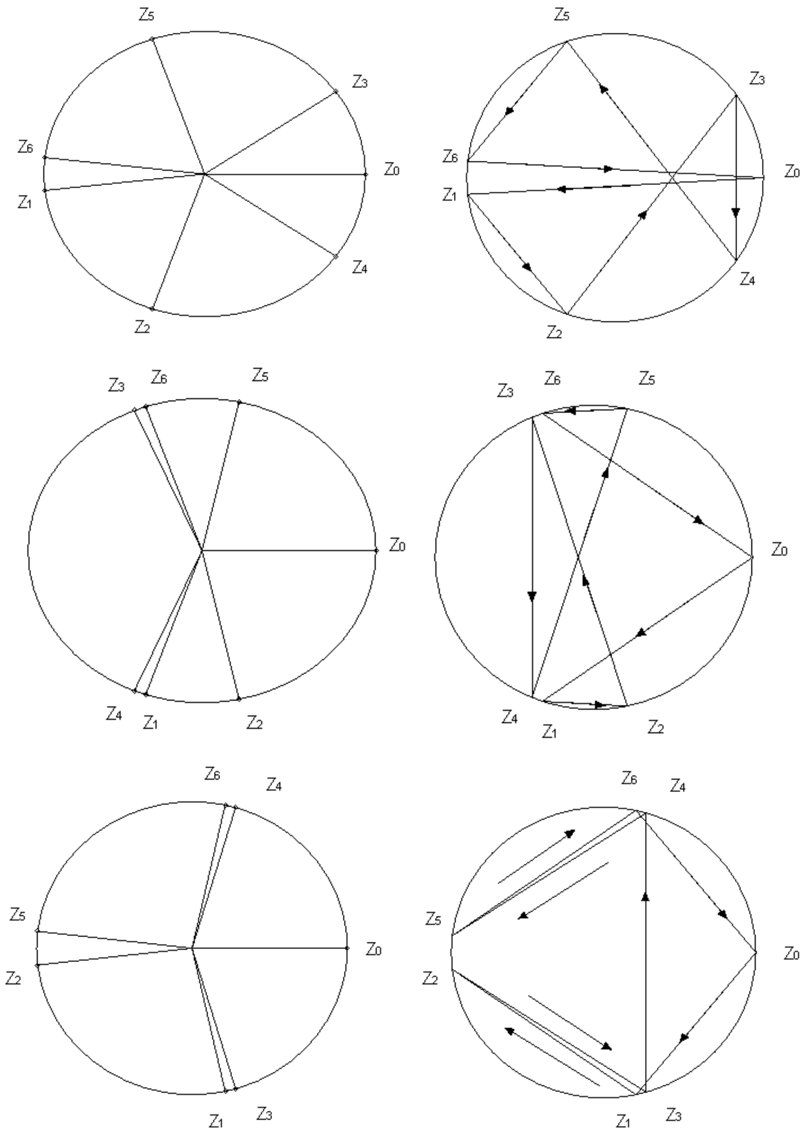
The equations for $N = 7$ in *Conjecture 1* are as follows:

$$\begin{aligned}
&z_0+z_1+z_2+z_3+z_4+z_5+z_6 = 0,\\
&z_0z_1+z_1z_2+z_2z_3+z_3z_4+z_4z_5+z_5z_6+z_6z_0 = 0,\\
&z_0z_1z_2+z_1z_2z_3+z_2z_3z_4+z_3z_4z_5+z_4z_5z_6+z_5z_6z_0+z_6z_0z_1 = 0,
\end{aligned}$$

By using $|z_k| = 1$, $(k = 0..6)$ the conjugation gives us three more equations:

$$\begin{aligned}
&\tfrac{1}{z_0}+\tfrac{1}{z_1}+\tfrac{1}{z_2}+\tfrac{1}{z_3}+\tfrac{1}{z_4}+\tfrac{1}{z_5}+\tfrac{1}{z_6} = 0,\\
&\tfrac{1}{z_0z_1}+\tfrac{1}{z_1z_2}+\tfrac{1}{z_2z_3}+\tfrac{1}{z_3z_4}+\tfrac{1}{z_4z_5}+\tfrac{1}{z_5z_6}+\tfrac{1}{z_6z_0} = 0,\\
&\tfrac{1}{z_0z_1z_2}+\tfrac{1}{z_1z_2z_3}+\tfrac{1}{z_2z_3z_4}+\tfrac{1}{z_3z_4z_5}+\tfrac{1}{z_4z_5z_6}+\tfrac{1}{z_5z_6z_0}+\tfrac{1}{z_6z_0z_1} = 0.
\end{aligned}$$

If we specialize $z_0 = 1$, $z_6 = \frac{1}{z_1}$, $z_5 = \frac{1}{z_2}$, $z_4 = \frac{1}{z_3}$ in the above six equations, we shall produce (in *Section 4.* by using Maple) explicit counterexamples to *Conjecture 1* in the case $N = 7$. The only three different (up to complex conjugation) such counterexamples are depicted (each in two views) in the following figures obtained by Maple:



In case $N = 7$ if we add one more quadratic condition $z_0z_2 + z_1z_3 + z_2z_4 + z_3z_5 + z_4z_6 + z_5z_7 + z_6z_1 = 0$ and its conjugate, then with weaker symmetry requirement $z_4 = \frac{1}{z_3}$ we get only regular solutions (i.e. only regular heptagons and regular heptagrams) as the following commented transcript of a Maple session shows:

```
>  Z[0]:= 1:  Z[1]:=a:  Z[2]:=b:  Z[3]:=c:  Z[4]:=1/c:
>  Z[5]:=e:  Z[6]:=d:
>  for j from 0 to 2 do F[j] :=
>  sort(add(mul(Z[`mod`(k+m,7)],m=0..j),k=0..6)); od;
```

$$F_0 := a + b + c + e + d + \frac{1}{c} + 1$$

$$F_1 := a\,b + b\,c + e\,d + a + d + \frac{e}{c} + 1$$

$$F_2 := a\,b\,c + a\,b + a\,d + e\,d + b + \frac{e\,d}{c} + e$$

```
> for j from 2 to 0 by -1 do F[5-j] :=
> add(1/mul(Z['mod'(k+m,7)],m=0..j),k=0..6); od;
```

$$F_3 := \frac{1}{a\,b} + \frac{1}{a\,b\,c} + \frac{1}{b} + \frac{1}{e} + \frac{c}{e\,d} + \frac{1}{e\,d} + \frac{1}{a\,d}$$

$$F_4 := 1 + \frac{1}{a} + \frac{1}{a\,b} + \frac{1}{b\,c} + \frac{c}{e} + \frac{1}{e\,d} + \frac{1}{d}$$

$$F_5 := 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + c + \frac{1}{e} + \frac{1}{d}$$

```
> F[6]:= sort(add(mul(Z['mod'(k+2*m,7)],m=0..1),k=0..6));
```

$$F_6 := a\,c + a\,d + c\,e + b + e + \frac{b}{c} + \frac{d}{c}$$

```
> F[7]:= sort(add(1/mul(Z['mod'(k+2*m,7)],m=0..1),k=0..6));
```

$$F_7 := \frac{c}{b} + \frac{c}{d} + \frac{1}{b} + \frac{1}{e} + \frac{1}{a\,c} + \frac{1}{a\,d} + \frac{1}{c\,e}$$

```
> Sols:={solve({seq(F[k]=0,k=0..7)},{a,b,c,d,e})}:
> nops(Sols);
```

$$1$$

```
> Sols;
```

$$\{\{b = \alpha^3,\, a = \alpha^5,\, d = \alpha^2,\, e = \alpha^4,\, c = \alpha\}\}$$

Now we state a theorem whose generalization for N prime (or even squearefree) could be a more feasible general conjecture.

Theorem 3. *Let $N = 5$ or 7. Let $z_0, z_1, \ldots, z_{N-1}$ be any N distinct complex numbers lying on the unit circle $\{|z| = 1\}$ and satisfying*

$$\sum_{0\le k<N} z_k = 0, \tag{A_0'}$$

$$\sum_{0\le k<N} z_k z_{k+r} = 0, (1 \le r \le n) \tag{A_r'}$$

(indices modulo N). Then these numbers represent vertices of a regular N–gon.

Proof. As before, let $(c_0, c_1, \ldots, c_{N-1})$ be the Fourier transform of the sequence $(z_0,\, z_1,\, \ldots\,,\, z_{N-1})$. Then

$$c_s = \sum_{0\le t<N} \omega^{st} z_t \iff N z_s = \sum_{0\le t<N} \overline{\omega}^{st} c_t.$$

Now, the conditions (A_0') and (A_r') $(1 \le r \le n)$ transform into:

$$c_0 = 0,\ \sum_{1\le t\le n} (\omega^{rt} + \overline{\omega}^{rt}) c_t c_{N-t} = 0,\ (1 \le r \le n) \tag{19}$$

The matrix $A=(a_{rs})$, $a_{rs}=\omega^{rs}+\overline{\omega}^{rs}$ of the system (19) is nonsingular since its square $A^2=\left(a_{rt}^{(2)}\right)$ given by

$$\begin{aligned} a_{rt}^{(2)} &= \sum_{1\le s\le n} a_{rs}a_{st} = \sum_{1\le s\le n}(\omega^{rs}+\overline{\omega}^{rs})(\omega^{st}+\overline{\omega}^{st}) \\ &= \sum_{1\le s\le n}(\omega^{s(r+t)}+\overline{\omega}^{s(r+t)}) = -2+\delta_{rt}N, \end{aligned} \tag{20}$$

is clearly nonsingular. Thus (19) is equivalent to the following much simpler system of equations:

$$c_0=0,\ c_1c_{N-1}=0,\ c_2c_{N-2}=0,\ \ldots,\ c_nc_{N-n}=0.$$

Now we take into account the system $(\widehat{B_s}),(s=1..n)$ (see p. 76). Then for $N=5$ or 7 it is easy to conclude that one of c_i's is nonzero. □

In the following section we continue our study of *Conjecture 1* in case $N=7$.

4. The problem of seven points on a circle

In the case $N=7$ of *Conjecture 1* we deal with the basic cyclic polynomial equations $A_k=0$, $k=1..3$, where

$$\begin{aligned} A_1 &= z_0+z_1+z_2+z_3+z_4+z_5+z_6, \\ A_2 &= z_0z_1+z_1z_2+z_2z_3+z_3z_4+z_4z_5+z_5z_6+z_6z_0, \\ A_3 &= z_0z_1z_2+z_1z_2z_3+z_2z_3z_4+z_3z_4z_5+z_4z_5z_6+z_5z_6z_0+z_6z_0z_1, \end{aligned}$$

which on the 7-torus $T^7=\{|z_k|=1,\ k=0..6\}$ imply also that $A_k=0$, $k=4..6$ where we set $A_{7-j}:=z_0z_1z_2z_3z_4z_5z_6\overline{A_j}$, $j\in\{1,2,3\}$, i.e.

$$\begin{aligned} A_4 &= z_0z_1z_2z_3+z_1z_2z_3z_4+z_2z_3z_4z_5+z_3z_4z_5z_6+z_4z_5z_6z_0+z_5z_6z_0z_1+ \\ &\quad +z_6z_0z_1z_2, \\ A_5 &= z_0z_1z_2z_3z_4+z_1z_2z_3z_4z_5+z_2z_3z_4z_5z_6+z_3z_4z_5z_6z_0+z_4z_5z_6z_0z_1+ \\ &\quad +z_5z_6z_0z_1z_2+z_6z_0z_1z_2z_3, \\ A_6 &= z_0z_1z_2z_3z_4z_5+z_1z_2z_3z_4z_5z_6+z_2z_3z_4z_5z_6z_0+z_3z_4z_5z_6z_0z_1+ \\ &\quad +z_4z_5z_6z_0z_1z_2+z_5z_6z_0z_1z_2z_3+z_6z_0z_1z_2z_3z_4. \end{aligned}$$

We shall call A_{7-k}, for $k=1..3$, *a conjugate* of A_k, for an obvious reason. To avoid infinitely many solutions, we set $z_0=1$. Several cases will be considered, all having

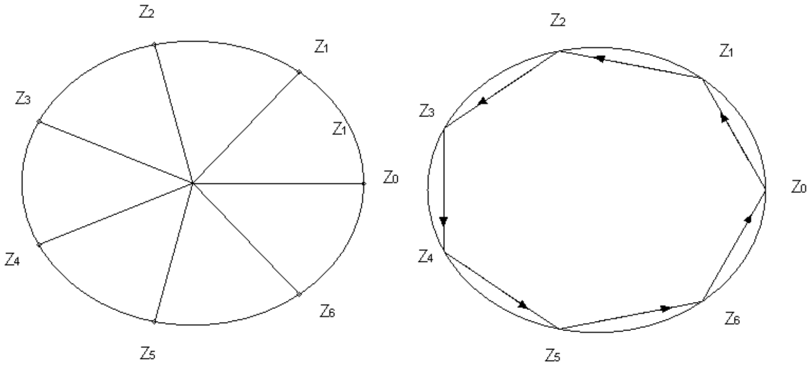
$$\{A_j=0\mid j=1..6\} \tag{21}$$

as the common set of equations.

All calculations and graphics in the sequel are done by Maple 7.

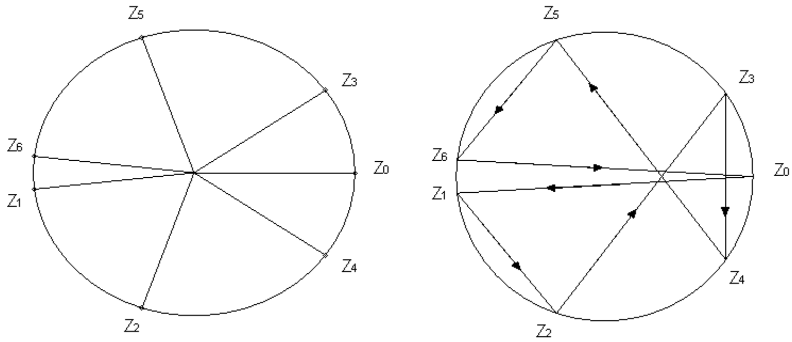
4.1. The easiest case yields irregular solutions

If we substitute $z_1=a$, $z_2=b$, $z_3=c$, $z_4=\frac{1}{c}$, $z_5=e$ and $z_6=d$ into the system (21), then it follows that $e=\frac{1}{b}$ and $d=\frac{1}{a}$, and twelve solutions are obtained, with a, b, c distinct. Six regular solutions emerge, with $z_0, z_1, \ldots, z_6$ sitting in the vertices of a regular heptagon

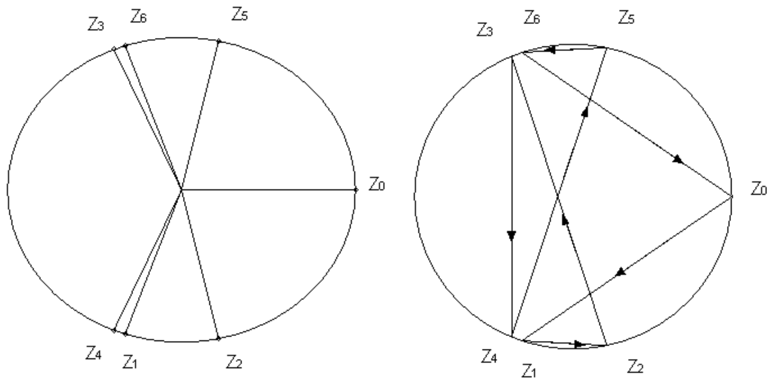


$(z_k)^{(l)} = (\cos \frac{2kl\pi}{7} + i \sin \frac{2kl\pi}{7}),\ l = 1..6$
(the figure above corresponds to the case $l = 1$)

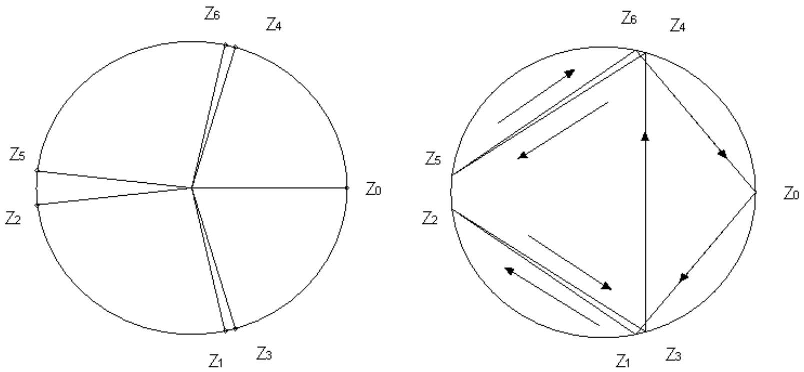
and three (up to complex conjugation) irregular solutions given by



$z_1 = a_3^-$, $z_2 = a_1^-$ and $z_3 = c_2^+$



$z_1 = a_1^-$, $z_2 = a_2^-$ and $z_3 = c_3^+$



$$z_1 = a_2^-,\ z_2 = a_3^- \text{ and } z_3 = c_1^-,$$

where for $k = 1..3$ we have:

$$\begin{aligned} a_k^\pm &= \mathfrak{a}_k/2 \pm i\sqrt{1 - \mathfrak{a}_k^2/4}, \\ c_k^\pm &= \mathfrak{c}_k/2 \pm i\sqrt{1 - \mathfrak{c}_k^2/4}; \end{aligned}$$

$$\begin{aligned} \mathfrak{a}_k &= -1 + (3 - \sqrt{21})\Re(\alpha^{2k}), \\ \mathfrak{c}_k &= (5 - \sqrt{21})/2 + (3 - \sqrt{21})\Re(\alpha^{2k}) \end{aligned}$$

and $\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

4.2. How we found irregular solutions for the case $N = 7$

When we substituted $z_1 = a$, $z_2 = b$, $z_3 = c$, $z_4 = \frac{1}{c}$, $z_5 = \frac{1}{b}$ and $z_6 = \frac{1}{a}$ into equations $A_1, \ldots, A_3$ of the case $N = 7$ of *Conjecture 1* we have obtained the following equations $\{E_k = 0 \mid k = 1..3\}$ where:

$$\begin{aligned} E_1 &:= 1 + a + b + c + \frac{1}{c} + \frac{1}{b} + \frac{1}{a}, \\ E_2 &:= a + ab + bc + 1 + \frac{1}{cb} + \frac{1}{ba} + \frac{1}{a}, \\ E_3 &:= ab + abc + b + \frac{1}{b} + \frac{1}{cba} + \frac{1}{ba} + 1, \end{aligned}$$

Then we have studied various resultants [GKZ]. By a sequence of curious substitutions and factorizations one can do this in Maple as follows.
Define the following algebraic numbers α, μ and λ, using Maple 7 syntax (using its *alias* abbreviation), by

$$\begin{aligned} \alpha &= \mathrm{RootOf}(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1), \\ \mu &= \mathrm{RootOf}(-2x^6 - 9x^5 + x^5\sqrt{21} + 12x^4 - 4x^4\sqrt{21} + 61x^3 \\ &\quad -15x^3\sqrt{21} + 12x^2 - 4x^2\sqrt{21} - 9x + x\sqrt{21} - 2), \\ \lambda &= \mathrm{RootOf}(x^{12} + 9x^{11} + 3x^{10} - 73x^9 - 177x^8 - 267x^7 - 315x^6 \\ &\quad -267x^5 - 177x^4 - 73x^3 + 3x^2 + 9x + 1). \end{aligned}$$

The polynomial $p_{12}(z)$ defining λ appears in many places in the long Maple's output when it solves the above system. The polynomial $q_6(z)$ defining μ is a factor (in the factorization of $p_{12}(z)$ over the quadratic field $\mathbb{Q}(\sqrt{21})$) having as roots all the roots of $p_{12}(z)$ of the unit modulus.

Solving the system $\{E_k = 0 \mid k = 1..3\}$ for variables a, b and c (using Maple's *solve* command) yields three families of solutions:

$$\{a = 1, c = RootOf(x^2 + 5 * x + 1), b = 1\},$$
$$\{b = \alpha, a = \alpha^4, c = \alpha^5\}$$

and $\{a, b, c\}$ where

$$\begin{aligned}
a &= -\tfrac{232}{63} - \tfrac{764}{63}\lambda - \tfrac{506}{63}\lambda^2 + \tfrac{2}{7}\lambda^3 + \tfrac{96}{7}\lambda^4 + \tfrac{88}{3}\lambda^5 + \tfrac{88}{3}\lambda^6 + \tfrac{530}{21}\lambda^7 + \tfrac{88}{7}\lambda^8 \\
&\quad -\tfrac{5}{63}\lambda^9 - \tfrac{88}{63}\lambda^{10} - \tfrac{10}{63}\lambda^{11}, \\
b &= \lambda, \\
c &= \tfrac{118}{9} + \tfrac{182}{9}\lambda - \tfrac{826}{9}\lambda^2 - \tfrac{5518}{21}\lambda^3 - \tfrac{8704}{21}\lambda^4 - 517\lambda^5 - 452\lambda^6 - \tfrac{931}{3}\lambda^7 \\
&\quad -140\lambda^8 + \tfrac{20}{9}\lambda^9 + \tfrac{1063}{63}\lambda^{10} + \tfrac{121}{63}\lambda^{11}.
\end{aligned}$$

Since in the solutions we are seeking all a, b, c should be distinct and of unit modulus, in particular $b(= \lambda)$, we can make further simplifications in the intermediate field $\mathbb{Q}(\sqrt{21}, \mu)$. Then e.g. collect(evala(subs(λ=μ,_),μ)) in Maple gives further simplification as follows:

$$\begin{aligned}
a &= \left(\frac{3}{2} + \frac{13}{42}\sqrt{21}\right)\mu^5 + \left(2 + \frac{2}{7}\sqrt{21}\right)\mu^4 + \left(\frac{5}{2} + \frac{25}{42}\sqrt{21}\right)\mu^3 \\
&\quad + \left(-\frac{1}{2} + \frac{59}{42}\sqrt{21}\right)\mu^2 + \left(-4 + \frac{12}{7}\sqrt{21}\right)\mu - \frac{1}{2} + \frac{29}{42}\sqrt{21} \\
b &= \mu, \\
c &= \left(\frac{11}{6} + \frac{5}{42}\sqrt{21}\right)\mu^5 + \left(\frac{35}{6} - \frac{25}{42}\sqrt{21}\right)\mu^4 + \left(-\frac{23}{3} + \frac{20}{7}\sqrt{21}\right)\mu^3 \\
&\quad + \left(-\frac{227}{6} + \frac{391}{42}\sqrt{21}\right)\mu^2 + \left(-\frac{7}{3} + \frac{23}{21}\sqrt{21}\right)\mu - \frac{11}{14}\sqrt{21} + \frac{37}{6}.
\end{aligned}$$

There are six possible values of b given by $b_k^{\pm} = \mu_k^{\pm} := \beta_k \pm i\sqrt{1-\beta_k^2}$ $(k = 1..3)$ (these are all solutions of $q_6(x) = 0$) where

$$\beta_k := -\frac{1}{2} + \frac{\left(3-\sqrt{21}\right)\left(\alpha^{2k} + \alpha^{-2k}\right)}{4} = \frac{\left(-1 + \left(3-\sqrt{21}\right)\cos\frac{4k\pi}{7}\right)}{2} \quad (k = 1..3)$$

we get altogether six irregular (three up to complex conjugation) heptagrams (with 1 as a vertex)

$$H_k^{\pm} := (1, a_k^{\pm}, b_k^{\pm}, c_k^{\pm}, \frac{1}{c_k^{\pm}}, \frac{1}{b_k^{\pm}}, \frac{1}{a_k^{\pm}})$$

inscribed into a unit circle and satisfying the three cyclic equations $A_1 = 0$, $A_2 = 0$, $A_3 = 0$ (see the Figures in the *Section 4.1.*).

4.3. Adding one more quadratic condition (and its conjugate)

If we wish to obtain a system having regular solutions only, more equations need to be added. For example Equation set (21) can be augmented by two equations, the equation $A_2^{(2)} = 0$ and its conjugate $A_5^{(2)} = 0$,

$$\begin{aligned} A_2^{(2)} &= z_0z_2 + z_1z_3 + z_2z_4 + z_3z_5 + z_4z_6 + z_5z_0 + z_6z_1, \\ A_5^{(2)} &= z_1z_3z_4z_5z_6 + z_0z_2z_4z_5z_6 + z_0z_1z_3z_5z_6 + z_0z_1z_2z_4z_6 + z_0z_1z_2z_3z_5 + \\ &\quad z_1z_2z_3z_4z_6 + z_0z_2z_3z_4z_5, \end{aligned}$$

(the second quadratic equation and its conjugate), and we can use the same substitutions as before, i.e. $z_1 = a$, $z_2 = b$, $z_3 = c$, $z_4 = \frac{1}{c}$, $z_5 = e$ and $z_6 = d$.

With a,b,c distinct there are only (six) regular solutions – corresponding to regular heptagons and regular heptagrams.

In the next section we present another algebraic characterization of regular heptagons and regular heptagrams.

5. Another algebraic characterization of regular heptagons and regular heptagrams

Here we present a system for $N = 7$ on 7–torus consisting of one linear

$$A_1 = 0, \text{ where } A_1 = z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6,$$

and five cubic cyclic equations

$$A_3[j,k] = 0, \text{ where } A_3[j,k] = \sum_{0 \le i < 7} z_i z_{i+j} z_{i+k} \text{ with } jk = 12, 24, 15, 13, 14$$

characterizing regular heptagons and regular heptagrams. The proof uses Fourier transform.

Here follows a commented (and slightly edited) transcript of a Maple session where all necessary calculations are performed by Maple 7.

Let $\alpha = exp(2\pi i/7)$.

```
> c[0]:=0;
```

$$c_0 := 0$$

This condition is equivalent to $A_1 = 0$. Now the conditions that all complex numbers z_k have equal moduli are equivalent to the following equations M_k ($k = 1..3$):

```
> for k to 3 do
>   M[k]:=sum('c[j]*conjugate(c[j+k mod 7])',j=0..6)=0
> od;
```

$$M_1 := c_1\,\overline{(c_2)} + c_2\,\overline{(c_3)} + c_3\,\overline{(c_4)} + c_4\,\overline{(c_5)} + c_5\,\overline{(c_6)} = 0$$

$$M_2 := c_1\,\overline{(c_3)} + c_2\,\overline{(c_4)} + c_3\,\overline{(c_5)} + c_4\,\overline{(c_6)} + c_6\,\overline{(c_1)} = 0$$

$$M_3 := c_1\,\overline{(c_4)} + c_2\,\overline{(c_5)} + c_3\,\overline{(c_6)} + c_5\,\overline{(c_1)} + c_6\,\overline{(c_2)} = 0$$

The following equations $E3_k$ $(k = 1..5)$ are the Fourier transforms of the five cubic cyclic equations $A_3[j,k] = 0$ with increments $jk = 12, 24, 15, 13, 14$ respectively:

```
> E3[1] :=
> (1+alpha^3+alpha^4)*(c[1]^2*c[5]+c[2]*c[6]^2)+(1+alpha^2+
> alpha^5)*(c[1]*c[3]^2+c[4]^2*c[6])+(1+alpha^1+alpha^6)*
> (c[2]^2*c[3]+c[4]*c[5]^2)-c[1]*c[2]*c[4]-c[3]*c[5]*c[6]=0;
```

$$E3_1 := (1+\alpha^3+\alpha^4)\,(c_1{}^2\,c_5 + c_2\,c_6{}^2) + (1+\alpha^2+\alpha^5)\,(c_1\,c_3{}^2 + c_4{}^2\,c_6)$$
$$+\,(1+\alpha+\alpha^6)\,(c_2{}^2\,c_3 + c_4\,c_5{}^2) - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$

```
> E3[2] :=
> (1+alpha^1+alpha^6)*(c[1]^2*c[5]+c[2]*c[6]^2)+(1+alpha^3+
> alpha^4)*(c[1]*c[3]^2+c[4]^2*c[6])+(1+alpha^2+alpha^5)*
> (c[2]^2*c[3]+c[5]^2*c[4])-c[1]*c[2]*c[4]-c[3]*c[5]*c[6]=0;
```

$$E3_2 := (1+\alpha+\alpha^6)\,(c_1{}^2\,c_5 + c_2\,c_6{}^2) + (1+\alpha^3+\alpha^4)\,(c_1\,c_3{}^2 + c_4{}^2\,c_6)$$
$$+\,(1+\alpha^2+\alpha^5)\,(c_2{}^2\,c_3 + c_4\,c_5{}^2) - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$

```
> E3[3] :=
> (1+alpha^2+alpha^5)*(c[1]^2*c[5]+c[2]*c[6]^2)+(1+alpha+
> alpha^6)*(c[1]*c[3]^2+c[4]^2*c[6])+(1+alpha^3+alpha^4)*
> (c[2]^2*c[3]+c[5]^2*c[4])-c[1]*c[2]*c[4]-c[3]*c[5]*c[6]=0;
```

$$E3_3 := (1+\alpha^2+\alpha^5)\,(c_1{}^2\,c_5 + c_2\,c_6{}^2) + (1+\alpha+\alpha^6)\,(c_1\,c_3{}^2 + c_4{}^2\,c_6)$$
$$+\,(1+\alpha^3+\alpha^4)\,(c_2{}^2\,c_3 + c_4\,c_5{}^2) - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$

```
> E3[4]
> :=(alpha+alpha^2+alpha^4)*(c[1]^2*c[5]+c[2]^2*c[3]+c[4]^2*
> c[6])+(alpha^3+alpha^5+alpha^6)*(c[1]*c[3]^2+c[4]*c[5]^2+
> c[2]*c[6]^2)+(alpha^3+alpha^5+alpha^6+3)*c[1]*c[2]*c[4]+
> (alpha+alpha^2+alpha^4+3)*c[3]*c[5]*c[6]=0;
```

$$E3_4 := (\alpha+\alpha^2+\alpha^4)\,(c_1{}^2\,c_5 + c_2{}^2\,c_3 + c_4{}^2\,c_6) + (\alpha^3+\alpha^5+\alpha^6)\,(c_1\,c_3{}^2 + c_4\,c_5{}^2 + c_2\,c_6{}^2)$$
$$+\,(\alpha^3+\alpha^5+\alpha^6+3)\,c_1\,c_2\,c_4 + (\alpha+\alpha^2+\alpha^4+3)\,c_3\,c_5\,c_6 = 0$$

```
> E3[5]
> :=(alpha^3+alpha^5+alpha^6)*(c[1]^2*c[5]+c[2]^2*c[3]+
> c[4]^2*c[6])+(alpha+alpha^2+alpha^4)*(c[1]*c[3]^2+
> c[4]*c[5]^2+c[2]*c[6]^2)+(alpha+alpha^2+alpha^4+3)*
> c[1]*c[2]*c[4]+(alpha^3+alpha^5+alpha^6+3)*
> c[3]*c[5]*c[6]=0;
```

$$E3_5 := (\alpha^3+\alpha^5+\alpha^6)\,(c_1{}^2\,c_5 + c_2{}^2\,c_3 + c_4{}^2\,c_6) + (\alpha+\alpha^2+\alpha^4)\,(c_1\,c_3{}^2 + c_4\,c_5{}^2 + c_2\,c_6{}^2)$$
$$+\,(\alpha+\alpha^2+\alpha^4+3)\,c_1\,c_2\,c_4 + (\alpha^3+\alpha^5+\alpha^6+3)\,c_3\,c_5\,c_6 = 0$$

This system of equations turns out to be equivalent to a simpler system F_k $(k = 1..5)$

which is further reduced to a system G_k $(k = 1..6)$

```
> for k to 5 do
> E[k]:=subs(c[1]^2*c[5]+c[2]*c[6]^2=x,
> c[1]*c[3]^2+c[4]^2*c[6]=y,
> c[2]^2*c[3]+c[5]^2*c[4]=z,
> c[1]^2*c[5]+c[2]^2*c[3]+c[4]^2*c[6]=u,
> c[1]*c[3]^2+c[4]*c[5]^2+c[2]*c[6]^2=v,E3[k]) od;
```

$$E_1 := (1+\alpha^3+\alpha^4)\,x + (1+\alpha^2+\alpha^5)\,y + (1+\alpha+\alpha^6)\,z - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$
$$E_2 := (1+\alpha+\alpha^6)\,x + (1+\alpha^3+\alpha^4)\,y + (1+\alpha^2+\alpha^5)\,z - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$
$$E_3 := (1+\alpha^2+\alpha^5)\,x + (1+\alpha+\alpha^6)\,y + (1+\alpha^3+\alpha^4)\,z - c_1\,c_2\,c_4 - c_3\,c_5\,c_6 = 0$$
$$E_4 := (\alpha+\alpha^2+\alpha^4)\,u + (\alpha^3+\alpha^5+\alpha^6)\,v + (\alpha^3+\alpha^5+\alpha^6+3)\,c_1\,c_2\,c_4 \\ + (\alpha+\alpha^2+\alpha^4+3)\,c_3\,c_5\,c_6 = 0$$
$$E_5 := (\alpha^3+\alpha^5+\alpha^6)\,u + (\alpha+\alpha^2+\alpha^4)\,v + (\alpha+\alpha^2+\alpha^4+3)\,c_1\,c_2\,c_4 \\ + (\alpha^3+\alpha^5+\alpha^6+3)\,c_3\,c_5\,c_6 = 0$$

Now let Maple solve the system $\{E_k \mid k = 1..5\}$:

```
> S1:=solve({seq(E[k],k=1..5)},{x,y,z,u,v});
```

$$\mathit{S1} := \{u = 6r+4s,\ v = 4r+6s,\ x = r+s,\ y = r+s,\ z = r+s\}$$
$$r := \tfrac{1}{2}c_1\,c_2\,c_4,\ \ s := \tfrac{1}{2}c_3\,c_5\,c_6$$

```
> for k to 5 do F[k]:=subs(x=c[1]^2*c[5]+c[2]*c[6]^2,
> y=c[1]*c[3]^2+c[4]^2*c[6],z=c[2]^2*c[3]+c[5]^2*c[4],
> u=c[1]^2*c[5]+c[2]^2*c[3]+c[4]^2*c[6],
> v=c[1]*c[3]^2+c[4]*c[5]^2+c[2]*c[6]^2,op(k,S1)) od;
```

$$F_1 := {c_2}^2\,c_3 + c_4\,{c_5}^2 = \frac{1}{2}\,c_1\,c_2\,c_4 + \frac{1}{2}\,c_3\,c_5\,c_6$$
$$F_2 := {c_1}^2\,c_5 + c_2\,{c_6}^2 = \frac{1}{2}\,c_1\,c_2\,c_4 + \frac{1}{2}\,c_3\,c_5\,c_6$$
$$F_3 := c_1\,{c_3}^2 + c_4\,{c_5}^2 + c_2\,{c_6}^2 = 2\,c_1\,c_2\,c_4 + 3\,c_3\,c_5\,c_6$$
$$F_4 := {c_1}^2\,c_5 + {c_2}^2\,c_3 + {c_4}^2\,c_6 = 3\,c_1\,c_2\,c_4 + 2\,c_3\,c_5\,c_6$$
$$F_5 := c_1\,{c_3}^2 + {c_4}^2\,c_6 = \frac{1}{2}\,c_1\,c_2\,c_4 + \frac{1}{2}\,c_3\,c_5\,c_6$$

```
> G[6]:=(F[3]+F[4]-(F[1]+F[2]+F[5]))*2/7;
```

$$G_6 := 0 = c_1\,c_2\,c_4 + c_3\,c_5\,c_6$$

```
> for k to 5 do
> G[k]:=subs(c[3]*c[5]*c[6]=-c[1]*c[2]*c[4],F[k]) od;
```

$$G_1 := {c_2}^2\,c_3 + c_4\,{c_5}^2 = 0$$

$$G_2 := {c_1}^2\, c_5 + c_2\, {c_6}^2 = 0$$
$$G_3 := c_1\, {c_3}^2 + c_4\, {c_5}^2 + c_2\, {c_6}^2 = -c_1\, c_2\, c_4$$
$$G_4 := {c_1}^2\, c_5 + {c_2}^2\, c_3 + {c_4}^2\, c_6 = c_1\, c_2\, c_4$$
$$G_5 := c_1\, {c_3}^2 + {c_4}^2\, c_6 = 0$$

We solve the system G_k ($k = 1..6$) first in the simpler case by adding a condition G_7; and then in the remaining cases by adding an inequality G_8:

```
> G[7]:=product('c[l]',l=1..6)=0;
```

$$G_7 := c_1\, c_2\, c_3\, c_4\, c_5\, c_6 = 0 \ \ \text{(at least one variable equals 0)}$$

```
> G[8]:=product('c[l]',l=1..6)<>0;
```

$$G_8 := c_1\, c_2\, c_3\, c_4\, c_5\, c_6 \neq 0 \ \ \text{(all variables nonzero)}$$

```
> S_triv:=solve({seq(G[k],k=1..7)},{seq(c[k],k=1..6)});
```

$$\begin{aligned}
&S_triv := \{c_1 = c_1,\, c_2 = c_2,\, c_6 = 0,\, c_4 = 0,\, c_5 = 0,\, c_3 = 0\},\\
&\{c_5 = c_5,\, c_2 = c_2,\, c_6 = 0,\, c_4 = 0,\, c_1 = 0,\, c_3 = 0\},\\
&\{c_3 = c_3,\, c_5 = c_5,\, c_6 = 0,\, c_4 = 0,\, c_1 = 0,\, c_2 = 0\},\\
&\{c_3 = c_3,\, c_4 = c_4,\, c_6 = 0,\, c_5 = 0,\, c_1 = 0,\, c_2 = 0\},\\
&\{c_4 = c_4,\, c_1 = c_1,\, c_6 = 0,\, c_5 = 0,\, c_3 = 0,\, c_2 = 0\},\\
&\{c_4 = c_4,\, c_2 = c_2,\, c_6 = 0,\, c_5 = 0,\, c_1 = 0,\, c_3 = 0\},\\
&\{c_3 = c_3,\, c_6 = c_6,\, c_4 = 0,\, c_5 = 0,\, c_1 = 0,\, c_2 = 0\},\\
&\{c_1 = c_1,\, c_6 = c_6,\, c_4 = 0,\, c_5 = 0,\, c_3 = 0,\, c_2 = 0\},\\
&\{c_5 = c_5,\, c_6 = c_6,\, c_4 = 0,\, c_1 = 0,\, c_3 = 0,\, c_2 = 0\}
\end{aligned}$$

```
> for k to 9 do evalc(subs(S_triv[k],[M[1],M[2],M[3]])) od;
```

$$[c_1\, c_2 = 0,\, 0 = 0,\, 0 = 0]$$
$$[0 = 0,\, 0 = 0,\, c_2\, c_5 = 0]$$
$$[0 = 0,\, c_3\, c_5 = 0,\, 0 = 0]$$
$$[c_3\, c_4 = 0,\, 0 = 0,\, 0 = 0]$$
$$[0 = 0,\, 0 = 0,\, c_1\, c_4 = 0]$$
$$[0 = 0,\, c_2\, c_4 = 0,\, 0 = 0]$$
$$[0 = 0,\, 0 = 0,\, c_3\, c_6 = 0]$$
$$[0 = 0,\, c_6\, c_1 = 0,\, 0 = 0]$$
$$[c_5\, c_6 = 0,\, 0 = 0,\, 0 = 0]$$

Now from the system M_k ($k = 1..3$) in each of nine cases we conclude that exactly one of c_k's is nonzero.

The discussion of the remaining system with the condition G_8 is not so easy. Its subsystem $\{G_1, G_2, G_5, G_8\}$ has nicely parameterized solution set $S1$ in terms of the seventh root (ζ) of -1:

```
> alias(zeta=RootOf(x^6-x^5+x^4-x^3+x^2-x+1,label=_L1));
```

$$\alpha, \zeta$$

```
> zeta=-alpha;
```

$$\zeta = -\alpha$$

```
> S1:=solve({G[1],G[2],G[5],G[8]},{seq(c[k],k=1..6)});
```

$$S1 := \{c_2 = -c_5,\, c_3 = -c_4,\, c_4 = c_4,\, c_5 = c_5,\, c_6 = c_6,\, c_1 = -c_6\},$$
$$\{c_4 = c_4,\, c_5 = c_5,\, c_6 = c_6,\, c_3 = -\frac{c_4}{\zeta^4},\, c_2 = -\zeta^2\, c_5,\, c_1 = \zeta\, c_6\}$$

```
> S1[1];
```

$$\{c_2 = -c_5,\, c_3 = -c_4,\, c_4 = c_4,\, c_5 = c_5,\, c_6 = c_6,\, c_1 = -c_6\}$$

```
> expand(subs(S1[1],[M[1],M[2],M[3]]));
```

$$[c_6\,\overline{(c_5)} + c_5\,\overline{(c_4)} - c_4\,\overline{(c_4)} + c_4\,\overline{(c_5)} + c_5\,\overline{(c_6)} = 0,$$
$$c_6\,\overline{(c_4)} - c_5\,\overline{(c_4)} - c_4\,\overline{(c_5)} + c_4\,\overline{(c_6)} - c_6\,\overline{(c_6)} = 0,$$
$$-c_6\,\overline{(c_4)} - c_5\,\overline{(c_5)} - c_4\,\overline{(c_6)} - c_5\,\overline{(c_6)} - c_6\,\overline{(c_5)} = 0]$$

```
> expand(subs(S1[1],-(M[1]+M[2]+M[3])));
```

$$c_4\,\overline{(c_4)} + c_6\,\overline{(c_6)} + c_5\,\overline{(c_5)} = 0$$

This immediately implies that $c_4 = c_5 = c_6 = 0$ and hence $c_1 = c_2 = c_3 = 0$ (from $S1_1$) but this is impossible because at least one of the Fourier coefficients should be nonzero.

Now we consider the second family $S1_2$. We substitute it into the system $\{M_1, M_2, M_3\}$ and show that the following combination of the equations (see below)

$$\zeta N3[1] + \zeta^5 N3[2] - \zeta^2 N3[3]$$

lead again to a remarkable condition

$$c_4\,\overline{(c_4)} + c_6\,\overline{(c_6)} + c_5\,\overline{(c_5)} = 0$$

which directly leads to the conclusion as in the first case.

```
> S1[2];
```

$$\{c_4 = c_4,\, c_5 = c_5,\, c_6 = c_6,\, c_3 = -\frac{c_4}{\zeta^4},\, c_2 = -\zeta^2\, c_5,\, c_1 = \zeta\, c_6\}$$

```
> subs(S1[2],[M[1],M[2],M[3]]);
```

$$[\zeta\, c_6\, \overline{(-\zeta^2\, c_5)} - \zeta^2\, c_5\, \overline{(-\frac{c_4}{\zeta^4})} - \frac{c_4\, \overline{(c_4)}}{\zeta^4} + c_4\, \overline{(c_5)} + c_5\, \overline{(c_6)} = 0,$$
$$\zeta\, c_6\, \overline{(-\frac{c_4}{\zeta^4})} - \zeta^2\, c_5\, \overline{(c_4)} - \frac{c_4\, \overline{(c_5)}}{\zeta^4} + c_4\, \overline{(c_6)} + c_6\, \overline{(\zeta\, c_6)} = 0,$$
$$\zeta\, c_6\, \overline{(c_4)} - \zeta^2\, c_5\, \overline{(c_5)} - \frac{c_4\, \overline{(c_6)}}{\zeta^4} + c_5\, \overline{(\zeta\, c_6)} + c_6\, \overline{(-\zeta^2\, c_5)} = 0]$$

```
> N:=(evala(expand(subs(S1[2],conjugate(zeta)^4*
> [M[1],M[2],M[3]]))));
```

$$N := [-\overline{\zeta}^6\, \zeta\, c_6\, \overline{(c_5)} + \zeta^2\, c_5\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \zeta^3\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \overline{(c_5)} + \overline{\zeta}^4\, c_5\, \overline{(c_6)} = 0,$$
$$-\zeta\, c_6\, \overline{(c_4)} - \overline{\zeta}^4\, \zeta^2\, c_5\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \zeta^3\, \overline{(c_5)} + \overline{\zeta}^4\, c_4\, \overline{(c_6)} + \overline{\zeta}^5\, c_6\, \overline{(c_6)} = 0,$$
$$\overline{\zeta}^4\, \zeta\, c_6\, \overline{(c_4)} - \overline{\zeta}^4\, \zeta^2\, c_5\, \overline{(c_5)} + \overline{\zeta}^4\, c_4\, \zeta^3\, \overline{(c_6)} + \overline{\zeta}^5\, c_5\, \overline{(c_6)} - \overline{\zeta}^6\, c_6\, \overline{(c_5)} = 0]$$

```
> N1:=factor((evala(expand(subs(S1[2],conjugate(zeta)^4*
> [M[1],M[2],M[3]])))));
```

$$N1 := [-\overline{\zeta}^6\, \zeta\, c_6\, \overline{(c_5)} + \zeta^2\, c_5\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \zeta^3\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \overline{(c_5)} + \overline{\zeta}^4\, c_5\, \overline{(c_6)} = 0,$$
$$-\zeta\, c_6\, \overline{(c_4)} - \overline{\zeta}^4\, \zeta^2\, c_5\, \overline{(c_4)} + \overline{\zeta}^4\, c_4\, \zeta^3\, \overline{(c_5)} + \overline{\zeta}^4\, c_4\, \overline{(c_6)} + \overline{\zeta}^5\, c_6\, \overline{(c_6)} = 0,$$
$$\overline{\zeta}^4\, (-c_6\, \overline{\zeta}^2\, \overline{(c_5)} + c_5\, \overline{\zeta}\, \overline{(c_6)} + \zeta\, c_6\, \overline{(c_4)} - \zeta^2\, c_5\, \overline{(c_5)} + c_4\, \zeta^3\, \overline{(c_6)}) = 0]$$

```
> N2:=evala(subs(conjugate(zeta)=zeta^(-1),N1));
```

$$N2 := [\zeta^2\, c_6\, \overline{(c_5)} + \zeta^2\, c_5\, \overline{(c_4)} + \%1\, c_4\, \overline{(c_4)} - c_4\, \zeta^3\, \overline{(c_5)} - \zeta^3\, c_5\, \overline{(c_6)} = 0,$$
$$-\zeta\, c_6\, \overline{(c_4)} + \zeta^5\, c_5\, \overline{(c_4)} + \%1\, c_4\, \overline{(c_5)} - c_4\, \zeta^3\, \overline{(c_6)} - \zeta^2\, c_6\, \overline{(c_6)} = 0,$$
$$-\zeta^3\, (c_6\, \zeta^5\, \overline{(c_5)} + c_5\, \%1\, \overline{(c_6)} + \zeta\, c_6\, \overline{(c_4)} - \zeta^2\, c_5\, \overline{(c_5)} + c_4\, \zeta^3\, \overline{(c_6)}) = 0]$$
$$\%1 := 1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^5$$

```
> N3:=subs(1-zeta+zeta^2-zeta^3+zeta^4-zeta^5=-zeta^6,N2);
```

$$N3 := [\zeta^2\, c_6\, \overline{(c_5)} + \zeta^2\, c_5\, \overline{(c_4)} - \zeta^6\, c_4\, \overline{(c_4)} - c_4\, \zeta^3\, \overline{(c_5)} - \zeta^3\, c_5\, \overline{(c_6)} = 0,$$
$$-\zeta\, c_6\, \overline{(c_4)} + \zeta^5\, c_5\, \overline{(c_4)} - \zeta^6\, c_4\, \overline{(c_5)} - c_4\, \zeta^3\, \overline{(c_6)} - \zeta^2\, c_6\, \overline{(c_6)} = 0,$$
$$-\zeta^3\, (c_6\, \zeta^5\, \overline{(c_5)} - c_5\, \zeta^6\, \overline{(c_6)} + \zeta\, c_6\, \overline{(c_4)} - \zeta^2\, c_5\, \overline{(c_5)} + c_4\, \zeta^3\, \overline{(c_6)}) = 0]$$

```
> evala(expand(zeta*N3[1]+zeta^5*N3[2]-zeta^2*N3[3]));
```

$$c_4\, \overline{(c_4)} + c_6\, \overline{(c_6)} + c_5\, \overline{(c_5)} = 0$$

Thus we have proved a nice joint characterization of regular heptagons and heptagrams in terms of the set

$$\{A_1 = 0,\ A_3[j,k] = 0 \text{ with increments } jk = 12, 24, 15, 13, 14\}$$

of cyclic invariants, one of degree one and five of degree three.

Of course one may conjecture a similar set of invariants for p–gon (p any odd prime).

Conjecture: The set $\{A_1 = 0,\ A_{(p-1)/2}[i_1, \ldots, i_{(p-3)/2}] = 0$, with all increments $0 < i_1 \leq i_2 \leq \ldots \leq i_{(p-3)/2} \leq p-1\}$ of cyclic invariants (one of degree one and others of degree $(p-1)/2$) characterizes regular p–gons and regular p–grams.

6. Acknowledgement

The authors wish to thank prof. D. Veljan for his helpful remarks.

Added in proof. After this paper was finished we learned (email by G. Björck, Sep. 2002.) that the topic of this paper is closely related to a conjecture by Enflo "There are no bi–unimodular sequences other than the classical ones (found by Gauss)." (cf. paper by G. Björck et al., *Fourier Transform on Z_p and "cyclic p–roots"*, Report, Matematiska Institutionen, Stockholms Universitet, 1989., No 9., p. 9; Uffe Haagerup, *MSRI–slides: Old and New Results on Spin Models (12/07/2000)*) and U Haagerup, *Orthogonal maximal abelian *-subalgebras of the $n \times n$ matrices and cyclic n-roots* in S. Doplicher et al., Operator algebras and quantum field theory, pCambridge, MA: International Press. 296–322 (1997).

Our *Theorems 1* and *2* give two different proofs to Enflo's conjecture (after Lovasz's) for $N = 5$. More explicit (compared to Björck's) counterexamples to the case $N = 7$ are given in *Section 4.2.*

References

[B] A. BJÖRNER, *Personal communication*, 1983.

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