GLASNIK MATEMATIČKI
Vol. 48(68)(2013), 31 – 48

ON THE DIOPHANTINE EQUATION $f(n) = u! + v!$

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ABSTRACT. In this paper, we show under the *abc* conjecture that the Diophantine equation $f(x) = u!+v!$ has only finitely many integer solutions (x, u, v) whenever $f(X) \in \mathbb{Q}[X]$ is a polynomial of degree at least three.

1. INTRODUCTION

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $d \geq 1$. The Diophantine equation

$$f(n) = u!$$

in integers n and $u \geq 0$ was investigated in many papers (see, for example, [2, 7]). Here, we look at the equation

$$f(n) = Au! + Bv!, \tag{1.1}$$

in integer unknowns n, $u \geq 0$, $v \geq 0$, where A, B are fixed nonzero integers. Our result is conditional upon the *abc* conjecture which we now recall. For a nonzero integer n put

$$N(n) = \prod_{p|n} p.$$

CONJECTURE 1.1 (The *abc* conjecture). *For all $\varepsilon > 0$, there exists $C = C_\varepsilon$ such that whenever a, b, c are nonzero integers with $a+b = c$ and $\gcd(a, b, c) = 1$, then*

$$\max\{|a|, |b|, |c|\} \leq C_\varepsilon N(abc)^{1+\varepsilon}.$$

Our result is the following.

2010 *Mathematics Subject Classification.* 11D85.
Key words and phrases. Factorials, polynomials, applications of the *abc* conjecture.

Theorem 1.2. *Assume that A, B are fixed nonzero integers and $f(x) \in \mathbb{Q}[x]$ is a polynomial of degree $d \geq 3$. Then, under the abc conjecture, equation (1.1) has only finitely many integer solutions (n, u, v) with $u \geq 0$, $v \geq 0$, except when $A+B=0$. In this last case, there are only finitely many solutions (n, u, v) with $u \neq v$.*

Particular cases of equation (1.1) have been studied before. For example, in was shown in [3] unconditionally that equation (1.1) has only finitely many solutions when $(A, B) = (1, 1)$, $(1, -1)$ and $f(x) = x^d$ and $d \geq 2$. Further, under the *abc* conjecture it was shown in [4] that equation (1.1) has only finitely many solutions (n, u, v) when $A = B = 1$ and

$$f(x) = c_0 x^d + c_1 x^{d-1} + c_2 x^{d-2} + \cdots + c_d \in \mathbb{Z}[x]$$

with $c_0 \neq 0$, and $c_1 = c_2 = 0$.

Throughout the paper, we use the Landau symbols O and o as well as the Vinogradov symbols $\ll$, $\gg$ and $\asymp$ with their regular meanings. Recall that $F = O(G)$, $F \ll G$ and $G \gg F$ are all equivalent and mean that the inequality $|F| \leq cG$ holds with some constant c, whereas $F \asymp G$ means that both inequalities $F \ll G$ and $G \ll F$ hold. The constants implied by these symbols depend on our data $f(x)$, A, B and some fixed $\varepsilon > 0$. Further, $F = o(G)$ means that $F/G \to 0$. For a polynomial $g(x) \in \mathbb{Q}[x]$, we write D_g for the minimal positive integer D such that $Dg(x) \in \mathbb{Z}[x]$.

2. Preliminary results

While the *abc* conjecture is an important ingredient in the proof of Theorem 1.2, it is not the only one. We shall need a few more facts about polynomials with rational coefficients and factorials which we collect in this section. For a polynomial $g(x) \in \mathbb{Q}[x]$ put

$$\mathcal{R}_g = \{p : g(n) \equiv 0 \pmod p \text{ does not admit an integer solution } n\}.$$

The following result is [1, Lemma 3].

Lemma 2.1. *If $g(x) \in \mathbb{Q}[x]$ is irreducible of degree $d \geq 2$, then $\mathcal{R}_g$ has a positive (relative) density r_g. Further, r_g is a rational number in the interval $[(d-1)/d!, 1 - 1/d]$.*

For a real number y we write $\lfloor y \rfloor$ and $\{y\}$ for the integer and fractional part of y, respectively. The next result is a particular case ($J = 1$) of [5, Lemma 5.1].

Lemma 2.2. *Fix $\varepsilon > 0$. Then there exists a constant $c > 0$ such that for $y \leq x$ there are*

$$\sigma_1 \pi(y) + O\left((y^{1-c(\log y)^2/\log x} + y^{3/2+\varepsilon} x^{-1/2})(\log x)^4\right)$$

primes $p \leq y$ such that $\{x/p\} < \sigma_1$.

LEMMA 2.3. *Let*

$$u! = \prod_{p \le u} p^{\alpha_p(u)}. \tag{2.1}$$

(i) *If* $g(x) \in \mathbb{Q}[x]$ *is irreducible of degree* $d \ge 2$, *then for* $u > u_0$ *we have*

$$\prod_{\substack{p \le u \\ p \in \mathcal{R}_g}} p^{\alpha_p(u)} > u!^{\delta}, \qquad \textit{where} \qquad \delta = \frac{d-1}{3d!}.$$

(ii) *If* $e > 1$ *is an integer, then for* $u > u_0$ *we have*

$$\prod_{\substack{p \le u \\ \alpha_p(u) \not\equiv 0 \pmod e}} p^{\alpha_p(u)} > u!^{\delta}, \qquad \textit{where} \qquad \delta = \frac{e-1}{3e}.$$

PROOF. Observe that

$$\alpha_p(u) = \left\lfloor \frac{u}{p} \right\rfloor \qquad \text{for all primes} \qquad p \in (\sqrt{u}, u].$$

Thus, if $\mathcal{S} \subset (\sqrt{u}, u]$ is a set of primes, then

$$\begin{aligned} \log \prod_{p \in \mathcal{S}} p^{\alpha_p(u)} &= \sum_{p \in \mathcal{S}} \alpha_p(u) \log p = \sum_{p \in \mathcal{S}} \left\lfloor \frac{u}{p} \right\rfloor \log p \\ &= \sum_{p \in \mathcal{S}} \left(\frac{u}{p} + O(1) \right) \log p = u \sum_{p \in \mathcal{S}} \frac{\log p}{p} + O\left(\sum_{p \in \mathcal{S}} \log p \right) \qquad (2.2) \\ &= u \sum_{p \in \mathcal{S}} \frac{\log p}{p} + O(u). \end{aligned}$$

Assume now that $\varepsilon > 0$ is arbitrarily small and that $\mathcal{S} \subset (\sqrt{u}, u^{1-\varepsilon})$ has a relative density $\eta > 0$ in $(\sqrt{u}, u^{1-\varepsilon})$; that is, if we put

$$\mathcal{S}(t) = \#\{p \le t : p \in \mathcal{S}\},$$

then the estimate

$$\mathcal{S}(t) = (\eta + o(1))\pi(t) \qquad \text{holds for} \qquad t \in (\sqrt{u}, u^{1-\varepsilon}).$$

Then, by the Prime Number Theorem, the estimate

$$\mathcal{S}(t) = \frac{\eta t}{\log t} + o\left(\frac{t}{\log u} \right) \qquad \text{holds for} \qquad t \in (\sqrt{u}, u^{1-\varepsilon})$$

as $u \to \infty$. By partial summation, we have

$$
\begin{aligned}
\sum_{p\in\mathcal{S}} \frac{\log p}{p} &= \int_{\sqrt{u}}^{u^{1-\varepsilon}} \frac{\log t}{t} d\left(\mathcal{S}(t)\right)\\
&= \frac{(\log t)\mathcal{S}(t)}{t}\Big|_{t=\sqrt{u}}^{t=u^{1-\varepsilon}} + \int_{\sqrt{u}}^{u^{1-\varepsilon}} \frac{(\log t-1)\mathcal{S}(t)dt}{t^2}\\
&= O(1) + \int_{\sqrt{u}}^{u^{1-\varepsilon}} \frac{(\log t-1)}{t^2}\left(\frac{\eta t}{\log t} + o\left(\frac{t}{\log u}\right)\right) dt \qquad (2.3)\\
&= O(1) + \eta\int_{\sqrt{u}}^{u^{1-\varepsilon}} \frac{\log t-1}{t\log t} dt + o\left(\int_{\sqrt{u}}^{u^{1-\varepsilon}} \frac{dt}{t}\right)\\
&= O(1) + \eta\left(\log t - \log\log t\right)\Big|_{t=\sqrt{u}}^{t=u^{1-\varepsilon}} + o(\log u)\\
&= \left((1/2-\varepsilon)\eta + o(1)\right)\log u \qquad \text{as} \qquad u\to\infty.
\end{aligned}
$$

Taking $\varepsilon > 0$ sufficiently small and then $u > u_0$ sufficiently large, we deduce from (2.2) and (2.3) that

$$\log \prod_{p\in\mathcal{S}} p^{\alpha_p(u)} \ge (\eta/3)u\log u > \log\left(u!^{\eta/3}\right),$$

so

$$\prod_{p\in\mathcal{S}} p^{\alpha_p(u)} > u!^{\eta/3}.$$

Now (i) follows with

$$\mathcal{S} = \mathcal{R}_g \cap (\sqrt{u}, u^{1-\varepsilon})$$

and Lemma 2.1 according to which η exists and satisfies $\eta \ge (d-1)/d!$. Thus, we can take $\delta = \eta/3 = (d-1)/3d!$. For (ii) we take

$$\mathcal{T} = \{\sqrt{u} < p \le u : \alpha_p(u) \equiv 0 \pmod e\} \quad \text{and} \quad \mathcal{S} = \{\sqrt{u} < p < u^{1-\varepsilon}\}\backslash\mathcal{T},$$

and note that $p \in \mathcal{T}$ if and only if $e \mid \lfloor u/p \rfloor$, which is equivalent to the inequality $\{(u/e)/p\} < 1/e$. By Lemma 2.2, $\mathcal{T}$ has relative density $1/e$ in $(\sqrt{u}, u^{1-\varepsilon})$ for any $\varepsilon > 0$, therefore $\mathcal{S}$ has relative density $\eta = (e-1)/e$ in $(\sqrt{u}, u^{1-\varepsilon})$, which leads to (ii) with $\delta = \eta/3 = (e-1)/3e$. $\square$

3. Unconditional results on equation (1.1)

In equation (1.1), we shall assume that $n \ge 0$. The case $n \le 0$ follows by replacing $f(x)$ by $f(-x)$. We shall also assume that the leading term of $f(x)$ is positive, otherwise we replace the triple $(f(x), A, B)$ by $(-f(x), -A, -B)$. We also assume that $u \le v$. If $u = v$, we then get

$$f(n) = (A+B)u!$$

If $A + B = 0$, then $f(n) = 0$. Hence, we get infinitely many solutions (n, u, v) but they all have $u = v$ and $f(n) = 0$. From now on, we do not consider such solutions. If $A + B \neq 0$, then we replace $f(x)$ by $g(x) = f(x)/(A + B) \in \mathbb{Q}[x]$ and equation (1.1) becomes

$$g(n) = v!$$

This equation has only finitely many solutions for $d \geq 2$ under the *abc* conjecture by the main result from [7]. So, from now on, we assume that $u < v$. We multiply both sides of equation (1.1) by D_f and get

$$D_f f(x) = A_1 u! + B_1 v!,$$

where $(A_1, B_1) = (D_f A, D_f, B)$. Hence, we may replace $f(x)$ by $D_f f(x)$, and (A, B) by (A_1, B_1), and therefore assume that $f(x) \in \mathbb{Z}[x]$.

Let K be any positive integer. If $u \leq K$, then we can give u the values $u = 0, 1, \ldots, K$, and replace $f(x) \in \mathbb{Q}[x]$ by $g(x) = (f(x) - Au!)/B \in \mathbb{Q}[x]$, so equation (1.1) reduces to equation

$$g(n) = v!,$$

which was already treated in [7]. Thus, only solutions (n, u, v) of equation (1.1) with a large u are of interest. To study them, it turns out that the factorization of $f(x) \in \mathbb{Z}[x]$ plays an important role. So, let us write

$$f(x) = f_1(x)^{e_1} \cdots f_s(x)^{e_s} \in \mathbb{Z}[x],$$

where $f_1(x), \ldots, f_s(x)$ are non associated irreducible polynomials of positive leading terms and positive degrees $d_1, \ldots, d_s$, respectively, and $e_1 \geq e_2 \geq \cdots \geq e_s \geq 1$. We have the following unconditional result concerning solutions of equation (1.1).

LEMMA 3.1. *Assume that $d \geq 2$. In equation (1.1) with $u < v$, the number u is bounded in any of the following instances:*

(i) $e_s > 1$;
(ii) $s = 1$.

PROOF. (i) Assume that u is sufficiently large such that $u^{5/6} > 3|A|$. Take $\mathcal{I} = (u^{5/6}, 3u^{5/6})$ and let $p_1 < p_2 < \cdots < p_t$ be all primes in $\mathcal{I}$. Since $p_j^2 > u$ for $j = 1, \ldots, t$, it follows that

$$\alpha_{p_j}(u) = \left\lfloor \frac{u}{p_j} \right\rfloor \qquad \text{for} \qquad j = 1, \ldots, t,$$

where $\alpha_{p_j}(u)$ is the exponent of p_j in the factorization of $u!$ (see formula (2.1)). Further, observe that for $j \in \{1, \ldots, t-1\}$, we have

$$\frac{u}{p_j} - \frac{u}{p_{j+1}} = \frac{u(p_{j+1} - p_j)}{p_j p_{j+1}} = O\left(\frac{p_{j+1} - p_j}{u^{2/3}}\right) = o(1)$$

as $u \to \infty$, where we used the known fact that $p_{j+1} - p_j = O(p_j^{0.6}) = O(u^{0.5})$ (see, for example, [6]). Thus, for large u, the numbers

$$\left\lfloor \frac{u}{p_1} \right\rfloor, \ldots, \left\lfloor \frac{u}{p_t} \right\rfloor$$

cover all the integers interval

$$\left[\lfloor u^{1/6}/3 \rfloor + 1, \lfloor u^{1/6} \rfloor - 1\right].$$

By Bertand's postulate, the above interval contains a prime if u is sufficiently large. Any such prime, call it q, satisfies $q > u^{1/6}/3$, so for large u, the prime q is coprime to $e_1 \cdots e_s$. Thus, if $p_j \in \mathcal{I}$ is such that $\alpha_{p_j}(u) = q$, then $p_j \mid A + B(u+1) \cdots v$, and since $p_j > u^{5/6}/3 > |A|$, it follows that $p_j \nmid A$, so $p_j \nmid B(u+1) \cdots v$. This shows that

$$v - u < p_j \leq 3u^{5/6}.$$

However, $p \| u!$ for all $p \in (u/2, u)$, therefore, by a similar argument, we have

$$\prod_{u/2 < p < u} p \mid A + B(u+1) \cdots v.$$

By the Prime Number Theorem, the number in the left–side above is of size $\exp((1/2 + o(1))u)$ as $u \to \infty$, and in particular it exceeds $\exp(u/3)$ for large enough u. However, the number on the right–side above is nonzero and its size is smaller than

$$2 \max\{|A|, |B|\} v^{v-u} = \exp(O(u^{5/6} \log u).$$

Putting together the above bounds we get

$$\exp(u/3) < \exp(O(u^{5/6} \log u)),$$

so $u = O(1)$, which is what we wanted. This takes case of (i).

(ii) Follows from (i) if $e_1 > 1$. If $e_1 = 1$, then $d_1 = d > 1$, and now (ii) follows from Lemma 2.1 and the fact that $r_{f_1} > 0$, which implies, in particular, that there are infinitely many primes p in $\mathcal{R}_{f_1}$ and obviously the smallest such cannot divide $f(n)$, therefore it exceeds u. ∎

4. The proof of Theorem 1.2

Given any positive constant K, there are only finitely many pairs of integers (u, v) with $0 \leq u < v \leq K$, so only finitely many elements in the set

$$\mathcal{F}_K = \{Au! + Bv! : 0 \leq u < v \leq K\}.$$

Since for a fixed z the equation $f(n) = z$ has at most d solutions, it follows that there are only finitely many n such that $f(n) \in \mathcal{F}_K$. Discarding such "small" solutions, from now on, we may assume that v is as large as we wish. In particular, we assume that v is sufficiently large such that

$$|Au! + Bv!| \geq |B|v! - |A|(v-1)! > v!/2.$$

The last inequality holds when $v > 2|A|/(2|B|-1)$. Thus,

$$v!/2 < |Au! + Bv!| < (|A| + |B|)v!. \tag{4.1}$$

Write

$$f(x) = c_0x^d + \cdots + c_d,$$

where $c_0, \dots, c_d \in \mathbb{Z}$ and $c_0 > 0$. Assume that n is sufficiently large such that the estimates

$$(c_0/2)n^d < |f(n)| < 2c_0n^d \tag{4.2}$$

hold. Comparing estimates (4.1) and (4.2), we get that

$$(c_0/2)n^d < |f(n)| = |Au! + Bbv!| < (|A| + |B|)v!$$
$$v!/2 < |Au! + Bv!| = |f(n)| < 2c_0n^d,$$

so $n^d \asymp v!$. In particular, n is as large as we wish. Since also $n \geq 0$, it follows that $f(n) > 0$, so $Au! + Bv! > 0$. Since $v > u$ and v is large, the sign of $Au! + Bv!$ is the same as the sign of B. Thus, we assume that $B > 0$. Finally, recall that we also assume that u is as large as we wish.

To continue, we distinguish several cases. We let $\varepsilon_0 > 0$ be some small number depending on d to be determined later.

4.1. *Solutions with small u.*

LEMMA 4.1. *Assume that $d \geq 2$. Under the abc conjecture, there are only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$ to equation* (1.1) *with $u! < n^{d-1-\varepsilon_0}$.*

PROOF. We multiply both sides of equation (1.1) by $d^dc_0^{d-1}$ obtaining

$$(dc_0n)^d + dc_1(dc_0n)^{d-1} + \cdots + c_dd^dc_0^{d-1} = (d^dc_0^{d-1}A)u! + (d^dc_0^{d-1}B)v!$$

Put $m = dc_0n + c_1$, $A_1 = d^dc_0^{d-1}A$, $B_1 = d^dc_0^{d-1}B$. Then the above relation can be rewritten as

$$m^d + (g(m) - A_1u!) = B_1v!, \tag{4.3}$$

where $g(x) \in \mathbb{Z}[x]$ has degree at most $d-2$. Since n and c_0d are positive and n is large, it follows that $m > 0$. Further,

$$|g(m)| \ll m^{d-2} \ll n^{d-2}, \qquad \text{so} \qquad |g(m) - A_1u!| \ll n^{d-2} + u! \ll n^{d-1-\varepsilon_0}.$$

We treat equation (4.3) as an *abc* equation, with

$$a = m^d, \qquad b = g(m) - A_1u!, \qquad \text{and} \qquad c = B_1v!.$$

We first need to insure that none of the above amounts a, b or c is 0. If $a = 0$, then $m = 0$, which is not the case. If $b = 0$, then $m^d = B_1v!$. Since the interval $(v/2, v)$ contains a prime for $v > 2$, it follows that if $v > 2B_1$, then $B_1 v!$ is divisible by a prime $p \in (v/2, v)$, but not by its square, so the equation $m^d = B_1 v!$ is not possible for such values of v because $d \geq 2$. Thus, $v = O(1)$, and there are only finitely many such solutions. Finally, $c \neq 0$, because $dc_0 B \neq 0$. Let $\Delta = \gcd(a, b, c)$ and put $a_1 = a/\Delta$, $b_1 = b/\Delta$, and $c_1 = c/\Delta$. Then

$$a_1 + b_1 = c_1. \tag{4.4}$$

We apply the *abc* conjecture with some small $\varepsilon > 0$ to equation (4.4) getting

$$\frac{m^d}{\Delta} = a_1 \leq C_\varepsilon N\left(a_1 b_1 c_1\right)^{1+\varepsilon}.$$

Clearly,

$$\begin{aligned}
&N(a_1) \leq N(m) \leq m \ll n; \\
&N(b_1) \leq |b_1| = \frac{|b|}{\Delta} \ll \frac{n^{d-1-\varepsilon_0}}{\Delta}; \\
&N(c_1) = N(B_1 v!) \ll \prod_{p \leq v} p = \exp((1+o(1))v) = v!^{O(1/\log v)} = n^{O(1/\log v)}.
\end{aligned}$$

Thus,

$$N(a_1 b_1 c_1) \ll \frac{n^{d-\varepsilon_0+O(1/\log v)}}{\Delta}. \tag{4.5}$$

We thus get that

$$\frac{n^d}{\Delta} \ll \frac{m^d}{\Delta} \ll_\varepsilon \left(\frac{n^{d-\varepsilon_0+O(1/\log v)}}{\Delta}\right)^{1+\varepsilon}$$

leading to

$$n^{d-(d-\varepsilon_0+O(1/\log v))(1+\varepsilon)} = O_\varepsilon(1). \tag{4.6}$$

Assume that κ_1 is the constant implied by the above O–symbol. We first choose a sufficiently large v such that $\kappa_1/\log v < \varepsilon_0/2$ (that is, $v > e^{2\kappa_1/\varepsilon_0}$), and then we choose $\varepsilon > 0$ sufficiently small such that

$$(d - \varepsilon_0/2)(1+\varepsilon) < d - \varepsilon_0/3.$$

Then the exponent of n in the left–hand side of inequality (4.6) exceeds $\varepsilon_0/3$. This in turn implies that $n = O(1)$, so $v = O(1)$. So, there are only finitely solutions (n, u, v) with $n \geq 0$ and $u < v$ in this case, and the lemma follows. $\square$

4.2. *Solutions with large u.*

LEMMA 4.2. *Assume that $d \geq 2$. Under the abc conjecture, there are only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$ of equation (4.3) with $u! > n^{d-1+\varepsilon_0}$.*

PROOF. Here, we work with the *abc* equation

$$m^d + g(m) = A_1 u! + B_1 v!,$$

where the notations are from the proof of Lemma 4.1. In particular, we have $m = dc_0 n + c_1$. We take

$$a = m^d, \qquad b = g(m), \qquad \text{and} \qquad c = A_1 u! + B_1 v!.$$

We already saw that the case $a = 0$ is impossible. In fact, $a > 0$ in our case. The case $b = 0$ leads to $g(m) = 0$. If $g(x)$ is not the zero polynomial, then $m = O(1)$, so $n = O(1)$ and again $v = O(1)$. If $g(x) = 0$, then $f(x) = f_1(x)^{e_1}$, with $f_1(x) = dc_0 x + c_1$ being linear, so $e_1 = d > 1$. Lemma 3.1 (i) now implies that $u = O(1)$, which in turn leads to only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$ under the *abc* conjecture as in [7]. This deals with the case $b = 0$. Finally, the case $c = 0$ is not possible because we are assuming that inequality (4.1) holds. So, we put, as in the proof of Lemma 4.1, $\Delta = \gcd(a, b, c)$, $a_1 = a/\Delta$, $b_1 = b/\Delta$, and $c_1 = c/\Delta$, and apply the *abc* conjecture to the equation

$$a_1 + b_1 = c_1$$

with some small $\varepsilon > 0$, getting

$$\frac{m^d}{\Delta} = a_1 \ll_\varepsilon N(a_1 b_1 c_1)^{1+\varepsilon}.$$

We now have

$$\begin{aligned}
N(a_1) &\leq N(m) \leq m \ll n;\\
N(b_1) &\leq |b_1| \leq \frac{|g(m)|}{\Delta} \ll \frac{m^{d-2}}{\Delta} \ll \frac{n^{d-2}}{\Delta};\\
N(c_1) &\leq N(u!)(|A_1| + B_1(u+1)\cdots v) \ll \left(\prod_{p<v} p\right)(v!/u!)\\
&\leq \exp((1+o(1))v) n^{1-\varepsilon_0} = v!^{O(1/\log v)} n^{1-\varepsilon_0}\\
&= n^{1-\varepsilon_0+O(1/\log v)},
\end{aligned}$$

where we used the fact that $v! \ll n^d$ and $u! \gg n^{d-1+\varepsilon_0}$ to conclude that $v!/u! \ll n^{1-\varepsilon_0}$. Thus,

$$N(a_1 b_1 c_1) \ll \frac{n^{d-\varepsilon_0+O(1/\log v)}}{\Delta},$$

which is the same as inequality (4.5). Now the argument finishes as in the proof of Lemma 4.1. $\square$

From now on, we assume that $\log u!/\log n \in [d-1-\varepsilon_0, d-1+\varepsilon_0]$. Now we look at the factorization of $f(x)$. The case $s=1$ leads to $u=O(1)$ by Lemma 3.1 (ii), so to only finitely many solutions (n,u,v) with $n \geq 0$ and $u<v$ of equation (1.1). From now on, we assume that $s \geq 2$.

4.3. *The case $s \geq 3$.*

LEMMA 4.3. *Under the abc conjecture, there are only finitely many solutions (n,u,v) with $n \geq 0$ and $u<v$ of equation* (1.1) *when $s \geq 3$.*

PROOF. Assume that $s \geq 3$ and let

$$p(x)=f_1(x)^{e_1}, \qquad q(x)=f_2(x)^{e_2}, \qquad \text{and} \qquad r(x)=\prod_{j=3}^{s} f_j(x)^{e_j}.$$

Write

$$a(n)=u_1v_1, \qquad b(n)=u_2v_2, \qquad \text{and} \qquad c(n)=u_3v_3,$$

where $u_1u_2u_3=u!$. Write

$$\begin{aligned} p(x) &= p_0x^d+p_1x^{d-1}+\cdots; \\ q(x) &= q_0x^e+q_1x^{e-1}+\cdots; \\ r(x) &= r_0x^f+r_1x^{f-1}+\cdots. \end{aligned}$$

Choose integers U,V,W not all zero such that

$$\begin{aligned} dU+eV+fW &= 0; \\ (p_1/p_0)U+(q_1/q_0)V+(r_1/r_0)W &= 0. \end{aligned}$$

Not all numbers U,V,W are positive. In fact, at least one is positive and one is negative since d,e,f are all positive. Up to relabeling the variables (U,V,W) and simultaneously changing the signs of (U,V,W), we may assume that $U>0$, $V<0$ and $W \leq 0$. Raise the relations

$$\begin{aligned} n^d+(p_1/p_0)n^{d-1}+\cdots &= u_1v_1/p_0; \\ n^e+(q_1/q_0)n^{e-1}+\cdots &= u_2v_2/q_0; \\ n^f+(r_1/r_0)n^{f-1}+\cdots &= u_3v_3/r_0 \end{aligned}$$

to powers U, $-V$ and $-W$ respectively, and note that

$$\begin{aligned} (u_1v_1/p_0)^U-(u_2v_2/q_0)^{-V}(u_3v_3/r_0)^{-W} &= (n^d+(p_1/p_0)n^{d-1}+\cdots)^U \\ -(n^e+(q_1/q_0)n^{e-1}+\cdots)^{-V} &\times (n^f+(r_1/r_0)n^{f-1}+\cdots)^{-W}=s(n), \end{aligned}$$

where $s(x) \in \mathbb{Q}[x]$ has degree $\leq dU-2$. We apply the *abc* conjecture to the above equation with

$$a=\Delta_1(u_1v_1/p_0)^U, \qquad b=-\Delta_1(u_2v_2/q_0)^{-V}(u_3v_3/r_0)^{-W}, \qquad c=\Delta_1 s(n),$$

where $\Delta_1 = p_0^U q_0^{-V} r_0^{-W}$. Next, we need to study the greatest common divisor of a and b. Note that

$$a = q_0^{-V} r_0^{-W} p(n)^U, \qquad b = -p_0^U q(n)^{-V} r(n)^{-W}.$$

Since $p(x)$, $q(x)$, and $r(x)$ are coprime any two as polynomials in $\mathbb{Q}[x]$, it follows that $\gcd(p(n), q(n)) = O(1)$ and $\gcd(p(n), r(n)) = O(1)$. Hence, we conclude that $\Delta = \gcd(a, b) = O(1)$. We now write $a = \Delta a_1$, $b = \Delta b_1$, and $c = \Delta c_1$, and apply the abc conjecture to the equation

$$a_1 + b_1 = c_1.$$

Since n is large and p_0, q_0, r_0 are positive, it follows that $a > 0$ and $b < 0$. Observe that $a_1 = a/\Delta \gg n^{dU}$. We thus get

$$n^{dU} \ll a_1 \ll_\varepsilon N(a_1 b_1 c_1)^{1+\varepsilon} \ll N(abc)^{1+\varepsilon}.$$

Clearly,

$$N(ab) \ll N(u!(A + B(u+1)\cdots v)) \ll N(u!)(v!/u!) \le n^{O(1/\log v)} n^{1+\varepsilon_0},$$

whereas

$$N(c) \le |c| \ll |s(n)| \ll n^{dU-2}.$$

Thus,

$$N(abc) \ll n^{dU-1+\varepsilon_0+O(1/\log v)}.$$

Hence, we get

$$n^{dU} \ll_\varepsilon n^{(dU-1+\varepsilon_0+O(1/\log v))(1+\varepsilon)},$$

and, as in the conclusion of the proofs of Lemma 4.1 and 4.2, we get that $n = O(1)$ provided that v is sufficiently large and $\varepsilon > 0$ is sufficiently small, which completes the proof of this lemma. ∎

So, from now on we assume that $s = 2$.

4.4. *The case $s = 2$ and $d_1 > 1$, $d_2 > 1$.*

Lemma 4.4. *Under the abc conjecture, equation* (1.1) *has only finitely many solutions (n, u, v) with $n \ge 0$ and $u < v$ in the case when $s = 2$ and $d_1 > 1$, $d_2 > 1$.*

Proof. Assume that $d_1 > 1$ and $d_2 > 1$. Write

$$p(x) = f_1(x)^{e_1}, \qquad \text{and} \qquad q(x) = f_2(x)^{e_2}.$$

Write also, as before,

$$p(n) = u_1 v_1, \qquad \text{and} \qquad q(n) = u_2 v_2,$$

where

$$u_1 = \gcd(p(n), u!), \qquad \text{and} \qquad u_2 = u!/u_1,$$

and $v_1 = p(n)/u_1,\ v_2 = q(n)/u_2$. Since $\gcd(p(n), q(n)) = O(1)$, it follows that $\gcd(v_1, v_2) = O(1)$. Since

$$v_1 v_2 \le |A| + B(u+1)\cdots v \ll v!/u! \ll n^{1+\varepsilon_0},$$

it follows that there exists $j \in \{1,2\}$ such that $v_j \ll n^{(1+\varepsilon_0)/2}$. To fix the notation, say $j = 1$, and write

$$p(x) = p_0 x^{d_1 e_1} + p_1 x^{d_1 e_1 - 1} + \cdots + p_{d_1 e_1}.$$

The equation $p(n) = u_1 v_1$ is then

$$p_0 n^{d_1 e_1} + p_1 n^{d_1 e_1 - 1} + \cdots + p_{d_1 e_1} = u_1 v_1.$$

Multiplying both sides of it by $U = p_0^{d_1 e_1 - 1}(d_1 e_1)^{d_1 e_1}$ and making the substitution $m = p_0 e_1 d_1 n + p_1$, we get

$$m^{d_1 e_1} + g(m) = U u_1 v_1,$$

where $g(x) \in \mathbb{Z}[x]$ is of degree at most $d_1 e_1 - 2$. Since n is large, and $p_0,\ e_1,\ d_1$ are positive, it follows that $m > 0$. We apply the *abc* conjecture to the above equation with

$$a = m^{d_1 e_1}, \qquad b = g(m), \qquad \text{and} \qquad c = U u_1 v_1.$$

We first check that $abc \ne 0$. The case $a = 0$, leads to $m = 0$, which is not the case we are considering. If $b = 0$, then either $g(m) = 0$ but $g(x)$ is not the constant zero polynomial, so $m = O(1)$, therefore $n = O(1)$, so only finitely many solutions (n, u, v) with $n \ge 0$ and $u < v$, or $g(x)$ is the constant zero polynomial but in this last case $p(x) = f_1(x)^{e_1}$ and $f_1(x) = p_0 e_1 d_1 x + p_1$ is linear, so $d_1 = 1$, which is not the case we are considering. Therefore $b \ne 0$. The fact that $c \ne 0$ is obvious. We let $\Delta = \gcd(a, b, c)$ and write $a = \Delta a_1,\ b = \Delta b_1$, and $c = \Delta c_1$. We apply the *abc* conjecture to the equation

$$a_1 + b_1 = c_1$$

with some $\varepsilon > 0$ and get

$$\frac{n^{d_1 e_1}}{\Delta} \ll \frac{m^{d_1 e_1}}{\Delta} = a_1 \ll_\varepsilon N(a_1 b_1 c_1)^{1+\varepsilon}.$$

Now

$$\begin{aligned}
&N(a_1) \le N(m) \le m \ll n;\\
&N(b_1) \le |b_1| = \frac{|b|}{\Delta} \ll \frac{|g(m)|}{\Delta} \ll \frac{m^{d_1 d_2 - 2}}{\Delta} \ll \frac{n^{d_1 d_2 - 2}}{\Delta};\\
&N(c_1) = N(U u_1 v_1) \ll \left(\prod_{p \le v} p\right) v_1 = \exp((1 + o(1))v) n^{(1+\varepsilon_0)/2}\\
&\qquad = v!^{O(1/\log v)} n^{(1+\varepsilon_0)/2} = n^{(1+\varepsilon_0)/2 + O(1/\log v)},
\end{aligned}$$

as $v \to \infty$. Thus,

$$N(a_1b_1c_1) \ll \frac{n^{d_1d_2-1+(1+\varepsilon_0)/2+O(1/\log v)}}{\Delta},$$

which leads to

$$n^{d_1d_2} \ll n^{(d_1d_2-1+(1+\varepsilon_0)/2+O(1/\log v))(1+\varepsilon)}.$$

This implies that $n = O(1)$ provided that $\varepsilon_0 \leq 1/2$, v is sufficiently large, say such that $\kappa_1/\log v < 1/10$, and then ε is chosen to be sufficiently small with respect to d_1e_1. This completes the proof of the lemma. $\square$

From now on, we assume that $s = 2$ and $d_1 = 1$. Discarding the condition that $e_1 \geq e_2$, we may assume that $f_1(x) = p_0x + p_1$ for some integers $p_0 > 0$ and p_1. We multiply both sides of equation (1.1) by p_0^{d-1}, replace (A, B) by $(A_1, B_1) = (p_0^{d-1}A, p_0^{d-1}B)$ and make the substitution $y = p_0x + p_1$. Thus, we may assume that $f_1(x) = x$. With the notations from the preceding subsection, we have

$$p(x) = x^{e_1}, \qquad \text{and} \qquad q(x) = f_2(x)^{e_2}.$$

Thus, we write

$$n^{e_1} = u_1v_1, \qquad \text{and} \qquad q(n) = u_2v_2, \tag{4.7}$$

where $u_1 = \gcd(p(n), u!)$, $u_2 = u!/u_1$, and $v_1v_2 = |A + B(u+1)\cdots(v+1)|$.

The remaining of the argument is split into two cases according to whether $e_1 > 1$ or $e_1 = 1$, respectively.

4.5. *The case when $s = 2$, $d_1 = 1$, and $e_1 > 1$.*

LEMMA 4.5. *Under the abc conjecture, equation* (1.1) *has only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$ in the case when $s = 2$, $d_1 = 1$, and $e_1 > 1$.*

PROOF. In this case, we may assume that $e_2 = 1$, otherwise Lemma 3.1 (i) implies right away that $u = O(1)$; hence only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$.

Assume next that $d_2 > 1$. Let k be a large integer, put $\ell = ke_1 + 1$ and consider primes $p \in (u/\ell + 1, u/\ell)$ in $\mathcal{R}_{f_2}$. Since $\mathcal{R}_{f_2}$ has positive density by Lemma 2.1, it follows that p exists if $u > u_0$. Assume further that $u > (\ell+1)^2$. Then $p > \sqrt{u}$, therefore $\alpha_p(u) = \lfloor u/p \rfloor = \ell = ke_1 + 1$. Since

$$n^{e_1} f_2(n) = Au! + Bv! = u!(A + B(u+1)\cdots v),$$

the exponent of p in $u!$ is not a multiple of e_1 and $p \nmid f_2(n)$, it follows that $p \mid A + B(u+1)\cdots v$. If $u > |A|\ell$, then $p > |A|$, so $p \nmid A$, therefore $p \nmid$

$B(u+1)\cdots v$. Hence, $v-u<u/\ell$. In particular, $v<2u$. However, since $v!\asymp n^d$, and $n^{d-1-\varepsilon_0}\ll u!\ll n^{d-1+\varepsilon_0}$, it follows that

$$(2u)^{v-u}>v^{v-u}\geq v!/u!\gg n^{1-\varepsilon_0}\gg u!^{\frac{1-\varepsilon_0}{d-1+\varepsilon_0}}\geq\left(\frac{u}{e}\right)^{\frac{u(1-\varepsilon_0)}{d-1+\varepsilon_0}}.$$

If $\varepsilon_0<1/(d+1)$, then the exponent on the right above exceeds u/d. Thus,

$$(2u)^{v-u}>\left(\frac{u}{e}\right)^{u/d},$$

which for $u>u_0$ implies that $v-u>u/(d+1)$. However, this last inequality contradicts $v-u<u/\ell$ with $\ell=ke_1+1$, provided that k is sufficiently large (say, $k>d/e_1$). This argument shows that if $d_2>1$, then $u=O(1)$, so we have only finitely many solutions (n,u,v) with $n\geq 0$ and $u<v$.

Assume now that $d_2=e_2=1$. Then $d_1=d-1$ and $q(x)=q_0x+q_1$. Since n and q_0 are positive and n is large, it follows that $q(n)>0$. Lemma 2.3 (ii) shows that if $u>u_0$, then

$$u_2>u!^{\delta}\gg n^{\delta(d-1-\varepsilon_0)},\qquad\text{where}\qquad\delta=\frac{d_1-1}{3d_1}=\frac{d-2}{3(d-1)}.$$

Put $\delta_1=(d-1)\delta=(d-2)/3$. Thus,

$$v_2=q(n)/u_2\ll n^{1-\delta_1+\delta\varepsilon_0}<n^{1-\delta_1+d\varepsilon_0}. \tag{4.8}$$

If $\delta_1\geq 1$, then since $v_2\geq 1$, we get that $n=O(1)$, and the lemma is proved. So, assume that $\delta_1<1$. In particular, $d\in\{3,4\}$, although we shall not need this information. Next write

$$v_1=v_1'v_1'',\qquad\text{where}\qquad v_1'=\prod_{\substack{p^{\delta_p}\|v_1\\ p\leq u}}p^{\delta_p}\qquad\text{and}\qquad v_1''=v_1/v_1'.$$

From the equation

$$n^{d-1}=u_1v_1=(u_1v_1')v_1'',$$

and the fact that all prime factors of v_1'' exceed u, we get $\gcd(u_1v_1',v_1'')=1$, therefore $v_1''=w_1^{d-1}$ for some integer w_1. Observe that

$$w_1=(v_1'')^{\frac{1}{d-1}}\leq v_1^{\frac{1}{d-1}}=\left(\frac{v_1v_2}{v_2}\right)^{\frac{1}{d-1}}\ll n^{\frac{1+\varepsilon_0}{d-1}}v_2^{-\frac{1}{d-1}}.$$

Further, $n=u'w_1$, where $(u')^{d-1}=u_1v_1'$, so u' is a positive integer all whose prime factors are at most u. We now apply the *abc* conjecture to the equation $q(n)=u_2v_2$ written under the form

$$q_0n+q_1=u_2v_2,$$

where

$$a=q_0n=q_0u'w_1,\qquad b=q_1,\qquad\text{and}\qquad c=u_2v_2.$$

Clearly, $a\neq 0$, and $b\neq 0$, since if $b=0$, then $f_2(x)=q_0x=q_0f_1(x)$, so $s=1$, which is impossible, and clearly $c\neq 0$. Put $\Delta=\gcd(a,b)$ and note that $\Delta \mid q_1$, so $\Delta = O(1)$. Write $a = \Delta a_1$, $b = \Delta b_1$, and $c = \Delta c_1$ and apply the *abc* conjecture to the equation

$$a_1 + b_1 = c_1$$

with some small $\varepsilon > 0$ to get that

$$n \ll a_1 \ll_\varepsilon N(a_1 b_1 c_1)^{1+\varepsilon} \ll N(abc)^{1+\varepsilon}. \tag{4.9}$$

Observe that

$$N(a) \le N(q_0 u' w_1) \ll N(v!) w_1 \ll v!^{O\left(\frac{1}{\log v}\right)} w_1 \ll n^{\frac{1+\varepsilon_0}{d-1} + O\left(\frac{1}{\log v}\right)} v_2^{-\frac{1}{d-1}};$$
$$N(b) = O(1);$$
$$N(c) = N(u_2 v_2) \le N(v!) v_2 \ll n^{O\left(\frac{1}{\log v}\right)} v_2.$$

Thus,

$$N(abc) \ll n^{\frac{1+\varepsilon_0}{d-1} + O\left(\frac{1}{\log v}\right)} v_2^{\left(1 - \frac{1}{d-1}\right)}.$$

Using inequality (4.8), we get that

$$N(abc) \ll n^{\frac{1+\varepsilon_0}{d-1} + (1-\delta_1+d\varepsilon_0)\left(1-\frac{1}{d-1}\right) + O\left(\frac{1}{\log v}\right)}.$$

The exponent of n above is smaller than

$$1 - \delta_1 \left(1 - \frac{1}{d-1}\right) + d\varepsilon_0 + O\left(\frac{1}{\log v}\right).$$

Thus, assuming that $\varepsilon_0 < \delta_1/(3d) = (d-2)/(9d)$, and then that v is large, the above expression is smaller than $1 - \delta_1/2$. Thus, for such large values of v, we have

$$n \ll_\varepsilon n^{(1-\delta_1/2)(1+\varepsilon)},$$

which for a sufficiently small $\varepsilon > 0$ implies that $n = O(1)$, which is what we wanted. ◻

4.6. *The case when $s = 2$, $d_1 = 1$, and $e_1 = 1$.*

LEMMA 4.6. *Under the abc conjecture, equation* (1.1) *has only finitely many solutions (n, u, v) with $n \ge 0$ and $u < v$ in the case when $s = 2$, $d_1 = e_1 = 1$, and $d \ge 3$.*

PROOF. In this case, $d_2 > 1$, for if $d_2 = 1$, then since $d_1 = e_1 = 1$ and $d \ge 3$, it follows that $e_2 = d - 1 > 1 = e_1$, and this is a case treated already (just reverse the roles of $f_1(x)$ and $f_2(x)$ in Lemma 4.5). In this case, the equations (4.7) are

$$n = u_1 v_1 \qquad \text{and} \qquad q(n) = u_2 v_2.$$

We distinguish two cases.

Case 1. $u_1 \le n^{2/3}$.

In this case, $v_1 \gg n^{1/3}$, therefore $v_2 \ll n^{1+\varepsilon_0}/v_1 \ll n^{2/3+\varepsilon_0}$. Write the equation $q(n) = u_2v_2$ as

$$q_0n^{d-1} + q_1n^{d-1} + \cdots + q_{d-1} = u_2v_2.$$

Multiply both sides of the above equation by $U = q_0^{d-2}(d-1)^{d-1}$ and put $m = q_0(d-1)n + q_1$. Since n is large, it follows that $m > 0$. Then the above equation can be rewritten as

$$m^{d-1} + g(m) = Uu_2v_2,$$

where $g(x) \in \mathbb{Q}[x]$ is of degree at most $d-3$. We apply the *abc* conjecture to the above relation with

$$a = m^{d-1}, \qquad b = g(m), \qquad \text{and} \qquad c = Uu_2v_2.$$

Since $m > 0$, it follows that $a > 0$. Note that $g(x) \neq 0$, for if not, then $q(x) = f_2(x)^{d-1}$, where $f_2(x) = q_0(d-1)x + q_1$, which is not the case we are considering. Thus, $g(x) \neq 0$, therefore if $b = 0$, then $g(m) = 0$, so $m = O(1)$, and we get only finitely many solutions (n, u, v) of equation (1.1) with $n \geq 0$ and $u < v$. This deals with the case $b = 0$. Finally, it is clear that $c \neq 0$. Put $\Delta = \gcd(a, b)$ and write $a = \Delta a_1$, $b = \Delta b_1$, and $c = \Delta c_1$. We apply the *abc* conjecture to the equation

$$a_1 + b_1 = c_1$$

with some small $\varepsilon > 0$ to get that

$$\frac{n^{d-1}}{\Delta} \ll \frac{m^{d-1}}{\Delta} = a \ll_\varepsilon N(a_1b_1c_1)^{1+\varepsilon}.$$

Clearly,

$$N(a_1) \leq N(m) \leq m \ll n;$$
$$N(b_1) \leq |b_1| = \frac{|b|}{\Delta} \ll \frac{|g(m)|}{\Delta} \ll \frac{n^{d-3}}{\Delta};$$
$$N(c_1) = N(u_2v_2) \ll \left(\prod_{p \leq v} p\right) v_2 = n^{O(1/\log v)+(2/3+\varepsilon_0)}.$$

Thus,

$$N(a_1b_1c_1) \ll \frac{n^{d-4/3+\varepsilon_0+O(1/\log v)}}{\Delta}. \tag{4.10}$$

We thus get that

$$n^{d-1} \ll_\varepsilon n^{(d-4/3+\varepsilon_0+O(1/\log v))(1+\varepsilon)}. \tag{4.11}$$

If $\varepsilon_0 < 1/6$ and v is sufficiently large, then

$$d - 4/3 + \varepsilon_0 + O(1/\log v) < d - 1.15,$$

so if additionally $\varepsilon > 0$ is sufficiently small, then the exponent of n on the right–hand side of inequality (4.11) above is $< d - 1.1$. In turn, the above inequality (4.11) then implies that $n = O(1)$, so only finitely many solutions (n, u, v) with $n \geq 0$ and $u < v$.

Case 2. $u_1 > n^{2/3}$.

In this case, $v_1 < n^{1/3}$. Further, equation (1.1) is

$$q(n)n = Au! + Bv!,$$

therefore

$$q_0 n^{d-1} v_1 + \left(q_1 n^{d-2} v_1 + \cdots + q_{d-1} - Au_2\right) = Bv!/u_1.$$

We apply the *abc* conjecture to the above equation with

$$a = q_0 n^{d-1} v_1, \qquad b = q_1 n^{d-2} v_1 + \cdots + q_{d-1} - Au_2, \qquad \text{and} \qquad c = Bv!/u_1.$$

It is clear that $a > 0$ and that $c \neq 0$. If $b = 0$, we then get $q_0 n^{d-1} = Bv!/u_1$, so $q_0 n^d = Bv!$, and we get $n = O(1)$ by invoking the Bertrand postulate concerning the existence of primes in the interval $(n/2, n)$. Thus, we may assume that $b \neq 0$. Put $\Delta = \gcd(a, b)$, write $a = \Delta a_1$, $b = \Delta b_1$, and $c = \Delta c_1$, and apply the *abc* conjecture to the equation

$$a_1 + b_1 = c_1$$

with some small $\varepsilon > 0$ to get that

$$\frac{n^{d-1}}{\Delta} \leq \frac{n^{d-1} v_1}{\Delta} \ll a_1 \ll_\varepsilon N(a_1 b_1 c_1)^{1+\varepsilon}.$$

Clearly,

$$\begin{aligned}
N(a_1) &\leq N(u_1 v_1) \leq N(u!) v_1 \ll n^{1/3 + O(1/\log v)};\\
N(b_1) &\leq |b_1| = \frac{|b|}{\Delta} \ll \frac{\max\{m^{d-2} v_1, u!/u_1\}}{\Delta}\\
&\ll \frac{\max\{n^{d-5/3}, n^{d-5/3+\varepsilon_0}\}}{\Delta} \ll \frac{n^{d-5/3+\varepsilon_0}}{\Delta};\\
N(c_1) &\leq N(Bv!) = n^{O(1/\log v)}.
\end{aligned}$$

Thus,

$$N(a_1 b_1 c_1) \ll \frac{n^{d-4/3+\varepsilon_0+O(1/\log v)}}{\Delta}. \tag{4.12}$$

We thus get that

$$n^{d-1} \ll_\varepsilon n^{(d-4/3+\varepsilon_0+O(1/\log v))(1+\varepsilon)}, \tag{4.13}$$

which is the same as estimate (4.11). The proof now ends as the proof in Case 1. Thus, the proof of this lemma and indeed of Theorem 1.2 is complete. ∎

ACKNOWLEDGEMENTS.

I thank Maciej Ulas for useful discussions and the referee for useful comments. Work supported in part by Projects PAPIIT 104512, CONACyT

Mexico–France 193539, CONACyT Mexico–India 163787 and a Marcos Moshinsky fellowship.

References

[1] C. Ballot and F. Luca, *Prime factors of* $a^{f(n)}-1$ *with an irreducible polynomial* $f(x)$, New York J. Math. **12** (2006), 39–45.
[2] D. Berend and J. Harmse, *On polynomial factorial diophantine equations*, Trans. Amer. Math. Soc. **358** (2005), 1741–1779.
[3] P. Erdős and R. Obláth, *Über diophantishe Gleichungen der Form* $n! = x^p \pm y^p$ *und* $n! \pm m! = x^p$, Acta Litt. Sci. Szeged **8** (1937), 241–255.
[4] M. Gawron, *On the equation* $P(z) = n! + m!$, preprint.
[5] A. Granville and O. Ramaré, *Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients*, Mathematika **43** (1996), 73–107.
[6] H. Iwaniec and J. Pintz, *Primes in short intervals,* Monatsh. Math. **98** (1984), 115–143.
[7] F. Luca, *The Diophantine equation* $P(x) = n!$ *and a result of M. Overholt*, Glas. Mat. Ser. III **37** (2002), 269–273.

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Received: 19.9.2012.