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WAŻEWSKI'S UNIVERSAL DENDRITE AS AN INVERSE LIMIT WITH ONE SET-VALUED BONDING FUNCTION

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Abstract. We construct a family of upper semi-continuous set-valued functions $f : [0,1] \to 2^{[0,1]}$ (belonging to the class of so-called comb functions), such that for each of them the inverse limit of the inverse sequence of intervals $[0,1]$ and f as the only bonding function is homeomorphic to Ważewski's universal dendrite. Among other results we also present a complete characterization of comb functions for which the inverse limits of the above type are dendrites.

1. Introduction

In 1923. T. Ważewski described an example of a dendrite in the plane which contains a topological copy of any dendrite ([26]). The described dendrite is now known as Ważewski's universal dendrite. In [20, p. 181] one can find a construction of Ważewski's universal dendrite using inverse limits. In particular, it is constructed as the inverse limit of an inverse sequence of planar dendrites D_n and monotone bonding mappings $f_n : D_{n+1} \to D_n$, where dendrites D_n are getting more and more complicated as n increases, and so do the functions f_n. In this paper we construct Ważewski's universal dendrite as the inverse limit $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$ of closed unit intervals $[0,1]$ and a single upper semi-continuous set-valued bonding function f. We believe that this new presentation of Ważewski's universal dendrite sheds new light on the classic dendrite and simultaniously shows the strength of the theory of the inverse systems with upper semi-continuous set-valued bonding functions.

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2. Definitions and Notation

Our definitions and notation mostly follow Nadler ([20]) and Ingram and Mahavier ([14]).

A *map* is a continuous function. For $i = 1, 2$, $\pi_i : [0,1] \times [0,1] \to [0,1]$ denotes the i-th projection from $[0,1] \times [0,1]$ onto the i-th factor, and for any positive integer i, $p_i : \Pi_{n=1}^{\infty}[0,1] \to [0,1]$ denotes the i-th projection from $\Pi_{n=1}^{\infty}[0,1]$ onto the i-th factor.

A *continuum* is a nonempty, compact and connected metric space. A *Peano continuum* is a locally connected continuum.

A *dendrite* is a Peano continuum which contains no simple closed curve.

Let D be a dendrite, $b \in D$, and β a cardinal number. We say that b is of order less than or equal to β in D, written $\mathrm{ord}(b, D) \leq \beta$, provided that for each open neighborhood U of b in D, there is an open neighborhood V of b in D, such that $b \in V \subseteq U$ and $|\,\mathrm{Bd}(V)| \leq \beta$. We say that b is of order β, $\mathrm{ord}(b, D) = \beta$, provided that $\mathrm{ord}(b, D) \leq \beta$ and $\mathrm{ord}(b, D) \not\leq \alpha$ for any cardinal number $\alpha < \beta$.

Points of order 1 in a dendrite D are called *end points* of D, the set of all end points of D is denoted by $E(D)$. Points of order $n > 2$ are called *ramification points* and the set of all ramification points of D is denoted by $R(D)$.

A *free arc* in a dendrite D is an arc such that all its points but its end points are of order 2 in D. In particular, a *maximal free arc* in a dendrite D is an arc A with end points x and y in D such that $A \cap (E(D) \cup R(D)) = \{x, y\}$.

A continuum S is a *star* if there is a point $c \in S$ such that S can be presented as the countable union $S = \bigcup_{n=1}^{\infty} B_n$ of arcs B_n, each having c as an end point and satisfying $\lim_{n\to\infty} \mathrm{diam}(B_n) = 0$, such that $B_n \cap B_m = \{c\}$ when $m \neq n$. The point c is uniquely determined and is called the *center* of S. The arcs B_n are called *beams* of S.

Let D_1 be a star in a compact metric space X. Let $c_A \notin R(D_1)$ denote a point in the maximal free arc A, for each maximal free arc A of D_1 (here maximal free arcs are precisely the beams of D_1). Let $C_1 = \{x_1, x_2, x_3, \ldots\}$ be any subset of the set $\{c_A \mid A$ is a maximal free arc in $D_1\}$. For each positive integer i, form a star S_i in X with the center x_i and otherwise disjoint from D_1, making sure that $S_i \cap S_j \neq \emptyset$ only when $i = j$ and that $\lim_{i\to\infty} \mathrm{diam}(S_i) = 0$. Let $D_2 = D_1 \cup (\bigcup_{i=1}^{\infty} S_i)$. Next define D_3 in a similar manner. Let $c_A \notin R(D_2)$ denote a point in the maximal free arc A in D_2, for each maximal free arc A of D_2. Let $C_2 = \{x_1, x_2, x_3, \ldots\}$ be any subset of the set $\{c_A \mid A$ is a maximal free arc in $D_2\}$. For each positive integer i, form a star S_i in X with the center x_i and otherwise disjoint from D_2, making sure that $S_i \cap S_j \neq \emptyset$ only when $i = j$ and that $\lim_{i\to\infty} \mathrm{diam}(S_i) = 0$. Let

$D_3 = D_2 \cup (\bigcup_{i=1}^{\infty} S_i)$. Continuing in this manner, we obtain a continuum D_n for each positive integer n. The following theorem (already implicitly used in the above inductive construction) is a well-known fact, see [20] for details.

Theorem 2.1. *For each positive integer n, D_n is a dendrite.*

The construction of the continuum, homeomorphic to Ważewski's universal dendrite in [20, p. 181] uses the above mentioned construction of a chain of dendrites $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$, then defines certain bonding maps $f_n : D_{n+1} \to D_n$, and then finally obtains Ważewski's universal dendrite as $\varprojlim\{D_k, f_k\}_{k=1}^{\infty}$.

Finally we state a result that is characterizing Ważewski's universal dendrite that will be needed in Section 4.

Theorem 2.2. *For any dendrite D, D is homeomorphic to Ważewski's universal dendrite if and only if its set of ramification points is dense in D and each of its ramification points is of infinite order.*

Proof. [7, p. 169], [26, p. 123] □

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X. Let for each $\varepsilon > 0$ and each $A \in 2^X$

$$N_d(\varepsilon, A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

The set 2^X will be always equipped with the *Hausdorff metric* H_d, which is defined by

$$H_d(H, K) = \inf\{\varepsilon > 0 \mid H \subseteq N_d(\varepsilon, K), K \subseteq N_d(\varepsilon, H)\},$$

for $H, K \in 2^X$. Then $(2^X, H_d)$ is a metric space, called the *hyperspace* of the space (X, d). For more details see [13, 20].

When we say that f is a set-valued function from X to Y, we mean that f is a single-valued function from X to 2^Y, i.e., $f : X \to 2^Y$. By a slight abuse of notation and terminology we will also say that function $f : X \to 2^Y$ is set-valued (without explicitly mentioning "from X to Y").

A function $f : X \to 2^Y$ is *surjective* set-valued function from X to Y if for each $y \in Y$ there is an $x \in X$, such that $y \in f(x)$.

The *graph* $\Gamma(f)$ of a set-valued function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

A function $f : X \to 2^Y$, where X and Y are compact metric spaces, is *upper semi-continuous* set-valued function from X to Y (abbreviated *u.s.c.*) if for each open set $V \subseteq Y$ the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X.

The following is a well-known characterization of u.s.c. functions between metric compacta (see [14, p. 120, Theorem 2.1]).

Theorem 2.3. *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a set-valued function. Then f is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

An *inverse sequence* of compact metric spaces X_k with u.s.c. bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$ for each k.

The *inverse limit* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ with u.s.c. bonding functions is defined to be the subspace of the product space $\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim\{X_k, f_k\}_{k=1}^{\infty}$.

Note that each inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ with continuous single-valued bonding functions can be interpreted as an inverse sequence with u.s.c. set-valued bonding functions and that the inverse limits obtained according to both interpretations coincide. Therefore we do not specially emphasize the status of bonding functions in inverse sequences we deal with.

The notion of the inverse limit of an inverse sequence with u.s.c. bonding functions was introduced by Mahavier in [18] and Ingram and Mahavier in [14]. Since the introduction of such inverse limits, there has been much interest in the subject and many papers appeared ([1–6, 8, 9, 11, 12, 15, 16, 21–25]).

The most important case in the present paper is the case when for each k, $X_k = [0,1]$ and $f_k = f$ for some $f : [0,1] \to 2^{[0,1]}$. In such case the inverse limit will be denoted by $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$.

On the product space $\prod\limits_{n=1}^{\infty} X_n$, where (X_n, d_n) is a compact metric space for each n, and the set of all diameters of (X_n, d_n) is majorized by 1, we use the metric

$$D(x,y) = \sup_{n \in \{1,2,3,\ldots\}} \left\{ \frac{d_n(x_n, y_n)}{2^n} \right\},$$

where $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$. It is well known that the metric D induces the product topology ([10, p. 190]).

3. The comb functions

Let $A \subseteq [0,1] \times [0,1]$ be defined by

$$A = \{(t,t) \in [0,1] \times [0,1] \mid t \in [0,1]\}.$$

For any positive integer n, let $\{(a_i, b_i)\}_{i=1}^{n}$ be a finite sequence in $[0,1] \times [0,1]$, such that $a_i < b_i$ for each $i = 1, 2, 3, \ldots, n$ and $a_i \neq a_j$ whenever $i \neq j$. Next denote by $A(a_i, b_i)_{i=1}^{n}$ the union

$$A(a_i, b_i)_{i=1}^{n} = \bigcup_{i=1}^{n} ([a_i, b_i] \times \{a_i\}) \subseteq [0,1] \times [0,1].$$

Then

$$G(a_i, b_i)_{i=1}^{n} = A \cup A(a_i, b_i)_{i=1}^{n}$$

is closed in $[0,1] \times [0,1]$, since it is a union of finitely many closed arcs. Furthermore $\pi_1(G(a_i, b_i)_{i=1}^{n}) = \pi_2(G(a_i, b_i)_{i=1}^{n}) = [0,1]$. Therefore by

Theorem 2.3 there is a surjective u.s.c. function $f_{(a_i,b_i)_{i=1}^n} : [0,1] \to 2^{[0,1]}$ such that its graph $\Gamma(f_{(a_i,b_i)_{i=1}^n})$ equals to $G(a_i,b_i)_{i=1}^n$.

Definition 3.1. *Let n be a positive integer and $\{(a_i,b_i)\}_{i=1}^n$ be a subset of $[0,1]\times[0,1]$, such that $0 < a_i < b_i$ for each $i = 1,2,3,\dots,n$ and $a_i \neq a_j$ whenever $i \neq j$. Then $f : [0,1] \to 2^{[0,1]}$ is called an n-comb function with respect to $\{(a_i,b_i)\}_{i=1}^n$, if $f = f_{(a_i,b_i)_{i=1}^n}$.*

We also say that $f : [0,1] \to 2^{[0,1]}$ is an n-comb function, if f is an n-comb function with respect to some $\{(a_i,b_i)\}_{i=1}^n$.

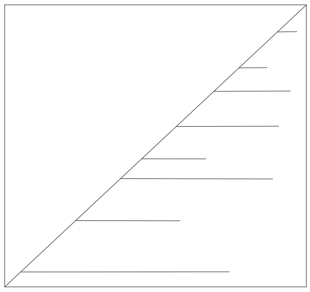

Figure 1. The graph of an 8-comb function

It is not necessary to eliminate the possibility $a_i = 0$ for some i (all the proofs in the paper would go through also in such case), but we have chosen to do so in order to reduce the number of cases that must be examined in the proofs and since the main result can be obtained with this restriction in place.

Definition 3.2. *Let for each j, i_j be a nonnegative integer. We use*

$$(a_1^{i_1}, a_2^{i_2}, a_3^{i_3}, \dots)$$

to denote the point $(\underbrace{a_1,a_1,\dots,a_1}_{i_1},\underbrace{a_2,a_2,\dots,a_2}_{i_2},\dots)$ *and*

$$(a_1^{i_1}, a_2^{i_2}, a_3^{i_3}, \dots, a_j^{i_j}, t^\infty)$$

to denote the point $(\underbrace{a_1,a_1,\dots,a_1}_{i_1},\underbrace{a_2,a_2,\dots,a_2}_{i_2},\dots,\underbrace{a_j,a_j,\dots,a_j}_{i_j},t,t,t,\dots)$.

Example 3.3. Let f be a 1-comb function with respect to $\{(a_i,b_i)\}_{i=1}^1$. Then $x \in \varprojlim\{[0,1],f\}_{k=1}^\infty$ if and only if

1. either $x = (t^\infty)$ for a $t \in [0,1]$ or
2. there is a positive integer n such that $x = (a_1^n, t^\infty)$ for a $t \in (a_1,b_1]$.

Therefore $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is the star with the center (a_1^{∞}) and beams $B_0 = \{(t^{\infty}) \mid t \in [0, a_1]\}$, $B_0' = \{(t^{\infty}) \mid t \in [a_1, 1]\}$ and $B_n = \{(a_1^n, t^{\infty}) \mid t \in [a_1, b_1]\}$, $n = 1, 2, 3, \ldots$

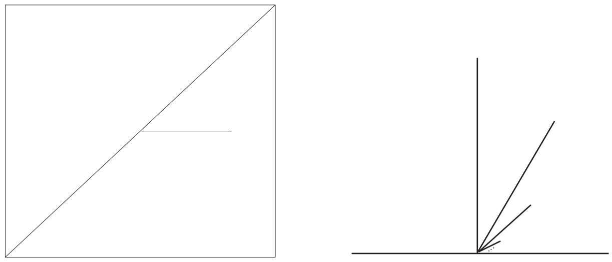

FIGURE 2. The graph of a 1-comb function and its inverse limit

EXAMPLE 3.4. Let f be a 2-comb function with respect to $\{(a_i, b_i)\}_{i=1}^{2}$, where $a_1 < a_2$. We distinguish the following two cases:

1. $b_1 < a_2$

   Then $x \in \varprojlim\{[0,1], f\}_{k=1}^{\infty}$ if and only if

   (a) either $x = (t^{\infty})$ for a $t \in [0, 1]$ or
   (b) there is a positive integer n such that $x = (a_1^n, t^{\infty})$ for a $t \in (a_1, b_1]$ or
   (c) there is a positive integer n such that $x = (a_2^n, t^{\infty})$ for a $t \in (a_2, b_2]$.

   Therefore $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is the union of two stars. The star S with the center (a_1^{∞}) and beams $B_0 = \{(t^{\infty}) \mid t \in [0, a_1]\}$, $B_0' = \{(t^{\infty}) \mid t \in [a_1, 1]\}$ and $B_n = \{(a_1^n, t^{\infty}) \mid t \in [a_1, b_1]\}$, $n = 1, 2, 3, \ldots$, and the star S_0 with the center (a_2^{∞}) and beams $C_n = \{(a_2^n, t^{\infty}) \mid t \in [a_2, b_2]\}$, $n = 1, 2, 3, \ldots$

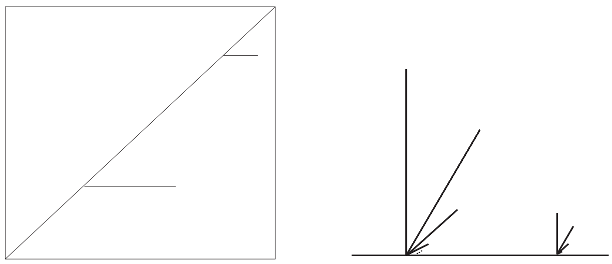

FIGURE 3. The graph of a 2-comb function and its inverse limit, $b_1 < a_2$

2. $b_1 \geq a_2$

Then $x \in \varprojlim\{[0,1], f\}_{k=1}^{\infty}$ if and only if

(a) either $x = (t^{\infty})$ for a $t \in [0,1]$ or

(b) there is a positive integer n such that $x = (a_1^n, t^{\infty})$ for a $t \in (a_1, b_1]$ or

(c) there is a positive integer n such that $x = (a_2^n, t^{\infty})$ for a $t \in (a_2, b_2]$ or

(d) there are positive integers n and m such that $x = (a_1^n, a_2^m, t^{\infty})$ for a $t \in (a_2, b_2]$.

Therefore $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is the union of countable many stars. The star S with the center (a_1^{∞}) and beams $B_0 = \{(t^{\infty}) \mid t \in [0, a_1]\}$, $B_0' = \{(t^{\infty}) \mid t \in [a_1, 1]\}$ and $B_n = \{(a_1^n, t^{\infty}) \mid t \in [a_1, b_1]\}$, $n = 1, 2, 3, \ldots$, the star S_0 with the center (a_2^{∞}) and beams

$$C_n = \{(a_2^n, t^{\infty}) \mid t \in [a_2, b_2]\},$$

$n = 1, 2, 3, \ldots$, and for each positive integer k the star S_k with the center (a_1^k, a_2^{∞}) and beams

$$C_n^k = \{(a_1^k, a_2^n, t^{\infty}) \mid t \in [a_2, b_2]\},$$

$n = 1, 2, 3, \ldots$

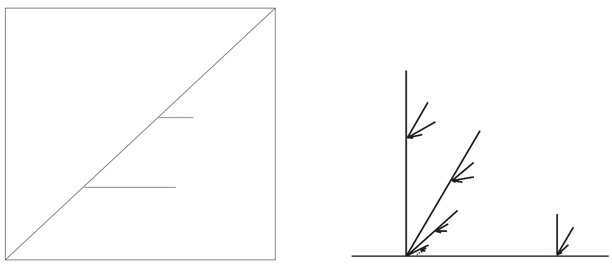

FIGURE 4. The graph of a 2-comb function and its inverse limit, $b_1 > a_2$

Note that if $b_1 = a_2$ the stars S_k, $k = 1, 2, 3, \ldots$, are attached at the end points (a_1^k, b_1^{∞}) of S, and if $b_1 > a_2$ the stars S_k, $k = 1, 2, 3, \ldots$, are attached at the interior points of the maximal free arcs $\{(a_1^k, t^{\infty}) \mid t \in [a_1, b_1]\}$ of S, $k = 1, 2, 3, \ldots$

In the following theorem we show that any inverse limit of intervals $[0,1]$ and a single n-comb function is a dendrite.

THEOREM 3.5. *Let n be any positive integer and let $f : [0,1] \to 2^{[0,1]}$ be any n-comb function. Then $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is a dendrite.*

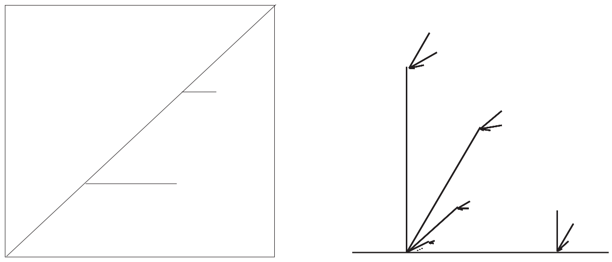

FIGURE 5. The graph of a 2-comb function and its inverse limit, $b_1 = a_2$

PROOF. We prove Theorem 3.5 by induction on n by proving the more precise claim that includes also information about maximal free arcs and ramification points in the dendrite. For each positive integer ℓ, let us introduce the following notation for certain statements that will be used in the inductive proof of the theorem:

(a)$_\ell$ The inverse limit $D_\ell = \varprojlim\{[0,1], f_{(a_i,b_i)_{i=1}^{\ell}}\}_{k=1}^{\infty}$ is a dendrite.

(b)$_\ell$ The points of the form $(x_1, x_2, x_3, \ldots, x_m, a_j^\infty) \in D_\ell$, $j \le \ell$, are exactly the ramification points of D_ℓ.

(c)$_\ell$ The points of the form $(x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_\ell$, $i \le \ell$, where $m \ge 1$, $a_i = x_m \ne b_i$, and $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_\ell\}$, are endpoints of D_ℓ.

(d)$_\ell$ All endpoints of D_ℓ are of such form, except endpoints (0^∞) and (1^∞).

(e)$_\ell$ The maximal free arc in D_ℓ having the point

$$x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty)$$

described in (c)$_\ell$ as one endpoint, has

$$(x_1, x_2, x_3, \ldots, x_m, a_\ell^\infty)$$

as the other endpoint if $a_\ell < b_i$; if $a_\ell > b_i$ then the maximal free arc in D_ℓ ending at x equals to the maximal free arc in $D_{\ell-1}$ ending at x.

(f)$_\ell$ The arc with endpoints (a_ℓ^∞) and (1^∞) is a maximal free arc in D_ℓ.

1. Let $n = 1$. There are $a_1, b_1 \in [0,1]$ such that $a_1 < b_1$ and $f = f_{(a_i,b_i)_{i=1}^{1}}$. In Example 3.3 it was shown that the inverse limit $D_1 = \varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is a star, and is therefore a dendrite. We see that (a_1^∞) is the only ramification point of D_1, and that maximal free arcs of D_1 are exactly the beams $B_0 = \{(t^\infty) \mid t \in [0, a_1]\}$, $B_0' = \{(t^\infty) \mid t \in [a_1, 1]\}$ and $B_k = \{(a_1^k, t^\infty) \mid t \in [a_1, b_1]\}$, $k = 1, 2, 3, \ldots$ of the star D_1. Note that (a)$_1$–(f)$_1$ hold true.

2. Let f be any n-comb function, $n \geq 2$. Without loss of generality we may assume that $f = f_{(a_i,b_i)_{i=1}^n}$, where $a_1 < a_2 < a_3 < \ldots < a_n$.

Let, as the inductive assumption, $(\mathrm{a})_{n-1}$–$(\mathrm{f})_{n-1}$ hold true for the function $f_{(a_i,b_i)_{i=1}^{n-1}}$.

We show that the inverse limit

$$\varprojlim\{[0,1],f\}_{k=1}^{\infty} = D_n = \varprojlim\{[0,1],f_{(a_i,b_i)_{i=1}^n}\}_{k=1}^{\infty}$$

satisfies all the above mentioned properties for $\ell = n$.

By the inductive assumption $D_{n-1} = \varprojlim\{[0,1],f_{(a_i,b_i)_{i=1}^{n-1}}\}_{k=1}^{\infty}$ is a dendrite.

Case 1. $a_n > b_i$ for each $i = 1,2,3,\ldots,n-1$.

In this case any $x \in D_n \setminus D_{n-1}$ is of the form $x = (a_n^k, t^\infty)$, where k is a positive integer and $t \in (a_n, b_n]$. Therefore

$$D_n = D_{n-1} \cup S,$$

where $S = \{(a_n^k, t^\infty) \mid k \in \mathbb{N}, t \in [a_n, b_n]\}$, and we see that S is a star with the center $(a_n^\infty) \in D_n \setminus R(D_{n-1})$. Obviously $(\mathrm{a})_n$-$(\mathrm{f})_n$ hold true.

Case 2. $a_n \leq b_i$ for some $i = 1,2,3,\ldots,n-1$.

In this case we show that

$$D_n = D_{n-1} \cup \left(\bigcup \mathcal{S}\right),$$

where

(a) $\mathcal{S} = \{S_1, S_2, S_3, \ldots\}$ is a countable family of stars with centers $c_1, c_2, c_3, \ldots$ respectively, where $c_1, c_2, c_3, \ldots \in D_n \setminus R(D_{n-1})$, and each of the maximal free arcs in D_{n-1} contains at most one of these centers,

(b) for each positive integer i, $S_i \cap D_{n-1} = \{c_i\}$,

(c) $S_i \cap S_j = \emptyset$ if $i \neq j$, and

(d) $\lim_{i\to\infty} \mathrm{diam}(S_i) = 0$,

and therefore it will follow that D_n is a dendrite by Theorem 2.1, using $(\mathrm{a})_{n-1}$. That will prove $(\mathrm{a})_n$.

Any point of $D_n \setminus D_{n-1}$ is of the form $(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty)$, where k is a positive integer, m is a nonnegative integer, $t \in (a_n, b_n]$, and $x_m \neq a_n$, and vice versa.

The set

$$\{(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty) \mid k \geq 1, t \in [a_n, b_n]\}$$

is a star centered in $(x_1, x_2, x_3, \ldots, x_m, a_n^\infty)$ having the beams

$$\{(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty) \mid t \in [a_n, b_n]\},$$

for each $k \geq 1$. Note that $\mathcal{S}$ is infinite since for each i such that $a_n \leq b_i$ the family $\mathcal{S}$ contains stars centered at (a_i^k, a_n^∞) for each positive integer k.

From $f^{-1}_{(a_i,b_i)_{i=1}^{n-1}}(a_n) = \{a_n\}$ it follows that if for $x \in D_{n-1}$ and for some positive integer m, $p_m(x) = a_n$, then $p_{m+1}(x) = a_n$. Therefore such x ends with the block a_n^∞. Let $X_1 = \{(a_n^\infty)\}$, and let for each positive integer $m \geq 2$,

$$X_m = \{x \in D_{n-1} \mid p_m(x) = a_n, p_{m-1}(x) \neq a_n\}.$$

Then X_m is a finite set for each m. Therefore $X = \bigcup_{m=1}^\infty X_m$ is a finite or countable infinite subset of $D_{n-1} \setminus R(D_{n-1})$ ((b)$_{n-1}$ is also used). Also, each maximal free arc of D_{n-1} contains at most one $x \in X$. To prove this, we shall for each $x \in X$ find the uniquely determined maximal free arc of D_{n-1} containing x. Let

$$x = (x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \in X,$$

where $x_m \neq a_n$. Then $x_m = a_i$ for some $i < n$. Note that since $a_i \in f_{(a_i,b_i)_{i=1}^{n-1}}(a_n)$, it follows that $a_n \in [a_i, b_i]$ and therefore $b_i \geq a_n$. Now we distinguish two cases, $b_i > a_n$ and $b_i = a_n$.

If $b_i > a_n$, then $b_i \notin \{a_{i+1}, a_{i+2}, \ldots, a_n\}$, hence the point $(x_1, x_2, \ldots, x_m, b_i^\infty)$ is an endpoint of D_{n-1} by (c)$_{n-1}$ and the arc

$$\{(x_1, x_2, x_3, \ldots, x_m, t^\infty) \mid t \in [a_{n-1}, b_i]\}$$

is a maximal free arc of D_{n-1} by (e)$_{n-1}$. Obviously x belongs to the arc, since $a_n \in [a_{n-1}, b_i]$.

If $b_i = a_n$, then x is an endpoint of D_{n-1} by (c)$_{n-1}$, and clearly it belongs to the maximal free arc $\{(x_1, x_2, x_3, \ldots, x_m, t^\infty) \mid t \in [a_{n-1}, b_i]\}$ of D_{n-1}, which is a maximal free arc in D_{n-1} by (e)$_{n-1}$.

Now, when we have the explicit description of all maximal free arcs in D_{n-1} containing elements of X, we see that each such maximal free arc contains exactly one point from X.

Take any $x = (x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \in X$, where $x_m \neq a_n$. Then $x_m = a_i$ for some $i < n$. For each positive integer k, let

$$B_k = \{(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty) \mid t \in [a_n, b_n]\}.$$

Obviously, B_k is an arc in D_n and $S(x) = \bigcup_{k=1}^\infty B_k$ is a star centered at x. The diameter of $S(x)$ satisfies

$$\operatorname{diam}(S(x)) \leq D((x_1, x_2, \ldots, x_m, 0^\infty), (x_1, x_2, \ldots, x_m, 1^\infty)) \leq \frac{1}{2^{m+1}}.$$

Since for each m there are only finitely many such points $x \in X$ (X_m is finite), it follows that the set $\mathcal{S} = \{S(x) \mid x \in X\}$ is finite or it can be presented as $\mathcal{S} = \{S_1, S_2, S_3, \ldots\}$. From the above upper bound for the diameters of the stars in the infinite case it follows that $\lim_{i\to\infty} \operatorname{diam}(S_i) = 0$.

Take any point $x \in D_n \setminus D_{n-1}$. As already noticed, it is of the form $x = (x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty)$, where k is a positive integer, m is a nonnegative integer, $t \in (a_n, b_n]$, and $x_m \neq a_n$. Therefore $x \in S(y)$, where $y = (x_1, x_2, \ldots, x_m, a_n^\infty) \in X$. Therefore

$$D_n \setminus D_{n-1} = \left(\bigcup_{x \in X} S(x) \right) \setminus X = \left(\bigcup \mathcal{S} \right) \setminus X,$$

and finally

$$D_n = D_{n-1} \cup (\bigcup_{x \in X} S(x)) = D_{n-1} \cup \left(\bigcup \mathcal{S} \right),$$

proving (a)$_n$.

To prove that the points of the form $x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n$, where $i \leq n$, $x_m = a_i$, and $b_i \notin \{a_{i+1}, a_{i+2}, \ldots, a_n\}$, are endpoints of D_n, we distinguish two cases. If $i \leq n-1$ then $x \in D_{n-1}$, and then x is an endpoint of D_{n-1} by (c)$_{n-1}$, since $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{n-1}\}$. Since $b_i \neq a_n$ the only star attached to the maximal free arc in D_{n-1} ending at x is centered at a point that differs from x, or no star is attached to that arc at all, it follows that $x \in E(D_n)$. If $i = n$, then $b_i = b_n$, and therefore x is an endpoint of a star from $\mathcal{S}$. That proves (c)$_n$.

Also each endpoint of D_n which belongs to D_{n-1}, is also an endpoint in D_{n-1}, therefore it is of the form

$$x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n,$$

where $x_m \neq b_i$ and $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{n-1}\}$, by (d)$_{n-1}$. Points of such form with $b_i = a_n$ are centers of the newly attached stars and therefore are not endpoints of D_n. It follows that $b_i \neq a_n$, and therefore $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\}$. Each endpoint of D_n, which belongs to $D_n \setminus D_{n-1}$, is necessarily an endpoint of a newly attached star and therefore is of the form $x = (x_1, x_2, x_3, \ldots, x_m, b_n^\infty)$, $a_n = x_m \neq b_n$. Additional condition $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\}$ is satisfied vacuously for $i = n$. Obviously (0^∞) and (1^∞) are endpoints of D_n, too. That proves (d)$_n$.

Let $x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n$ be any endpoint of D_n mentioned in (c)$_n$, where $a_i = x_m \neq b_i$ and $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\}$. If $i < n$ then by (c)$_{n-1}$ x is an endpoint of D_{n-1}. If $a_n < b_i$ then we have already proved that a new star centered at $(x_1, x_2, x_3, \ldots, x_m, a_n^\infty)$ is attached to the maximal free arc of D_{n-1} ending at x, and since no other star was attached to this arc it follows that the point $(x_1, x_2, x_3, \ldots, x_m, a_n^\infty)$ is the other endpoint of the maximal free arc of D_n ending at x. If $a_n > b_i$ then no star was attached to the maximal free arc of D_{n-1} ending at $x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty)$, and therefore it remained a maximal free arc of D_n as well. This proves (e)$_n$.

By $(\mathrm{f})_{n-1}$ the maximal free arc of D_{n-1} having (1^∞) as one endpoint has (a_{n-1}^∞) as the other endpoint. Since a star centered at (a_n^∞) was attached to D_{n-1}, and since no other star was attached to the above mentioned arc, $(\mathrm{f})_n$ follows.

Finally $(\mathrm{b})_n$ follows from $(\mathrm{b})_{n-1}$ and from the fact that at each point of the form $(x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \in D_n$ a new star was attached to D_{n-1}, as shown above.

□

In the following remark we extract certain parts of the above proof for later use.

Remark 3.6. Let n be a positive integer.

1. For each positive integer n and for each $y \in D_n$, y is either of the form $y = (t^\infty)$, $t \in [0,1]$, or of the form $y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, t^\infty)$, where m is a positive integer and for each $\ell \le m$ it holds that $i_\ell \le n$, $k_\ell > 0$, $a_{i_\ell} < a_{i_{\ell+1}} \le b_{i_\ell}$, and $a_{i_m} \le t \le b_{i_m}$.
2. Any point of $D_{n+1} \setminus D_n$ is of the form
$$(x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, t^\infty),$$
where k is a positive integer, m is a nonnegative integer, $t \in (a_{n+1}, b_{n+1}]$, and $x_m \neq a_{n+1}$.
3. $x \in D_n$ is a ramification point in D_n if and only if there are positive integers m and $j \le n$, such that $p_k(x) = a_j$ for each positive integer $k \ge m$.

Definition 3.7. *We will use D_n to denote the dendrite*
$$D_n = \varprojlim \{[0,1], f_{(a_i,b_i)_{i=1}^n}\}_{k=1}^\infty.$$

Next we define functions that we shall use later in proof of the main result.

Definition 3.8. *We define the function $f_n : D_{n+1} \to D_n$ by*
$$f_n(x) = \begin{cases} g_n(x) & ; \quad x \in \mathrm{Cl}(D_{n+1} \setminus D_n), \\ x & ; \quad x \in D_n, \end{cases}$$
where $g_n : \mathrm{Cl}(D_{n+1} \setminus D_n) \to D_n$ is defined as follows. Any point of $\mathrm{Cl}(D_{n+1} \setminus D_n)$ is of the form
$$x = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, t^\infty),$$
where k is a positive integer, m is a nonnegative integer, $t \in [a_{n+1}, b_{n+1}]$, and $x_m \neq a_{n+1}$ (see Remark 3.6), and we define
$$g_n(x) = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^\infty).$$
Note that f_n is continuous for each n by [19, Theorem 7.3, p. 108].

Lemma 3.9. *Let* $x \in D_n$.

1. *If*
$$x = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \dots, a_{i_j}^{k_j}, t^\infty) \in D_n,$$
where $j > 0$, $i_1, i_2, i_3, \dots, i_j \leq n$, $a_{i_1} < a_{i_2} < \dots < a_{i_j}$, $k_1, k_2, \dots, k_j > 0$, *and* $t \in [a_{i_j}, b_{i_j}]$, *then for each*
$$y \in f_n^{-1}(x)$$
and for each $i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1$ *it holds that* $p_i(x) = p_i(y)$.
2. *If* $x = (t^\infty)$, $t \in [0,1]$, *then for each*
$$y \in f_n^{-1}(x)$$
it holds that $p_1(x) = p_1(y) = t$.

Proof. If $y \in D_n$, then $y = x$ and the claim is obviously true. Note that in Case 1. from $t = a_{i_j}$ it follows that $y \in D_n$.

If $y \in D_{n+1} \setminus D_n$, then by 2. from Remark 3.6 y is of the form
$$(x_1, x_2, x_3, \dots, x_m, a_{n+1}^k, s^\infty),$$
where k is a positive integer, m is a nonnegative integer, $s \in (a_{n+1}, b_{n+1}]$, and $x_m \neq a_{n+1}$. Then $x = f_n(y) = g_n(y) = (x_1, x_2, x_3, \dots, x_m, a_{n+1}^\infty)$.

In Case 1. in the remaining case $t \neq a_{i_j}$ it follows that $m = k_1 + k_2 + k_3 + \ldots + k_j$ and $t = a_{n+1}$. Therefore $(x_1, x_2, x_3, \dots, x_m, a_{n+1}) = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \dots, a_{i_j}^{k_j}, t)$.

In Case 2. it follows that $m = 0$ and $t = a_{n+1}$. ∎

Lemma 3.10. *Let* $x = (x_1, x_2, x_3, \dots, x_m, a_{n+1}^\infty) \in D_n$, *where* n *is a positive integer,* m *is a nonnegative integer, and* $x_m \neq a_{n+1}$. *Then* $f_n^{-1}(x)$ *is a star centered in* x.

Proof. From what we have seen in the proof of Lemma 3.9 it follows that
$$f_n^{-1}(x) = \bigcup_{k=1}^{\infty} \{(x_1, x_2, x_3, \dots, x_m, a_{n+1}^k, s^\infty) \mid s \in [a_{n+1}, b_{n+1}]\},$$
and for each k the set
$$B_k = \{(x_1, x_2, x_3, \dots, x_m, a_{n+1}^k, s^\infty) \mid s \in [a_{n+1}, b_{n+1}]\}$$
is an arc with endpoints x and $(x_1, x_2, x_3, \dots, x_m, a_{n+1}^k, b_{n+1}^\infty)$, $B_i \cap B_j = \{x\}$ for any $i \neq j$, and $\lim_{k \to \infty} \operatorname{diam}(B_k) = 0$. ∎

Let $\{(a_n, b_n)\}_{n=1}^\infty$ be any sequence in $[0,1] \times [0,1]$, such that $a_n < b_n$ for each positive integer n, and $a_i \neq a_j$ whenever $i \neq j$. Next denote by $A(a_n, b_n)_{n=1}^\infty$ the union
$$A(a_n, b_n)_{n=1}^\infty = \bigcup_{n=1}^{\infty} ([a_n, b_n] \times \{a_n\}) \subseteq [0,1] \times [0,1].$$

and by $G(a_n,b_n)_{n=1}^{\infty}$ the subset of $[0,1]\times[0,1]$, defined by

$$G(a_n,b_n)_{n=1}^{\infty} = A \cup A(a_n,b_n)_{n=1}^{\infty},$$

where $A=\{(t,t) \mid t\in[0,1]\}$ as above.

It is easy to see that $\pi_1(G(a_i,b_i)_{i=1}^{n}) = \pi_2(G(a_i,b_i)_{i=1}^{n}) = [0,1]$.

Obviously $G(a_n,b_n)_{n=1}^{\infty}$ is not necessarily closed in $[0,1]\times[0,1]$. The following theorem gives a whole family of sets $G(a_n,b_n)_{n=1}^{\infty}$ that are closed in $[0,1]\times[0,1]$.

THEOREM 3.11. *Let* $\{(a_n,b_n)\}_{n=1}^{\infty}$ *be any sequence in* $[0,1]\times[0,1]$, *such that*

1. $a_n < b_n$ *for each positive integer* n,
2. $a_i \neq a_j$ *whenever* $i\neq j$,
3. $\lim\limits_{n\to\infty}(b_n-a_n)=0$.

Then $G(a_n,b_n)_{n=1}^{\infty}$ *is a closed subset of* $[0,1]\times[0,1]$.

PROOF. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in $G(a_n,b_n)_{n=1}^{\infty}$, which is convergent in $[0,1]\times[0,1]$ with the limit $x_0\in[0,1]\times[0,1]$. We show that $x_0\in G(a_n,b_n)_{n=1}^{\infty}$.

If there are positive integers k and n_0, such that $x_n\in[a_k,b_k]\times\{a_k\}$ for each $n\geq n_0$, then, since $[a_k,b_k]\times\{a_k\}$ is compact, $x_0\in[a_k,b_k]\times\{a_k\}$ and therefore $x_0\in G(a_n,b_n)_{n=1}^{\infty}$. Otherwise there are strictly increasing sequences $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ of integers, such that $x_{i_n}\in([a_{j_n},b_{j_n}]\times\{a_{j_n}\})\cup A$, where $A=\{(t,t)\in[0,1]\times[0,1] \mid t\in[0,1]\}$, for each positive integer n. Since $\lim\limits_{n\to\infty}(b_n-a_n)=0$, it follows that $x_0\in A$ and therefore $x_0\in G(a_n,b_n)_{n=1}^{\infty}$. ∎

Therefore by Theorem 2.3 it follows that for any sequence $\{(a_n,b_n)\}_{n=1}^{\infty}$ satisfying

1. $a_n < b_n$ for each positive integer n,
2. $a_i \neq a_j$ whenever $i\neq j$,
3. $\lim\limits_{n\to\infty}(b_n-a_n)=0$,

there is a surjective u.s.c. function $f_{(a_n,b_n)_{n=1}^{\infty}} : [0,1]\to 2^{[0,1]}$ such that its graph $\Gamma(f_{(a_n,b_n)_{n=1}^{\infty}})$ equals to $G(a_n,b_n)_{n=1}^{\infty}$, since $G(a_n,b_n)_{n=1}^{\infty}$ is a closed subset of $[0,1]\times[0,1]$ by Theorem 3.11, and since

$$\pi_1(G(a_i,b_i)_{i=1}^{n}) = \pi_2(G(a_i,b_i)_{i=1}^{n}) = [0,1].$$

DEFINITION 3.12. *Let* $\{(a_n,b_n)\}_{n=1}^{\infty}$ *be any sequence in* $[0,1]\times[0,1]$, *such that*

1. $a_n < b_n$ *for each positive integer* n,
2. $a_i \neq a_j$ *whenever* $i\neq j$,
3. $\lim\limits_{n\to\infty}(b_n-a_n)=0$.

Then $f_{(a_n,b_n)_{n=1}^\infty}$ *is called the comb function with respect to* $\{(a_n,b_n)\}_{n=1}^\infty$.

We also say that $f:[0,1]\to 2^{[0,1]}$ *is a comb function, if* f *is the comb function with respect to some sequence* $\{(a_n,b_n)\}_{n=1}^\infty$ *in* $[0,1]\times[0,1]$ *satisfying 1., 2. and 3.*

THEOREM 3.13. *Let* $f:[0,1]\to 2^{[0,1]}$ *be the comb function with respect to the sequence* $\{(a_n,b_n)\}_{n=1}^\infty$. *Then*

$$\varprojlim\{[0,1],f\}_{k=1}^\infty = \mathrm{Cl}\left(\bigcup_{n=1}^\infty D_n\right).$$

PROOF. Obviously, since $\varprojlim\{[0,1],f\}_{k=1}^\infty$ is closed in $\prod_{n=1}^\infty[0,1]$,

$$\varprojlim\{[0,1],f\}_{k=1}^\infty \supseteq \mathrm{Cl}\left(\bigcup_{n=1}^\infty D_n\right).$$

Next we show that

$$\varprojlim\{[0,1],f\}_{k=1}^\infty \subseteq \mathrm{Cl}\left(\bigcup_{n=1}^\infty D_n\right).$$

Let $x\in\varprojlim\{[0,1],f\}_{k=1}^\infty\setminus\bigcup_{n=1}^\infty D_n$. Obviously x is of the form

$$x=(a_{i_1},a_{i_2},a_{i_3},\ldots),$$

where $\{a_{i_n}\mid n=1,2,3,\ldots\}$ is an infinite subset of $\{a_n\mid n=1,2,3,\ldots\}$.

Take any open ball $U=B(x,\varepsilon)$ in $\prod_{n=1}^\infty[0,1]$ with respect to the metric D. Let m be a positive integer such that $\frac{1}{2^m}<\varepsilon$. Then

$$(a_{i_1},a_{i_2},\ldots,a_{i_{m-1}},a_{i_m}^\infty)\in U\cap D_{i_m}.$$

□

In the above proof we noticed that any $x\in\varprojlim\{[0,1],f\}_{k=1}^\infty\setminus\bigcup_{n=1}^\infty D_n$ is of the form $x=(a_{i_1},a_{i_2},a_{i_3},\ldots)$, where $\{a_{i_n}\mid n=1,2,3,\ldots\}$ is an infinite subset of $\{a_n\mid n=1,2,3,\ldots\}$. We can make this statement more precise taking into account the definitions of inverse limits and comb functions as follows:

REMARK 3.14. Let $f:[0,1]\to 2^{[0,1]}$ be the comb function with respect to the sequence $\{(a_n,b_n)\}_{n=1}^\infty$. Any point $x\in\varprojlim\{[0,1],f\}_{k=1}^\infty\setminus\bigcup_{n=1}^\infty D_n$ is of the form

$$(a_{i_1}^{k_1},a_{i_2}^{k_2},a_{i_3}^{k_3},\ldots),$$

where for each ℓ it holds that $k_\ell>0$ and $a_{i_\ell}<a_{i_{\ell+1}}\le b_{i_\ell}$.

In Examples 3.15 and 3.16 we show that there are comb functions f, such that the inverse limits $\varprojlim\{[0,1],f\}_{k=1}^\infty$ are not dendrites.

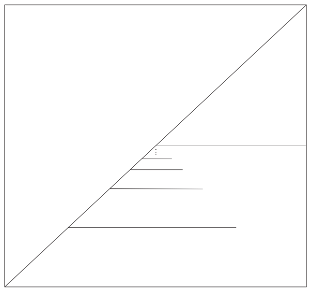

FIGURE 6. The graph of the comb function from Example 3.15

EXAMPLE 3.15. Let $(a_1,b_1)=(\frac{1}{2},1)$, and let for each positive integer $n\geq 2$, $(a_n,b_n)=(\frac{1}{2}-\frac{1}{n+1},\frac{1}{2}+\frac{1}{n+1})$. We show that $\varprojlim\{[0,1],f_{(a_n,b_n)_{n=1}^\infty}\}_{k=1}^\infty$ is not locally connected, and therefore it is not a dendrite. Let

$$x_0=(\frac{1}{2},\frac{1}{2},1^\infty)\in\varprojlim\{[0,1],f_{(a_n,b_n)_{n=1}^\infty}\}_{k=1}^\infty$$

and $\varepsilon=\min\{d(x_0,K),\frac{1}{2^3\cdot 6}\}>0$, where $K=\{(t^\infty)\mid t\in[0,1]\}$. Let $r\leq\varepsilon$

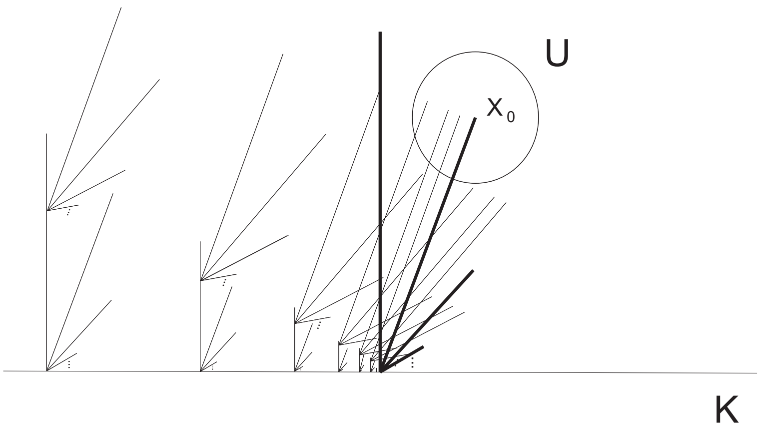

FIGURE 7. The continuum from Example 3.15

and $y=(y_1,y_2,y_3,\ldots)\in B(x_0,r)\cap\varprojlim\{[0,1],f_{(a_n,b_n)_{n=1}^\infty}\}_{k=1}^\infty$ be arbitrarily chosen. Then, since $r>D(x_0,y)\geq\frac{1-y_3}{2^3}$, it follows that $y_3>1-2^3r$. Therefore $y_3>1-2^3r\geq 1-\frac{2^3}{6\cdot 2^3}=\frac{5}{6}$, and hence $y_i=y_3$ for each $i\geq 3$. Furthermore, $y_2\in f(y_3)=\{\frac{1}{2},y_3\}$. If $y_2=y_3$, then

$$D(x_0,y)\geq\frac{y_2-\frac{1}{2}}{2^2}>\frac{\frac{5}{6}-\frac{1}{2}}{2^2}=\frac{1}{12}>r,$$

a contradiction. Therefore $y_2 = \frac{1}{2}$, and hence $y_1 \in f(\frac{1}{2})$. Clearly there is a positive integer n, such that $y_1 = a_n$ and $\frac{\frac{1}{2}-a_n}{2} = \frac{x_1-y_1}{2} < r$.

Therefore for each $r \le \varepsilon$, $y \in B(x_0, r)$ if and only if there is a positive integer n, such that $y = (a_n, \frac{1}{2}, t^\infty)$, where $\frac{\frac{1}{2}-a_n}{2} < r$ and $t > 1 - 2^3 r$.

Therefore for each $r \le \varepsilon$ the intersection $B(x_0, r) \cap \varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^\infty}\}_{k=1}^\infty$ is the union of countably many mutually disjoint intervals

$$\{(a_n, \frac{1}{2}, t^\infty) \mid t \in (1 - 2^3 r, 1]\},$$

where $\frac{\frac{1}{2}-a_n}{2} < r$. See Fig. 7.

Example 3.16. Let $(a_1, b_1) = (\frac{1}{2}, 1)$, and let for each positive integer $n \ge 2$, $(a_n, b_n) = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2})$. A similar argument as in Example 3.15 shows

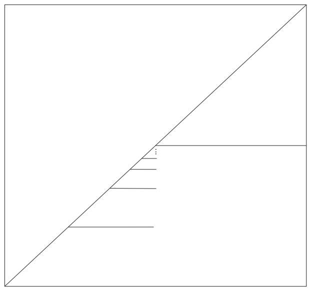

Figure 8. The graph of the comb function in Example 3.16

that the inverse limit $\varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^\infty}\}_{k=1}^\infty$ is not locally connected, and therefore it is not a dendrite.

In Theorem 3.20 we prove that under rather weak additional assumptions the inverse limit of a comb function is a dendrite. Essentially, the conditions say that the only comb functions for which the inverse limits are not dendrites are similar to those from Examples 3.15 and 3.16. Before stating and proving the theorem we introduce the necessary notion of admissible sequences and prove a few lemmas.

Definition 3.17. *The sequence $\{(a_n, b_n)\}_{n=1}^\infty$ in $[0,1] \times [0,1]$ is admissible if for each positive integer n there is a positive integer $\mu(n) \ge n$, such that for each $m \ge \mu(n)$ it holds that if $a_m < a_n$, then $b_m < a_n$.*

Lemma 3.18. *Let $f : [0,1] \to 2^{[0,1]}$ be any comb function with respect to a sequence $\{(a_n, b_n)\}_{n=1}^\infty$, and let*

$$x = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_j}^{k_j}, t^\infty) \in D_n,$$

$j \geq 0$, $i_1, i_2, i_3, \ldots, i_j \leq n$, $a_{i_1} < a_{i_2} < \cdots < a_{i_j}$, $k_1, k_2, \ldots, k_j > 0$, and $t \in [a_{i_j}, b_{i_j}]$. Let f_ℓ be the functions defined in Definition 3.8. Then for each

$$y \in \mathrm{Cl}(\bigcup_{k \geq n} (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x))$$

and for each $i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1$ it holds that $p_i(x) = p_i(y)$ (where x and y are interpreted as elements of $\Pi_{n=1}^{\infty}[0,1]$).

Proof. By induction on $k - n$ we prove the following claim:
for each

$$y \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)$$

and for each $i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1$ it holds that $p_i(x) = p_i(y)$.

For $k - n = 0$ the statement holds true by Lemma 3.9 (part 1. for $j > 0$ and part 2. for $j = 0$).

Let $k - n = \ell$ and assume that the claim is true for $\ell - 1$. Since

$$(f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x) = \bigcup_{z \in (f_n \circ f_{n+1} \circ \ldots \circ f_{k-1})^{-1}(x)} f_k^{-1}(z)$$

for any $y \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)$ we choose $z \in (f_n \circ f_{n+1} \circ \ldots \circ f_{k-1})^{-1}(x)$ such that $y \in f_k^{-1}(z)$. By the induction assumption $p_i(x) = p_i(z)$ for each $i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1$, and by Lemma 3.9 $p_i(y) = p_i(z)$ again for each $i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1$. This completes the proof since the limits of sequences of points with the required property have the property. ∎

We will also need the following lemma about point preimages.

Lemma 3.19. *Let $f : [0,1] \to 2^{[0,1]}$ be the comb function with respect to any admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. For each $\varepsilon > 0$ there is a positive integer k such that*

$$\mathrm{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) < \varepsilon$$

for each $p \in D_k$, where maps f_n are defined as in Definition 3.8.

Proof. Let $\varepsilon > 0$ and m be a positive integer such that $\frac{1}{2^{m-1}} < \varepsilon$. Also let $n_0 > m$ be any positive integer such that for each $n \geq n_0$, it holds that $b_n - a_n < \frac{\varepsilon}{m}$. For each positive integer ℓ, let $\mu(\ell)$ be a positive integer such that for each $n \geq \mu(\ell)$ it holds that if $a_n < a_\ell$, then $b_n < a_\ell$ (here we use the admissibility of the sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$).

Let

$$\begin{aligned}
k_0 &= \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(n_0)\},\\
k_1 &= \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_0)\},\\
k_2 &= \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_1)\},\\
&\vdots\\
k_m &= \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_{m-1})\}.
\end{aligned}$$

Then we show that

$$k = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_m)\}$$

is a positive integer, such that

$$\operatorname{diam}(\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) < \varepsilon$$

for each $p \in D_k$.

Take any $p \in D_k$. Then by 1. from Remark 3.6 p is either of the form $p = (t^\infty)$, $t \in [0,1]$, or of the form $p = (p_1, p_2, p_3, \ldots, p_j, t^\infty)$, where $p_j = a_s$ for some $s \leq k$ and $t \in (a_s, b_s]$.

Clearly, it holds that

$$\operatorname{diam}(\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) \leq \operatorname{diam}(\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty))$$

since

$$\begin{aligned}
&(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}((p_1, p_2, p_3, \ldots, p_j, t^\infty))\\
&\quad = \{(p_1, p_2, p_3, \ldots, p_j, x_1, x_2, x_3, \ldots) \mid (x_1, x_2, x_3, \ldots)\\
&\quad \in (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)\}.
\end{aligned}$$

If $t \neq a_i$ for all $i > k$, then $\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots f_n)^{-1}(t^\infty) = \{(t^\infty)\}$, and therefore $\operatorname{diam}(\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots f_n)^{-1}(t^\infty)) = 0$.

If $t = a_i$ for some $i > k$, then we shall prove that

$$\operatorname{diam}(\bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)) < \varepsilon$$

by proving that

$$D(y, (t^\infty)) = D(y, (a_i^\infty)) < \frac{\varepsilon}{2}$$

for arbitrary $y = (y_1, y_2, y_3, \ldots, y_m, \ldots) \in \bigcup_{n\geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)$.

Since $y_1 = t = a_i$ by Lemma 3.18 (and therefore $\frac{y_1 - a_i}{2} = 0$), and since $y_1 \leq y_2 \leq y_3 \leq \ldots$, it follows that

$$D(y, (a_i^\infty)) \leq \sup\{\frac{y_2 - a_i}{2^2}, \frac{y_3 - a_i}{2^3}, \ldots, \frac{y_m - a_i}{2^m}, \frac{1}{2^{m+1}}\}.$$

Let $j \in \{2, 3, 4, \ldots, m\}$ be arbitrary. We show that

$$\frac{y_j - a_i}{2^j} < \frac{\varepsilon}{2}.$$

First we show that for each $s \in \{2, 3, 4, \ldots, j\}$ there is a positive integer $\ell > n_0$ such that $y_s, y_{s-1} \in [a_\ell, b_\ell]$.

For $s = 2$, the claim is true since $y_2 \in [a_i, b_i]$, $y_1 = a_i$, and $i > k \geq n_0$.

If $y_2 \notin \{a_n \mid n = 1, 2, 3, \ldots\}$, then $y_2 = y_3 = y_4 = \cdots$ and therefore for each $s \in \{3, 4, 5, \ldots, j\}$, $y_s = y_{s-1} = y_2 \in [a_i, b_i]$.

In the rest of the proof we consider the case $y_2 = a_{i_0}$ for some positive integer i_0. If $i_0 \leq k_m$, then $\mu(i_0) \in \{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_m)\}$, and therefore $\mu(i_0) \leq k$. Since $k < i$, it follows that $\mu(i_0) < i$. Therefore from $y_2 = a_{i_0} > a_i$ it follows that $a_{i_0} > b_i$, and this contradicts the fact that $a_{i_0} = y_2 \in [a_i, b_i]$. So in this case $i_0 > k_m \geq n_0$, and the claim for $s = 3$ follows, since $y_3, y_2 \in [a_{i_0}, b_{i_0}]$.

If $y_3 \notin \{a_n \mid n = 1, 2, 3, \ldots\}$, then $y_3 = y_4 = y_5 = \cdots$, and therefore for each $s \in \{4, 5, 6, \ldots, j\}$, $y_s = y_{s-1} = y_3 \in [a_{i_0}, b_{i_0}]$.

In the rest of the proof we consider the case $y_3 = a_{i_1}$ for some positive integer i_1. If $i_1 \leq k_{m-1}$, then $\mu(i_1) \in \{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_{m-1})\}$, and therefore $\mu(i_1) \leq k_m$. Since $k_m < i_0$, it follows that $\mu(i_1) < i_0$. Therefore from $y_3 = a_{i_1} > a_{i_0}$ it follows that $a_{i_1} > b_{i_0}$, and this contradicts the fact that $a_{i_1} = y_3 \in [a_{i_0}, b_{i_0}]$. So in this case $i_1 > k_{m-1} \geq n_0$, and the claim follows for $s = 4$, since $y_4, y_3 \in [a_{i_1}, b_{i_1}]$.

Repeating this reasoning m times proves that for each $s \in \{2, 3, 4, \ldots, j\}$ there is a positive integer $\ell > n_0$ such that $y_s, y_{s-1} \in [a_\ell, b_\ell]$.

It follows that

$$d(y_j, a_i) \leq d(y_j, y_{j-1}) + \ldots + d(y_3, y_2) + d(y_2, a_i) \leq (j-1)\frac{\varepsilon}{m} < \varepsilon,$$

since $y_s, y_{s-1} \in [a_\ell, b_\ell]$ for each s, for some $\ell > n_0$. Therefore $\frac{y_j - a_i}{2^m} < \frac{\varepsilon}{2}$. □

THEOREM 3.20. *Let $f : [0, 1] \to 2^{[0,1]}$ be the comb function with respect to any admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Then $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is a dendrite.*

PROOF. We show that $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to the inverse limit of an inverse sequence of dendrites with monotone bonding functions, which is by [20, Theorem 10.36, p. 180] a dendrite, and therefore the inverse limit $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is a dendrite, too.

More specifically we prove that the inverse limit $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to $\varprojlim\{D_n, f_n\}_{n=1}^{\infty}$, where $f_n : D_{n+1} \to D_n$ is the mapping defined in Definition 3.8 and that each f_n is monotone.

For fixed $x = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^{\infty})$, $x_m \neq a_{n+1}$, and fixed k, let

$$B_k(x) = \{(x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, t^{\infty}), t \in [a_{n+1}, b_{n+1}]\}.$$

Then each

$$S(x) = \bigcup_{k=1}^{\infty} B_k(x),$$

is the star with the center x and beams $B_k(x)$, $k = 1, 2, 3, \ldots$

Using Remark 3.6 we see that

1. $f_n^{-1}(x) = \{x\}$ for each $x \in D_n \setminus \mathrm{Cl}(D_{n+1} \setminus D_n)$, and
2. $f_n^{-1}(x)$ is the star $S(x)$ for each $x \in D_n \cap \mathrm{Cl}(D_{n+1} \setminus D_n)$.

Therefore $f_n : D_{n+1} \to D_n$ is monotone for each n, and by [20, Theorem 10.36, p. 180]

$$\varprojlim\{D_n, f_n\}_{n=1}^{\infty}$$

is a dendrite.

Next we show that by

$$F(x_1, x_2, x_3, \ldots) = \lim_{n\to\infty} x_n$$

a homeomorphism

$$F : \varprojlim\{D_n, f_n\}_{n=1}^{\infty} \to \varprojlim\{[0,1], f\}_{n=1}^{\infty}$$

is defined.

1. First we show that $F : \varprojlim\{D_n, f_n\}_{n=1}^{\infty} \to \varprojlim\{[0,1], f\}_{n=1}^{\infty}$ is a well defined function. Take any point $(x_1, x_2, x_3, \ldots)$ in $\varprojlim\{D_n, f_n\}_{n=1}^{\infty} \subseteq \Pi_{i=1}^{\infty} D_i$. If there is a positive integer n_0, such that $x_n = x_{n_0}$ for each $n \geq n_0$, then $\lim_{n\to\infty} x_n = x_{n_0}$ and $x_{n_0} \in D_{n_0} \subseteq \varprojlim\{[0,1], f\}_{n=1}^{\infty}$. Therefore $F(x_1, x_2, x_3, \ldots) \in \varprojlim\{[0,1], f\}_{n=1}^{\infty}$. If there is no such n_0, then let $i_1 < i_2 < i_3 < \ldots$ be the sequence of all such integers that $x_{i_n} \neq x_{i_n+1}$ for each n. Then $x_{i_{n+1}} = x_{i_n+1} \in f_{i_n}^{-1}(x_{i_n})$, where $f_{i_n}^{-1}(x_{i_n})$ is the star $S(x_{i_n}) \subseteq D_{i_n+1}$ with center x_{i_n}. Therefore x_{i_n} is of the form

$$x_{i_n} = (y_1, y_2, y_3, \ldots, y_{m_n}, a_{i_n+1}^{\infty}).$$

Similarly, $x_{i_{n+1}}$ is of the form

$$x_{i_{n+1}} = (z_1, z_2, z_3, \ldots, z_{m_{n+1}}, a_{i_n+2}^{\infty}).$$

Since $x_{i_{n+1}} \in S(x_{i_n})$, it follows that $m_n < m_{n+1}$ and $y_i = z_i$ for each $i = 1, \ldots, m_n$. From $m_n < m_{n+1}$ for each n, it follows that $m_n \geq n$ for each n. Therefore $D(x_{i_n}, x_{i_{n+1}}) \leq \frac{1}{2^{m_n}} \leq \frac{1}{2^n}$. It follows that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathrm{Cl}\left(\bigcup_{n=1}^{\infty} D_n\right)$, and hence by Theorem 3.13 it converges to a point in $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$.
2. We show that F is continuous.

Take any $x = (x_1, x_2, x_3, \ldots) \in \varprojlim\{D_n, f_n\}_{n=1}^{\infty}$ and any $\varepsilon > 0$. Choose a positive integer k (given by Lemma 3.19), such that

$$\operatorname{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) < \varepsilon$$

for each $p \in D_k$.

Let $B = \{z \in \varprojlim\{[0,1], f\}_{n=1}^{\infty} \mid d(z, F(x)) < \varepsilon\}$, and let

$$V = P_k^{-1}(B \cap D_k),$$

where $P_k : \varprojlim\{D_n, f_n\}_{n=1}^{\infty} \to D_k$ is the projection map to the k-th factor. Since $B \cap D_k$ is open in D_k, V is open in $\varprojlim\{D_n, f_n\}_{n=1}^{\infty}$. Since $x \in \varprojlim\{D_n, f_n\}_{n=1}^{\infty}$ and $x_k \in D_k$, it follows from the definition of F that $F(x) \in \mathrm{Cl}(\bigcup_{n \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(x_k))$. From the definition of functions f_j it follows that $x_k \in \mathrm{Cl}(\bigcup_{n \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(x_k))$. Therefore $d(x_k, F(x)) < \varepsilon$, hence $x_k \in B$. It follows that $x_k \in B \cap D_k$, and thus $x \in V$. Let $y = (y_1, y_2, y_3, \ldots) \in V$. It follows that $y_k \in B$, and therefore $d(y_k, F(x)) < \varepsilon$. Since $F(y), y_k \in \mathrm{Cl}(\bigcup_{n \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(y_k))$, it follows that $d(y_k, F(y)) < \varepsilon$. Hence,

$$d(F(x), F(y)) \leq d(F(x), y_k) + d(y_k, F(y)) < 2\varepsilon.$$

Therefore F is continuous.

3. We show that F is a surjection. Let

$$y = (y_1, y_2, y_3, \ldots) \in \varprojlim\{[0,1], f\}_{n=1}^{\infty}$$

be arbitrarily chosen. We define a sequence $\{x_n\}_{n=1}^{\infty}$, such that
   (a) for each n, $x_n \in D_n$,
   (b) for each n, $f_n(x_{n+1}) = x_n$,
   (c) $\lim_{n \to \infty} x_n = y$.

If $y \notin D_n$ for each n, then by Remark 3.14 y is of the form $y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots)$, where for each ℓ, it holds that $a_{i_\ell} < a_{i_{\ell+1}} \leq b_{i_\ell}$ and that k_ℓ is a positive integer. In this case we define

$$x_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, a_{i_{m+1}}^{\infty}),$$

where $i_\ell \leq n$ for each $\ell = 1, 2, 3, \ldots, m$, and $i_{m+1} > n$. If $y \in D_m$ for some m, then define $x_n = y$ for $n \geq m$ and $x_n = (f_n \circ \cdots \circ f_{m-1})(y)$ for $n < m$.

Obviously the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies (a), (b) and (c), and therefore $F(x_1, x_2, x_3, \ldots) = y$.

4. Finally we show that F is an injection. Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be any points in $\varprojlim\{D_n, f_n\}_{n=1}^{\infty}$ such that $x \neq y$. Let k be a positive integer such that $x_k \neq y_k$. Since $x_k, y_k \in D_k$, it follows that

$$x_k = (a_{i_1}^{q_1}, a_{i_2}^{q_2}, a_{i_3}^{q_3}, \ldots, a_{i_j}^{q_j}, t^{\infty})$$

and

$$y_k = (a_{\ell_1}^{r_1}, a_{\ell_2}^{r_2}, a_{\ell_3}^{r_3}, \ldots, a_{\ell_m}^{r_m}, s^\infty),$$

where $i_1, i_2, \ldots, i_j, \ell_1, \ell_2, \ldots, \ell_m \leq k$, $t \in (a_{i_j}, b_{i_j}]$ and $s \in (a_{\ell_m}, b_{\ell_m}]$, by 1. from Remark 3.6. Let $q = q_1 + q_2 + q_3 + \ldots + q_j$, $r = r_1 + r_2 + r_3 + \ldots + r_m$. Assume that $q \leq r$. Also, let n be the smallest integer such that $p_n(x_k) \neq p_n(y_k)$. If $n \leq q$ then for each $z_1 \in \mathrm{Cl}(\bigcup_{i \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(x_k))$ and each $z_2 \in \mathrm{Cl}(\bigcup_{i \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(y_k))$ by Lemma 3.18 it follows that $p_n(z_1) = p_n(x_k)$ and $p_n(z_2) = p_n(y_k)$, and therefore

$$D(z_1, z_2) \geq \frac{d(p_n(x_k), p_n(y_k))}{2^n}.$$

Since

$$F(x) \in \mathrm{Cl}(\bigcup_{i \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(x_k))$$

and

$$F(y) \in \mathrm{Cl}(\bigcup_{i \geq k}(f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(y_k))$$

it follows that $F(x) \neq F(y)$.

If $n > q$, then y_k is of the form

$$y_k = (a_{i_1}^{q_1}, a_{i_2}^{q_2}, a_{i_3}^{q_3}, \ldots, a_{i_j}^{q_j}, a_{i_j}^{p}, a_{\ell_{j+1}}^{r_{j+1}}, a_{\ell_{j+2}}^{r_{j+2}}, a_{\ell_{j+3}}^{r_{j+3}}, \ldots, a_{\ell_m}^{r_m}, s^\infty),$$

since $r \geq q$.

We consider several cases.

Case 1. If $p \geq 1$, then $n = q+1$, since $p_{q+1}(x_k) = t$ and $p_{q+1}(y_k) = a_{i_j}$, and by Lemma 3.18 $p_n(F(x)) = p_n(x_k) = t \neq a_{i_j} = p_n(y_k) = p_n(F(y))$, hence $F(x) \neq F(y)$.

Case 2. If $p + r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m = 0$, then $n = q + 1$ and by Lemma 3.18 $p_n(F(x)) = p_n(x_k) = t \neq s = p_n(y_k) = p_n(F(y))$, hence $F(x) \neq F(y)$.

Case 3. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if there is a positive integer $i \leq k$ such that $t = a_i$, then $F(x) = x_k$ and $n \leq r+1$, and it follows that $p_n(F(x)) = p_n(x_k) \neq p_n(y_k) = p_n(F(y))$, where the last equality follows by Lemma 3.18.

Case 4. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if there is a positive integer $i > k$ such that $t = a_i$, then $n = q + 1$ since $p_{q+1}(y_k) = a_{l_{j+1}}$ and $l_{j+1} \leq k$, while $p_{q+1}(x_k) = t = a_i$, $i > k$. Therefore $p_n(F(x)) = p_n(x_k) = a_i \neq a_{l_{j+1}} = p_n(y_k) = p_n(F(y))$, by Lemma 3.18.

Case 5. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if $t \neq a_i$ for each positive integer i, then $F(x) = x_k$ and $n = q + 1 < r + 1$, and we continue as in Case 3.

Since $F : \varprojlim\{K_n, f_n\}_{n=1}^{\infty} \to \varprojlim\{[0,1], f\}_{n=1}^{\infty}$ is a continuous bijection from a compact space onto a metric space, it is by [19, Theorem 5.6, p. 167] a homeomorphism. ∎

4. Ważewski's universal dendrite as an inverse limit with one bonding function

The following example shows that the conditions of Theorem 3.20 are not sufficient to guaranty that the corresponding inverse limit is homeomorphic to Ważewski's universal dendrite.

Example 4.1. Let for each positive integer n, $(a_n, b_n) = (1 - \frac{1}{2^{2n}}, 1 - \frac{1}{2^{2n+1}})$. By Theorem 3.20, $\varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^{\infty}}\}_{k=1}^{\infty}$ is a dendrite. Since $a_n < b_n < a_{n+1}$ for each positive integer n, using Lemma 4.4 and 3. from Remark 3.6, we see that $x \in \varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^{\infty}}\}_{k=1}^{\infty}$ is a ramification point if and only if there is a positive integer m, such that $x = (a_m^{\infty})$. Therefore the set of all ramification points is not dense in $\varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^{\infty}}\}_{k=1}^{\infty}$. Hence, by Theorem 2.2, $\varprojlim\{[0,1], f_{(a_n,b_n)_{n=1}^{\infty}}\}_{k=1}^{\infty}$ is not homeomorphic to Ważewski's universal dendrite.

In Theorem 4.5 we show that with the additional condition that the set $\{a_n \mid n = 1,2,3,\ldots\}$ is dense in $[0,1]$, it follows that the inverse limit $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite. In Theorem 4.6 we show that in fact this additional condition characterizes inverse limits $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ that are homeomorphic to Ważewski's universal dendrite.

First we prove the following lemmas.

Lemma 4.2. *Let $f : [0,1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Let*

$$y \in \varprojlim\{[0,1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n$$

be arbitrarily chosen. Then for each $x \in \bigcup_{n=1}^{\infty} D_n$ the uniquely determined arc L from x to y satisfies the condition

$$L \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n.$$

Proof. By Remark 3.14 y is of the form $y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots)$, where $a_{i_\ell} < a_{i_{\ell+1}} \leq b_{i_\ell}$ for each ℓ. We use the same sequence $\{x_n\}_{n=1}^{\infty}$ as in the proof of surjectivity of F in the proof of Theorem 3.20, i.e.,

$$x_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, a_{i_{m+1}}^{\infty}) \in D_n,$$

where $i_\ell \leq n$ for each $\ell = 1,2,3,\ldots,m$, and $i_{m+1} > n$. Since D_{n+1} is a dendrite, there is a unique arc $[x_n, x_{n+1}]$ from x_n to x_{n+1} in D_{n+1} if $x_n \neq$

x_{n+1}. If $x_n = x_{n+1}$ let $[x_n, x_{n+1}]$ denote $\{x_n\}$. Then $A = \bigcup_{n=1}^{\infty}[x_n, x_{n+1}] \cup \{y\}$ is an arc from x_1 to y, since $[x_n, x_{n+1}] \setminus \{x_n\} \in D_{n+1} \setminus D_n$ and since $\lim_{n\to\infty} x_n = y$, as shown in the above mentioned proof of Theorem 3.20. Obviously

$$A \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n.$$

Next, take the unique arc B from x_1 to x in $\bigcup_{n=1}^{\infty} D_n$ (the existence of such an arc follows from the fact that there is an integer m such that $x_1, x \in D_m$). Then $\mathrm{Cl}((A \setminus B) \cup (B \setminus A))$ is an arc from x to y in $\varprojlim\{[0,1], f_{(a_i,b_i)_{i=1}^{\infty}}\}_{k=1}^{\infty}$. Since $\varprojlim\{[0,1], f_{(a_i,b_i)_{i=1}^{\infty}}\}_{k=1}^{\infty}$ is a dendrite, it follows that $\mathrm{Cl}((A \setminus B) \cup (B \setminus A)) = L$. Obviously

$$L \setminus \{y\} = \mathrm{Cl}((A \setminus B) \cup (B \setminus A)) \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n.$$

□

LEMMA 4.3. *Let $f : [0,1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Then each*

$$y \in \varprojlim\{[0,1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n$$

is an endpoint of $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ (and hence it is not a ramification point).

PROOF. Assuming that y is not an endpoint, using Lemma 4.2, one easily gets a simple closed curve in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$. This contradicts the fact that $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is a dendrite by Theorem 3.20. □

LEMMA 4.4. *Let $f : [0,1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Let $x \in \varprojlim\{[0,1], f\}_{k=1}^{\infty}$. The following statements are equivalent.*

1. *x is a ramification point in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$.*
2. *x is a ramification point in D_n for some positive integer n.*

PROOF. It is obvious that if there is a positive integer n, such that x is a ramification point in D_n, then x is a ramification point in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ (since $D_n \subseteq \varprojlim\{[0,1], f\}_{k=1}^{\infty}$). Suppose that x is a ramification point in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$. Since no point of

$$\varprojlim\{[0,1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n$$

is a ramification point in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$, by Lemma 4.3, it follows that $x \in D_{n_0}$ for some positive integer n_0. Let $[x, x_i]$, $i = 1, 2, 3$, be any three arcs in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$, such that $[x, x_1] \cup [x, x_2] \cup [x, x_3]$ is a simple triod.

Without loss of generality we may assume that $x_i \in \bigcup_{n=1}^{\infty} D_n$, $i = 1, 2, 3$, since if $x_i \notin \bigcup_{n=1}^{\infty} D_n$, we may replace $[x, x_i]$ by $[x, y_i]$, where $y_i \in (x, x_i)$, by Lemma 4.2. For each $i = 1, 2, 3$ let n_i be a positive integer such that $x_i \in D_{n_i}$. Let $n = \max\{n_0, n_1, n_2, n_3\}$. Obviously$[x, x_1] \cup [x, x_2] \cup [x, x_3]$ is a simple triod in D_n, and therefore x is a ramification point in D_n. ∎

THEOREM 4.5. *Let $f : [0, 1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0, 1]$. Then $\varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite.*

PROOF. By Theorem 3.20, $D = \varprojlim\{[0, 1], f\}_{k=1}^{\infty}$ is a dendrite. We show that the set of ramification points of D is dense in D and that each ramification point of D is of infinite order in D, and therefore by Theorem 2.2 D is homeomorphic to Ważewski's universal dendrite.

Let $y = (y_1, y_2, y_3, \ldots) \in D$ be arbitrarily chosen, such that y is not a ramification point. We show that there is a sequence of ramification points $\{z_n\}_{n=1}^{\infty}$ in D, such that $\lim_{n\to\infty} z_n = y$.

If $y \in D_n$ for some positive integer n, then by 1. and 3. from Remark 3.6 (taking into account that by Lemma 4.4 y is not a ramification point in D_ℓ for each ℓ) there are a positive integer m and a real number $t \in [0, 1] \setminus \{a_1, a_2, a_3, \ldots\}$, such that

$$y = (y_1, y_2, y_3, \ldots, y_{m-1}, t^{\infty}),$$

where $y_{m-1} = a_k$ for some $k \le n$, and $t \in (a_k, b_k]$. Since the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0, 1]$, there is a strictly increasing sequence $\{i_n\}_{n=1}^{\infty}$ of positive integers, such that $\lim_{n\to\infty} a_{i_n} = t$ and $a_{i_n} \in (a_k, b_k]$. Therefore

$$\{(y_1, y_2, y_3, \ldots, y_{m-1}, a_{i_n}^{\infty})\}_{n=1}^{\infty}$$

is a sequence of ramification points in D, which converges to y.

If $y \in D \setminus \bigcup_{n=1}^{\infty} D_n$, then by Remark 3.14

$$y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots),$$

where for each ℓ it holds that $k_\ell > 0$ and $a_{i_\ell} < a_{i_{\ell+1}} \le b_{i_\ell}$. Then the sequence $\{z_n\}_{n=1}^{\infty}$, where

$$z_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_{n-1}}^{k_{n-1}}, a_{i_n}^{\infty})$$

for each n, is a sequence of ramification points in D, which converges to y.

Next we show that each of the ramification points is of infinite order in D. Let $x \in D$ be any ramification point. Then by Lemma 4.4 and 3. from Remark 3.6 there are positive integers m and j, such that $p_k(x) = a_j$ for each positive integer $k \ge m$. Without loss of generality we may assume that $p_k(x) \ne a_j$ for each $k < m$.

Since

$$x \in f_{j-1}^{-1}(x) \subseteq D$$

and $f_{j-1}^{-1}(x)$ is a star with the center x by Lemma 3.10, it follows that x is of infinite order in D. □

THEOREM 4.6. *Let $f : [0,1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Then $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite if and only if the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0,1]$.*

PROOF. Taking Theorem 4.5 into account it suffices to prove that if the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is not dense in $[0,1]$, then $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is not homeomorphic to Ważewski's universal dendrite. If there is an interval $J = (a_j, a_k) \subseteq [0,1]$ containing no a_n, let $t = \frac{a_j + a_k}{2}$ and $\delta = \frac{a_k - a_j}{2}$. For any ramification point x of $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ $D(x, (t^{\infty})) \geq \frac{d(p_1(x), t)}{2} > \delta$, since $p_1(x) = a_n$ for some n. Therefore the open ball in $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ centered at (t^{∞}) with the radius δ contains no ramification points and hence by Theorem 2.2 $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is not homeomorphic to Ważewski's universal dendrite. □

THEOREM 4.7. *There is a comb function f such that $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite.*

PROOF. Let $\{a_n \mid n \in \mathbb{N}\}$ be any dense subset of $(0,1)$. Inductively we define sequence $\{b_n\}_{n=1}^{\infty}$ in such a way that $\{(a_n, b_n)\}_{n=1}^{\infty}$ would be admissible which would by Theorem 4.5 guaranty that $\varprojlim\{[0,1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski's universal dendrite. For each positive integer n, let

$$b_n = \frac{1}{2}\left(a_n + \min\{1, a_i \mid i < n, a_i > a_n\}\right).$$

First we show that $\lim_{n\to\infty}(b_n - a_n) = 0$. Let $\varepsilon > 0$ be arbitrary; choose a positive integer k such that $\frac{1}{k} < \varepsilon$. For each $j \leq k$ choose i_j, such that $a_{i_j} \in (\frac{j-1}{k}, \frac{j}{k})$, and let $n_0 = \max\{i_j \mid j = 1, 2, 3, \ldots, k\}$. For any $n > n_0$ let $a < b$ be two consecutive elements of the set $\{0, 1, a_{i_j} \mid j = 1, 2, 3, \ldots, k\}$ such that $a_n \in (a, b)$. Then $b_n - a_n \leq \frac{a_n + b}{2} - a_n = \frac{b - a_n}{2} < \frac{b - a}{2} < \varepsilon$.

Since for each positive integer n for each $m \geq n$ it holds that if $a_m < a_n$, then $b_m < \frac{1}{2}(a_m + a_n) < a_n$, it follows that the sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ is admissible. □

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