GLASNIK MATEMATIČKI
Vol. 48(68)(2013), 91 – 96

COMMUTING AUTOMORPHISMS OF SOME FINITE GROUPS

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Abstract. Let G be a group. An automorphism α of G is called a commuting automorphism if $xx^{\alpha} = x^{\alpha}x$ for all $x \in G$. We denote the set of all commuting automorphisms of G by $\mathcal{A}(G)$. Moreover a group G is called an AC-group if the centralizer of every non-central element of G is abelian. In this paper we show that $\mathcal{A}(G)$ is a subgroup of the automorphism group of G for all finite AC-groups, p-groups of maximal class, and metacyclic p-groups.

1. Introduction

Let G be a group and $\mathrm{Aut}(G)$ be the group of all automorphisms of G. Following [4], we define $\mathcal{A}(G) = \{\alpha \in \mathrm{Aut}(G) | xx^{\alpha} = x^{\alpha}x \text{ for all } x \in G\}$ and any element of this set is called a commuting automorphism. This definition first was considered for rings, see [2], [5] and [11]. Also I. N. Herstein in [8] posed a question: when $\mathcal{A}(G) = 1$? Then T. J. Laffey ([9]) and M. Pettet ([12]) provided extensions of Herstein's result. Moreover we see that $\mathcal{A}(G)$ is a subset of $\mathrm{Aut}(G)$ and $\mathrm{Aut}_c(G)$, the group of central automorphisms of G is a subset of $\mathcal{A}(G)$. This observation suggests a question which was considered by Deaconescu, Silberberg and Walls in [4]:

Is it true that the set $\mathcal{A}(G)$ is always a subgroup of $\mathrm{Aut}(G)$?

Obviously $\mathrm{Aut}_c(G) = \mathcal{A}(G) = \mathrm{Aut}(G)$ when G is abelian. Moreover it is shown in [4] that $\mathcal{A}(G)$ is not always a subgroup of $\mathrm{Aut}(G)$. They constructed a finite non-abelian 2-group G of order 2^5 such that $\mathcal{A}(G)$ is not a subgroup of $\mathrm{Aut}(G)$. In this paper we answer this question in some families of groups. Specially we show that if G is a finite AC-group, a finite p-group of maximal

2010 *Mathematics Subject Classification.* 20F28, 20D15.
Key words and phrases. Commuting automorphisms, AC-groups, minimal non-abelian p-groups, metacyclic p-groups, p-groups of maximal class.

class, or a finite metacyclic p-group, then $\mathcal{A}(G)$ is a subgroup of $\mathrm{Aut}(G)$ and in some cases $\mathcal{A}(G) = \mathrm{Aut}_c(G)$. We note that a group G is called an AC-group if the centralizer of every non-central element of G is abelian. Therefore we deduce that $\mathcal{A}(G)$ is also a subgroup of $\mathrm{Aut}(G)$ when G is a finite minimal non-abelian group, a p-group with the central quotient of order less than p^4, a p-group of order less that p^5 or a finite p-group with a cyclic maximal subgroup.

Throughout this paper the following notation is used. All groups are assumed to be finite. The letter p denotes a prime number. $\mathcal{C}_G(x)$ is the centralizer of an element x in a group G. The nilpotency class of a group G is denoted by $\mathrm{cl}(G)$. A p-group of maximal class is a non-abelian group G of order p^n with $\mathrm{cl}(G) = n - 1$. The terms of the lower central series of G are denoted by $\gamma_i(G)$. If α is an automorphism of G and x is an element of G, we write x^α for the image of x under α and $[x, \alpha]$ is $x^{-1}x^\alpha$. Also $\mathrm{Aut}_Z^Z(G)$ is the group of central automorphisms of G, which fix $Z(G)$ elementwise. We write $[a, b]$ for $a^{-1}b^{-1}ab$ when $a, b \in G$. Finally ${\mathbb{Z}_m}^n$ is the direct product of n copies of the cyclic group of order m.

2. AC-groups

In this section we prove that $\mathcal{A}(G) \leq \mathrm{Aut}(G)$ for any AC-group G. As a consequence we see that if G is a minimal non-abelian group, a non-abelian p-group with $|G/Z(G)| \leq p^3$, a p-group ($p > 2$) with a cyclic maximal subgroup or a p-group of order less that p^5 , then $\mathcal{A}(G) \leq \mathrm{Aut}(G)$. Moreover in some cases we see that $\mathcal{A}(G) = \mathrm{Aut}_c(G)$. First we state two following lemmas that are needed for the main results of the paper.

Lemma 2.1 ([4, Lemma 2.1]). *If $\alpha \in \mathcal{A}(G)$ and $x, y \in G$, then $[x^\alpha, y] = [x, y^\alpha]$.*

Lemma 2.2 ([4, Lemma 2.4 (vi)]). *Let G be a group and α, $\beta \in \mathcal{A}(G)$. Then $\alpha\beta \in \mathcal{A}(G)$ if and only if $[x^\alpha, x^\beta] = 1$ for all $x \in G$.*

Lemma 2.3. *If G is an AC-group, then $\mathcal{A}(G) \leq \mathrm{Aut}(G)$.*

Proof. Let $\alpha, \beta \in \mathcal{A}(G)$. Since $\mathcal{A}(G)$ is finite, it is enough to prove that $\alpha\beta \in \mathcal{A}(G)$ or equivalently $[x^\alpha, x^\beta] = 1$ for all $x \in G$ by Lemma 2.2. First if $x \in G \setminus Z(G)$, then $\mathcal{C}_G(x)$ is abelian and so $[x^\alpha, x^\beta] = 1$. Also if $x \in Z(G)$, then $x^\alpha, x^\beta \in Z(G)$, as desired. ∎

Lemma 2.4. *Let G be a non-abelian p-group with $|G/Z(G)| \leq p^3$. Then G is an AC-group.*

Proof. Let g be a non-central element of G. Then $Z(G) < Z(\mathcal{C}_G(g)) \leq \mathcal{C}_G(g) < G$ since $g \in Z(\mathcal{C}_G(g)) \setminus Z(G)$. This implies that $|\frac{\mathcal{C}_G(g)}{Z(\mathcal{C}_G(g))}|$ divides p. Hence $Z(\mathcal{C}_G(g)) = \mathcal{C}_G(g)$, as desired. ∎

LEMMA 2.5 ([3, Theorem 1.2]). *Let G be a group of order p^n with a cyclic maximal subgroup. Then G has one of the following presentations:*

(i) $M_{p^n} = \langle a, b | a^{p^{n-1}} = b^p = 1, a^b = a^{1+p^{n-2}} \rangle$, *where $n \geq 4$ if $p = 2$.*
(ii) D_{2^n}, *the dihedral group.*
(iii) Q_{2^n}, *the generalized quaternion group.*
(iv) SD_{2^n} $(n > 3)$, *the semi dihedral group.*

COROLLARY 2.6. *For any of the following groups, $\mathcal{A}(G)$ is a subgroup of $\mathrm{Aut}(G)$.*

(i) *G is a non-abelian p-group with $|G/Z(G)| \leq p^3$.*
(ii) *G is a p-group of order less than p^5.*
(iii) *G is a p-group with a cyclic maximal subgroup, where $p > 2$.*
(iv) *G is a minimal non-abelian group.*

PROOF. (i)-(ii) This follows from lemmas 2.3 and 2.4.
(iii) It is easy to see that $|G/Z(G)| = p^2$ by Lemma 2.5(i). The rest follows from (i).
(iv) This is clear by the fact G is an AC-group and Lemma 2.3. □

LEMMA 2.7. *If $G = M_{p^n}$, then $\mathcal{A}(G) = \mathrm{Aut}_c(G)$.*

PROOF. By Lemma 2.5(i), we see that $\mathcal{C}_G(a) = \langle a \rangle$, $|G'| = p$ and $Z(G) = \Phi(G) = \langle a^p \rangle$. Let $\alpha \in \mathcal{A}(G)$, then we may write $a^\alpha = a^i b^j$ and $b^\alpha = a^r b^s$, where $0 \leq s, j < p$ and $0 \leq i, r < p^{n-1}$. Since $[a^\alpha, a] = 1$ we deduce that $b^j \in \mathcal{C}_G(a)$. Hence $b^j \in \langle a \rangle \cap \langle b \rangle = 1$ and so $(i, p) = 1$. Also we have $1 = [b^\alpha, b] = [a^r, b] = [a, b]^r$ which implies that p divides r. Therefore $a^r \in Z(G)$. Now by applying [7, Proposition 3, p. 44] and the third relation of the presentation of G we deduce that $1 = [a, b]^{i(s-1)}$ and so p divides $s - 1$ since $(i, p) = 1$. Therefore $s = 1$. Moreover by Lemma 2.1, we have $[a^\alpha, b] = [a, b^\alpha]$, which yields that $[a, b]^{i-1} = 1$ or equivalently p divides $i - 1$. Therefore $\alpha \in \mathrm{Aut}_c(G)$, completing the proof. □

3. p-GROUPS OF MAXIMAL CLASS

Let G be a p-group of maximal class and order p^n, where $n \geq 4$. In this section we show that $\mathcal{A}(G) = \mathrm{Aut}_c(G)$. First we give some properties of p-groups of maximal class.

LEMMA 3.1. *Let G be a p-group of maximal class and order p^n. Then*

(i) *G is purely non-abelian,*
(ii) $\mathrm{Aut}_c(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

PROOF. (i) Assume by the way of contradiction that $G = A \times B$, where A is a non-trivial abelian subgroup of G and B is a purely non-abelian subgroup of G. Then $\mathrm{cl}(G) = \mathrm{cl}(B) = n - 1$, which is a contradiction since $|B|$ divides p^{n-1}.

(ii) By (i) and [1, Theorem 1], we have $|\mathrm{Aut}_c(G)| = p^2$ since $G/G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $Z(G) \cong \mathbb{Z}_p$. Moreover

$$\mathrm{Aut}_Z^Z(G) \cong \mathrm{Hom}(G/Z(G), Z(G)) \cong \mathrm{Hom}(\frac{G/Z(G)}{(G/Z(G))'}, Z(G)) \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

by [13, Result 1.1], which completes the proof since $\mathrm{Aut}_Z^Z(G) \leq \mathrm{Aut}_c(G)$. ∎

Let G be a p-group of maximal class and order p^n $(n \geq 4)$, where p is a prime. Following [10], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for G and $P_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$ for $1 \leq i \leq n-1$. We note that $P_{n-1} = Z(G)$.

For the rest of this section we fix the above notation.

Lemma 3.2 ([6, Hilfssatz III. 14.13]). *If G is a p-group of maximal class of order p^n and $s \notin K_i$ for $2 \leq i \leq n-2$, then $\mathcal{C}_G(s) = \langle s\rangle P_{n-1}(G)$ and $s^p \in P_{n-1}$.*

Now we state the following Lemma from [4] that will be used in the sequel.

Lemma 3.3 ([4, Lemma 2.2]). *If $\alpha \in \mathcal{A}(G)$ and $x \in G$, then $[x, \alpha] \in \mathcal{C}_G(G')$.*

Theorem 3.4. *Let G be a p-group of maximal class and order p^n, where $n \geq 4$. Then $\mathcal{A}(G) = \mathrm{Aut}_c(G)$.*

Proof. Let $\alpha \in \mathcal{A}(G)$, then we may write $s^\alpha = sx$ and $s_1^\alpha = s_1 y$, where $x, y \in \mathcal{C}_G(G')$ by Lemma 3.3. Now by considering $[s^\alpha, s] = 1$ we see that $x \in \mathcal{C}_G(s)$ and so we may assume that $x = s^i z$, where $z \in Z(G)$ and $0 \leq i < p$ by Lemma 3.2. We claim that $i = 0$. Otherwise, since $x \in \mathcal{C}_G(G')$, we have $1 = [s^i z, s_2] = [s^i, s_2]$. Hence $s_3 = [s_2, s] = 1$ since $(i, p) = 1$. Therefore $P_3(G) = 1$ and so $|G| = p^3$, which is impossible. Therefore $x \in Z(G)$. Moreover $[s^\alpha, s_1] = [s, s_1^\alpha]$ by Lemma 2.1, which implies that $y \in \mathcal{C}_G(s)$ by using the fact that $y \in \mathcal{C}_G(G')$. Hence by the same argument as above we conclude that $y \in Z(G)$ and so $\alpha \in \mathrm{Aut}_c(G)$, as desired. ∎

Corollary 3.5. *Let G be a group of order p^n with a cyclic maximal subgroup. Then $\mathcal{A}(G) = \mathrm{Aut}_c(G)$.*

Proof. If $G = D_8$ or Q_8, then it is easy to check that $\mathcal{A}(G) = \mathrm{Aut}_c(G)$. Therefore the proof follows from lemmas 2.5, 2.7 and Theorem 3.4. ∎

4. Metacyclic p-groups

Let G be a non-abelian metacyclic p-group. In this section we show that $\mathcal{A}(G) \leq \mathrm{Aut}(G)$. We know that there exists a normal cyclic subgroup $\langle a \rangle$ of G such that $G/\langle a \rangle$ is cyclic. Therefore we may choose an element $b \in G$ and a number $1 \leq k < |a|$ such that $G = \langle b, a \rangle$ and $b^{-1}ab = a^k$ and so any element of G has the form $b^j a^i$ for $j, i \geq 0$.

For the rest of the paper we fix the above notation.

Lemma 4.1. *Let G be a non-abelian metacyclic p-group.*

(i) $k \equiv 1 \pmod p$,

(ii) $[a^i, b] = [a, b]^i = a^{(k-1)i}$ *and* $[b^n, a] = [b, a]^{1+k+\cdots+k^{n-1}}$ *for* $i, n \geq 1$,

(iii) $G' = \langle [a, b] \rangle$,

(iv) *if* $b^{s-1} \in \mathcal{C}_G(G')$, *where* $s \geq 1$, *then* $[b^{ns}, a] = [b^s, a]^{1+k+\cdots+k^{n-1}}$ *for any* $n \geq 1$.

Proof. (i) Obviously $G' \leq \langle a \rangle$ and $\langle a^{k-1} \rangle \leq G'$. Now if $(p, k-1) = 1$, then $G' = \langle a \rangle$ and so G/G' is cyclic which is a contradiction.

(ii) This follows from $b^{-1}ab = a^k$.

(iii) We have $G' = \langle [x, y] | x, y \in G \rangle$ which completes the proof by using (ii).

(iv) We use induction on n and the fact that $[b^{ns}, a]^b = [b^{ns}, a]^k$ since $[b^{ns}, a] \in \langle a \rangle$ and $b^{-1}ab = a^k$. □

Theorem 4.2. *Let G be a non-abelian metacyclic p-group. Then*

$$\mathcal{A}(G) \leq \mathrm{Aut}(G).$$

Proof. Let $\alpha \in \mathcal{A}(G)$, then we may write $a^\alpha = a^i b^j$ and $b^\alpha = b^s a^l$. Therefore b^j, $a^l \in Z(G)$ by the definition of $\mathcal{A}(G)$. Hence we may assume that $a^\alpha = a^i z_1$ and $b^\alpha = b^s z_2$, where $z_1, z_2 \in Z(G)$. Consequently for $\beta \in \mathcal{A}(G)$ we have $a^\beta = a^{i'} z_1'$ and $b^\beta = b^{s'} z_2'$, where $z_1', z_2' \in Z(G)$. Now if $g \in G$, then we may write $g = b^r a^t$. Therefore $[g^\alpha, g^\beta] = [b^{sr}, a]^{i't} [b^{s'r}, a]^{-it}$ by Lemma 4.1(ii). Moreover by Lemma 3.3, $b^{-1}b^\alpha$ and $b^{-1}b^\beta \in \mathcal{C}_G(G')$ or equivalently $b^{s-1}, b^{s'-1} \in \mathcal{C}_G(G')$. Also by Lemma 2.1, we see that $[b, a^i] = [b^s, a]$ and $[b, a^{i'}] = [b^{s'}, a]$. This implies that $[g^\alpha, g^\beta] = 1$ by Lemma 4.1(iv) and (ii), which completes the proof by Lemma 2.2. □

References

[1] J. E. Adney and T. Yen, *Automorphisms of a p-group,* Illinois J. Math. **9** (1965), 137–143.

[2] H. E. Bell and W. S. Martindale, *Centralizing mappings of semiprime rings,* Canad. Math. Bull. **30** (1987), 92–101.

[3] Y. Berkovich, Groups of prime power order. Vol. 1, Walter de Gruyter, Berlin, 2008.

[4] M. Deaconescu, G. Silberberg and G. L. Walls, *On commuting automorphisms of groups,* Arch. Math. (Basel) **79** (2002), 423–429.

[5] N. Divinsky, *On commuting automorphisms of rings,* Trans. Roy. Soc. Canada. Sect. III. **49** (1955), 19–22.
[6] B. Huppert, Endliche Gruppen. I, Springer-Verlag, Berlin–New York, 1967.
[7] D. L. Johnson, Presentation of groups, 2nd ed., Cambridge University Press, Cambridge, 1997.
[8] I. N. Herstein, J. L. Brenner and W. A. Newcomb, *Problems and solutions: elementary problems: E3039-E3040,* Amer. Math. Monthly **91** (1984), 203.
[9] I. N. Herstein, T. J. Laffey, and J. Thomas, *Problems and solutions: solutions of elementary problems: E3039,* Amer. Math. Monthly **93** (1986), 816–817.
[10] C. R. Leedham-Green and S. McKay, The structure of groups of prime power order, Oxford University Press, Oxford, 2002.
[11] J. Luh, *A note on commuting automorphisms of rings,* Amer. Math. Monthly **77** (1970), 61–62.
[12] M. Pettet, Personal communication.
[13] P. Schmid, *Normal p-subgroups in the group of outer automorphisms of a finite p-group,* Math. Z. **147** (1976), 271–277.

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Received: 13.8.2012.