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ON LOGARITHMIC CONVEXITY FOR GIACCARDI'S DIFFERENCE

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ABSTRACT. In this paper, the Giaccardi's difference is considered in some special cases. By utilizing two classes of the convex functions, the logarithmic convexity of the Giaccardi's difference is proved. The positive semi-definiteness of the matrix generated by Giaccardi's difference is shown. Related means of Cauchy type are defined and monotonicity property of these means is proved. The related mean value theorems of Cauchy type are also given.

1. Introduction and preliminaries

The well known Giaccardi's inequality[1] is given in the following result (see also [5, page 153, 155]).

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, $(p_1, ..., p_n)$ be a non-negative n-tuple, $(x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and*

(1) $$(x_i - x_0)(\tilde{x}_n - x_i) \geq 0 \ \ for \ i = 1, ..., n, \ \ \tilde{x}_n \neq x_0.$$

If f is a convex function, then

(2) $$\sum_{k=1}^{n} p_k f(x_k) \leq Af(\tilde{x}_n) + Bf(x_0)$$

holds, where

(3) $$A = \frac{\sum_{i=1}^{n} p_i(x_i - x_0)}{\tilde{x}_n - x_0}, \quad B = \frac{\left(\sum_{i=1}^{n} p_i - 1\right) \tilde{x}_n}{\tilde{x}_n - x_0}.$$

Remark 1.2. *Condition that f is convex function can be replaced with $(f(x) - f(x_0))/(x - x_0)$ is an increasing function, then inequality* (2) *is also valid* [5, pages 152-153].

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Moreover, we proved the following result in [3].

Theorem 1.3. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, $(p_1, ..., p_n)$ be a non-negative n-tuple, $(x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and* (1) *is satisfied. If $f(x)/(x - x_0)$ is an increasing function on $I \backslash \{x_0\}$, then*

$$\sum_{k=1}^{n} p_k f(x_k) \leq A f(\tilde{x}_n) \tag{4}$$

holds, where A is the same as in Theorem 1.1.

In the same paper ([3]), we considered the following Giaccardi's type difference,

$$\Upsilon_t = \begin{cases} \frac{1}{t-1}\left\{A(\tilde{x}_n - x_0)^t - \sum\limits_{k=1}^{n} p_k (x_k - x_0)^t\right\}, & t \neq 1, \\ A(\tilde{x}_n - x_0)\log(\tilde{x}_n - x_0) - \sum\limits_{k=1}^{n} p_k (x_k - x_0)\log(x_k - x_0), & t = 1, \end{cases} \tag{5}$$

where $t \in \mathbb{R}$ and

$$\tilde{x}_n \geq x_i > x_0 \quad for \;\; i = 1, ..., n. \tag{6}$$

If $t \mapsto \Upsilon_t$ is a positive valued function, then for $r, s, t \in \mathbb{R}$ such that $r \leq s \leq t$, we proved

$$(\Upsilon_s)^{t-r} \leq (\Upsilon_r)^{t-s} (\Upsilon_t)^{s-r}. \tag{7}$$

An interesting question is to remove the condition (6) i.e. consider the case when for some i we have $x_i \leq x_0$ and for some i, $x_i > x_0$. To give an answer to this question, we can consider the case of convex function instead of that in Theorem 1.3. Namely, we can use the following theorem [2].

Theorem 1.4. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, $(p_1, ..., p_n)$ be a real n-tuple, $(x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and*

$$\begin{aligned} &\sum_{i=1}^{k} p_i (\tilde{x}_n - x_i) \geq 0 \quad (1 \leq k \leq m), \\ &\sum_{i=k}^{n} p_i (\tilde{x}_n - x_i) \leq 0 \quad (m + 1 \leq k \leq n). \end{aligned} \tag{8}$$

Then, for every convex function f on I, the inequality (2) *holds. If the reverse inequalities hold in* (8)*, then the reverse inequality holds in* (2)*.*

Let us consider the following Giaccardi's difference:

$$\Phi(\mathbf{x};\mathbf{p};f) = Af(\tilde{x}_n) - \sum_{k=1}^{n} p_k f(x_k) + Bf(x_0). \tag{9}$$

In this paper, we consider two classes of parameterized convex functions and give logarithmic convexity of the difference defined in (9), as a function of parameter. We introduce a symmetric matrix generated by the difference and prove that this matrix is positive semi-definite. We construct certain Cauchy type means and show that these means are monotone in each variable. Also we prove related mean value theorems of Cauchy type.

2. Main results

Lemma 2.1. *For all $t \in \mathbb{R}$, let $\phi_t : (0,\infty) \to \mathbb{R}$ be the function defined as*

$$\phi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0,1; \\ -\log x, & t = 0; \\ x \log x, & t = 1. \end{cases}$$

Then $\phi_t(x)$ is convex on $(0,\infty)$.

Proof. Since

$$\phi_t''(x) = x^{t-2} > 0 \quad \text{for all} \quad x \in (0,\infty), t \in \mathbb{R},$$

therefore $\phi_t(x)$ is convex on $(0,\infty)$ for every $t \in \mathbb{R}$. □

Lemma 2.2. [4] *A positive function f is log-convex in the Jensen sense on an interval I, that is, for each $s,t \in I$*

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right),$$

if and only if the relation

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0$$

holds for each real u,w and $s,t \in I$.

The following lemma is equivalent to the definition of convex function [5, page 2].

Lemma 2.3. *If $x_1, x_2, x_3 \in I$ are such that $x_1 \leq x_2 \leq x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if inequality*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0$$

holds.

Theorem 2.4. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *be a real* n*-tuple,* $\mathbf{x} = (x_1, ..., x_n)$ *be a positive* n*-tuple and* $x_0 \in (0, \infty)$ *such that* $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in (0, \infty)$ *and* (8) *is satisfied. Also let* $\Phi(\mathbf{x}; \mathbf{p}; \phi_t)$*, where* ϕ_t *is defined by* Lemma 2.1, be a positive valued function. Then $t \mapsto \Phi(\mathbf{x}; \mathbf{p}; \phi_t)$ is log-convex on $\mathbb{R}$. Also for $r \leq s \leq t$, where $r, s, t \in \mathbb{R}$, we have

$$(\Phi(\mathbf{x}; \mathbf{p}; \phi_s))^{t-r} \leq (\Phi(\mathbf{x}; \mathbf{p}; \phi_r))^{t-s} (\Phi(\mathbf{x}; \mathbf{p}; \phi_t))^{s-r}. \tag{10}$$

Proof. Let $f(x) = u^2\phi_s(x) + 2uw\phi_r(x) + w^2\phi_t(x)$ where $r, s, t \in \mathbb{R}$ such that $r = \frac{s+t}{2}$ and $u, w \in \mathbb{R}$. Then we have

$$\begin{aligned} f''(x) &= u^2 x^{s-2} + 2uwx^{r-2} + w^2 x^{t-2}, \\ &= \left(ux^{(s-2)/2} + wx^{(t-2)/2}\right)^2 \geq 0 \text{ for } x \in (0, \infty). \end{aligned}$$

This implies that f is convex on $(0, \infty)$.
By Theorem 1.4, we have

$$Af(\tilde{x}_n) - \sum_{k=1}^{n} p_k f(x_k) + Bf(x_0) \geq 0$$

and hence we obtain

$$\begin{aligned} u^2 &\left(A\phi_s(\tilde{x}_n) - \sum_{k=1}^{n} p_k\phi_s(x_k) + B\phi_s(x_0)\right) \\ &+ 2uw\left(A\phi_r(\tilde{x}_n) - \sum_{k=1}^{n} p_k\phi_r(x_k) + B\phi_r(x_0)\right) \\ &+ w^2\left(A\phi_t(\tilde{x}_n) - \sum_{k=1}^{n} p_k\phi_t(x_k) + B\phi_t(x_0)\right) \geq 0. \end{aligned}$$

Since

$$\Phi(\mathbf{x}; \mathbf{p}; \phi_t) = A\phi_t\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} p_k\phi_t(x_k) + B\phi_t(x_0),$$

therefore

$$u^2\Phi(\mathbf{x}; \mathbf{p}; \phi_s) + 2uw\Phi(\mathbf{x}; \mathbf{p}; \phi_r) + w^2\Phi(\mathbf{x}; \mathbf{p}; \phi_t) \geq 0.$$

Now by Lemma 2.2, we have that $\Phi(\mathbf{x}; \mathbf{p}; \phi_t)$ is log-convex in the Jensen sense.
Since $\lim_{t\to 1}\Phi(\mathbf{x}; \mathbf{p}; \phi_t) = \Phi(\mathbf{x}; \mathbf{p}; \phi_1)$ and $\lim_{t\to 0}\Phi(\mathbf{x}; \mathbf{p}; \phi_t) = \Phi(\mathbf{x}; \mathbf{p}; \phi_0)$ it follows that $\Phi(\mathbf{x}; \mathbf{p}; \phi_t)$ is continuous for all $t \in \mathbb{R}$, therefore it is a

log-convex function [5, page 6], i.e. $\log \Phi(\mathbf{x};\mathbf{p};\phi_t)$ is convex.
Now by Lemma 2.3, we have that, for $r \le s \le t$ with $f = \log \Phi$,

$$(t-s)\log \Phi(\mathbf{x};\mathbf{p};\phi_r) + (r-t)\log \Phi(\mathbf{x};\mathbf{p};\phi_s) + (s-r)\log \Phi(\mathbf{x};\mathbf{p};\phi_t) \ge 0.$$

This is equivalent to inequality (10). □

Remark 2.5. *We keep the assumption that the function $t \mapsto \Phi(\mathbf{x};\mathbf{p};\phi_t)$, is positive valued in the rest of the paper.*

We shall denote a square matrix of order l by $[a_{ij}]$, with elements a_{ij}, $i,j = 1,...,l$. Recall that a real symmetric matrix is said to be positive semi-definite provided the quadratic form

$$Q(x) = \sum_{i,j=1}^{l} a_{ij}x_i x_j$$

is non-negative for all non-trivial sets of the real variable x_i, i.e. for $(x_1,...,x_l) \neq (0,...,0)$.

Theorem 2.6. *Let $\mathbf{p} = (p_1,...,p_n)$ be a real n-tuple, $\mathbf{x} = (x_1,...,x_n)$ be a positive n-tuple and $x_0 \in (0,\infty)$ such that $x_1 \ge ... \ge x_m \ge x_0 \ge x_{m+1} \ge ... \ge x_n$ $(m \in \{0,1,2,...,n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in (0,\infty)$ and (8) is satisfied. For $l \in \mathbb{N}$, let $r_1,...,r_l$ be arbitrary real numbers, then the matrix*

$$\left[\Phi\left(\mathbf{x};\mathbf{p};\phi_{\frac{r_i+r_j}{2}}\right)\right], \quad \textit{where } 1 \le i,j \le l,$$

is a positive semi-definite matrix. Particularly

$$\det\left(\left[\Phi\left(\mathbf{x};\mathbf{p};\phi_{\frac{r_i+r_j}{2}}\right)\right]_{i,j=1}^{k}\right) \ge 0 \quad \textit{for all } k = 1,...,l.$$

Proof. Define a $l \times l$ matrix $M = \left[\phi_{\frac{r_i+r_j}{2}}\right]$, where $i,j = 1,...,l$, and let $\mathbf{v} = (v_1,...,v_l)$ be a nonzero arbitrary vector from $\mathbb{R}^l$.

Consider a function

$$\lambda(x) = \mathbf{v}M\mathbf{v}^{\tau} = \sum_{i,j=1}^{l} v_i v_j \phi_{\frac{r_i+r_j}{2}}(x).$$

Now we have

$$\begin{aligned}\lambda''(x) &= \sum_{i,j=1}^{l} v_i v_j x^{\frac{r_i+r_j}{2}-2} \\ &= \left(\sum_{i=1}^{l} v_i x^{\frac{r_i-2}{2}}\right)^2 \ge 0 \text{ for } x \in (0,\infty).\end{aligned}$$

This implies λ is convex on $(0,\infty)$. Now applying Theorem 1.4 for function λ, we have

$$0 \le \Phi\left(\mathbf{x};\mathbf{p};\lambda\right) = \sum_{i,j=1}^{l} v_i v_j \Phi\left(\mathbf{x};\mathbf{p};\phi_{\frac{r_i+r_j}{2}}\right)$$

so the given matrix is positive semi-definite matrix.
Using well-known Sylvester criterion, we get

$$\text{(11)}\qquad \begin{vmatrix} \Phi\left(\mathbf{x};\mathbf{p};\phi_{r_1}\right) & \cdots & \Phi\left(\mathbf{x};\mathbf{p};\phi_{\frac{r_1+r_k}{2}}\right) \\ \vdots & \ddots & \vdots \\ \Phi\left(\mathbf{x};\mathbf{p};\phi_{\frac{r_k+r_1}{2}}\right) & \cdots & \Phi\left(\mathbf{x};\mathbf{p};\phi_{r_k}\right) \end{vmatrix} \ge 0$$

for all $k = 1, ..., l$. □

Remark 2.7. *Note that, we can again deduce that* $t \mapsto \Phi(\mathbf{x};\mathbf{p};\phi_t)$ *is log-convex function by taking* $k = 2$ *in* (11).

Definition 1. Let $\mathbf{p} = (p_1, ..., p_n)$ be a real n-tuple, $\mathbf{x} = (x_1, ..., x_n)$ be a positive n-tuple and $x_0 \in (0,\infty)$ such that $x_1 \ge ... \ge x_m \ge x_0 \ge x_{m+1} \ge ... \ge x_n$ $(m \in \{0,1,2,...,n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in (0,\infty)$ and (8) is satisfied. Then for $t, r \in \mathbb{R}$, we define

$$\begin{aligned} M_{t,r}(\mathbf{x};\mathbf{p}) &= \left(\frac{\Phi(\mathbf{x};\mathbf{p};\phi_t)}{\Phi(\mathbf{x};\mathbf{p};\phi_r)}\right)^{1/(t-r)}, \quad t \ne r, \\ M_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{2r-1}{r(r-1)} - \frac{\Phi(\mathbf{x};\mathbf{p};\phi_r\phi_0)}{\Phi(\mathbf{x};\mathbf{p};\phi_r)}\right), \quad r \ne 0,1, \\ M_{0,0}(\mathbf{x};\mathbf{p}) &= \exp\left(1 - \frac{\Phi(\mathbf{x};\mathbf{p};\phi_0^2)}{\Phi(\mathbf{x};\mathbf{p};\phi_0)}\right), \\ M_{1,1}(\mathbf{x};\mathbf{p}) &= \exp\left(-1 - \frac{\Phi(\mathbf{x};\mathbf{p};\phi_0\phi_1)}{\Phi(\mathbf{x};\mathbf{p};\phi_1)}\right). \end{aligned}$$

Remark 2.8. *Note that* $\lim_{t\to r} M_{t,r}(\mathbf{x};\mathbf{p}) = M_{r,r}(\mathbf{x};\mathbf{p})$, $\lim_{r\to 1} M_{r,r}(\mathbf{x};\mathbf{p}) = M_{1,1}(\mathbf{x};\mathbf{p})$ *and* $\lim_{r\to 0} M_{r,r}(\mathbf{x};\mathbf{p}) = M_{0,0}(\mathbf{x};\mathbf{p})$.

To prove the monotonicity of $M_{t,r}(\mathbf{x};\mathbf{p})$, we shall use the following Lemma [4].

Lemma 2.9. *Let f be a log-convex function and assume that if* $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$. *Then the following inequality is valid:*

$$\text{(12)}\qquad \left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \le \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}.$$

Theorem 2.10. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *be a real* n*-tuple,* $\mathbf{x} = (x_1, ..., x_n)$ *be a positive* n*-tuple and* $x_0 \in (0, \infty)$ *such that* $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$*,* $\tilde{x}_n \in (0, \infty)$ *and* (8) *is satisfied. Also let* $t, r, v, u \in \mathbb{R}$ *such that* $r \leq u$*,* $t \leq v$*. Then*

$$M_{t,r}(\mathbf{x}; \mathbf{p}) \leq M_{v,u}(\mathbf{x}; \mathbf{p}). \tag{13}$$

Proof. As it is proved in Theorem 2.4 that $\Phi(\mathbf{x}; \mathbf{p}; \phi_t)$ is log-convex, so taking $x_1 = r$, $x_2 = t$, $y_1 = u$, $y_2 = v$, where $r \neq t$, $v \neq u$ and $f(t) = \Phi(\mathbf{x}; \mathbf{p}; \phi_t)$ in Lemma 2.9, we have

$$\left(\frac{\Phi(\mathbf{x}; \mathbf{p}; \phi_t)}{\Phi(\mathbf{x}; \mathbf{p}; \phi_r)}\right)^{1/(t-r)} \leq \left(\frac{\Phi(\mathbf{x}; \mathbf{p}; \phi_v)}{\Phi(\mathbf{x}; \mathbf{p}; \phi_u)}\right)^{1/(v-u)}. \tag{14}$$

This is equivalent to (13) for $r \neq t$ and $v \neq u$.
From Remark 2.8, we get (14) is also valid for $r = t$ or $v = u$. □

Similarly, we can consider another class of functions:

Lemma 2.11. *For all* $t \in \mathbb{R}$*, let* $\tilde{\phi}_t : \mathbb{R} \to \mathbb{R}$ *be the function defined as*

$$\tilde{\phi}_t(x) = \begin{cases} \frac{e^{tx}}{t^2}, & t \neq 0; \\ \frac{1}{2}x^2, & t = 0. \end{cases}$$

Then $\tilde{\phi}_t(x)$ *is convex on* $\mathbb{R}$*.*

Proof. Since

$$\tilde{\phi}_t''(x) = e^{tx} > 0 \quad \text{for all} \quad x, t \in \mathbb{R},$$

therefore $\tilde{\phi}_t(x)$ is convex on $\mathbb{R}$ for each $t \in \mathbb{R}$. □

Theorem 2.12. *Let* $\mathbf{p} = (p_1, ..., p_n)$*,* $\mathbf{x} = (x_1, ..., x_n)$ *be two real* n*-tuples,* $x_0 \in \mathbb{R}$ *such that* $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$*,* $\tilde{x}_n := \sum_{k=1}^n p_k x_k \in \mathbb{R}$ *and* (8) *is satisfied. Also let* $\Phi(\mathbf{x}; \mathbf{p}; \tilde{\phi}_t)$*, where* $\tilde{\phi}_t$ *is defined by* Lemma 2.11, be a positive valued function. Then $t \mapsto \Phi(\mathbf{x}; \mathbf{p}; \tilde{\phi}_t)$ is log-convex. Also for $r \leq s \leq t$, where $r, s, t \in \mathbb{R}$, we have

$$\left(\Phi(\mathbf{x}; \mathbf{p}; \tilde{\phi}_s)\right)^{t-r} \leq \left(\Phi(\mathbf{x}; \mathbf{p}; \tilde{\phi}_r)\right)^{t-s} \left(\Phi(\mathbf{x}; \mathbf{p}; \tilde{\phi}_t)\right)^{s-r}. \tag{15}$$

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.13. *Let* $\mathbf{p} = (p_1, ..., p_n)$*,* $\mathbf{x} = (x_1, ..., x_n)$ *be two real* n*-tuples,* $x_0 \in \mathbb{R}$ *such that* $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$*,* $\tilde{x}_n := \sum_{k=1}^n p_k x_k \in \mathbb{R}$ *and* (8) *is satisfied. Also let* $r_1, ..., r_l$ *be arbitrary real numbers. Then the matrix*

$$\left[\Phi\left(\mathbf{x}; \mathbf{p}; \tilde{\phi}_{\frac{r_i + r_j}{2}}\right)\right], \quad \text{where } 1 \leq i, j \leq l,$$

is a positive semi-definite matrix. Particularly

$$\det\left(\left[\Phi\left(\mathbf{x};\mathbf{p};\tilde{\phi}_{\frac{r_i+r_j}{2}}\right)\right]_{i,j=1}^{k}\right)\geq 0 \quad \textit{for all } k=1,...,l.$$

Proof. The proof is similar to the proof of Theorem 2.6. □

Definition 2. Let $\mathbf{p}=(p_1,...,p_n)$, $\mathbf{x}=(x_1,...,x_n)$ be two real n-tuples, $x_0\in\mathbb{R}$ such that $x_1\geq ...\geq x_m\geq x_0\geq x_{m+1}\geq ...\geq x_n$ $(m\in\{0,1,2,...,n\})$, $\tilde{x}_n:=\sum_{k=1}^{n}p_kx_k\in\mathbb{R}$ and (8) is satisfied. Then for $t,r\in\mathbb{R}$,

$$\begin{aligned}\tilde{M}_{t,r}(\mathbf{x};\mathbf{p}) &= \left(\frac{\Phi(\mathbf{x};\mathbf{p};\tilde{\phi}_t)}{\Phi(\mathbf{x};\mathbf{p};\tilde{\phi}_r)}\right)^{\frac{1}{t-r}}, \quad t\neq r,\\ \tilde{M}_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{2}{r}+\frac{\Phi(\mathbf{x};\mathbf{p};x\tilde{\phi}_r)}{\Phi(\mathbf{x};\mathbf{p};\tilde{\phi}_r)}\right), \quad r\neq 0,\\ \tilde{M}_{0,0}(\mathbf{x};\mathbf{p}) &= \exp\left(\frac{\Phi(\mathbf{x};\mathbf{p};x\tilde{\phi}_0)}{3\Phi(\mathbf{x};\mathbf{p};\tilde{\phi}_0)}\right).\end{aligned}$$

Remark 2.14. *Note that* $\lim_{t\to r}\tilde{M}_{t,r}(\mathbf{x};\mathbf{p})=\tilde{M}_{r,r}(\mathbf{x};\mathbf{p})$ *and* $\lim_{r\to 0}\tilde{M}_{r,r}(\mathbf{x};\mathbf{p})=\tilde{M}_{0,0}(\mathbf{x};\mathbf{p})$.

Theorem 2.15. *Let* $\mathbf{p}=(p_1,...,p_n)$, $\mathbf{x}=(x_1,...,x_n)$ *be two real n-tuples,* $x_0\in\mathbb{R}$ *such that* $x_1\geq ...\geq x_m\geq x_0\geq x_{m+1}\geq ...\geq x_n$ $(m\in\{0,1,2,...,n\})$, $\tilde{x}_n:=\sum_{k=1}^{n}p_kx_k\in\mathbb{R}$ *and (8) is satisfied. Also let* $r,t,v,u\in\mathbb{R}$ *such that* $r\leq u$, $t\leq v$. *Then we have*

$$\tilde{M}_{t,r}(\mathbf{x};\mathbf{p})\leq\tilde{M}_{v,u}(\mathbf{x};\mathbf{p}). \tag{16}$$

Proof. The proof is similar to the proof of the Theorem 2.10. □

3. Mean value theorems

Lemma 3.1. *Let* $f\in C^2(I)$, *where* $I\subseteq\mathbb{R}$ *is an interval, such that*

$$m\leq f''(x)\leq M \quad \textit{for all} \quad x\in I. \tag{17}$$

Consider the functions ρ_1, ρ_2 *defined as,*

$$\rho_1(x)=\frac{Mx^2}{2}-f(x),$$

$$\rho_2(x)=f(x)-\frac{mx^2}{2}.$$

Then $\rho_i(x)$ *for* $i=1,2$ *are convex on* I.

Proof. We have that

$$\rho_1''(x) = M - f''(x) \geq 0,$$
$$\rho_2''(x) = f''(x) - m \geq 0,$$

that is, $\rho_i(x)$ for $i = 1, 2$ are convex on I. □

Theorem 3.2. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, $\mathbf{p} = (p_1, ..., p_n)$ be a real n-tuple and $\mathbf{x} = (x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and (8) is satisfied. If $f \in C^2(I)$, then there exists $\xi \in I$ such that*

$$\Phi(\mathbf{x}; \mathbf{p}; f) = f''(\xi)\Phi(\mathbf{x}; \mathbf{p}; \phi_2), \tag{18}$$

where $\phi_2(x) = x^2/2$.

Proof. Suppose that f'' is bounded and that $m = \inf f''$ and $M = \sup f''$ $(-\infty < m < M < \infty)$.
In Theorem 1.4, setting $f = \rho_1$ and $f = \rho_2$ respectively as defined in Lemma 3.1, we get the following inequalities

$$\Phi(\mathbf{x}; \mathbf{p}; f) \leq M\Phi(\mathbf{x}; \mathbf{p}; \phi_2), \tag{19}$$

$$\Phi(\mathbf{x}; \mathbf{p}; f) \geq m\Phi(\mathbf{x}; \mathbf{p}; \phi_2). \tag{20}$$

Since $\Phi(\mathbf{x}; \mathbf{p}; \phi_2)$ is positive by assumption, therefore combining (19) and (20), we get,

$$m \leq \frac{\Phi(\mathbf{x}; \mathbf{p}; f)}{\Phi(\mathbf{x}; \mathbf{p}; \phi_2)} \leq M. \tag{21}$$

Now by condition (17), there exists $\xi \in I$ such that

$$\frac{\Phi(\mathbf{x}; \mathbf{p}; f)}{\Phi(\mathbf{x}; \mathbf{p}; \phi_2)} = f''(\xi).$$

This implies (18). Moreover (19) is valid if (for example) f'' is bounded from above and hence (18) is valid.
Of course (18) is obvious if f'' is not bounded. □

Theorem 3.3. *Let $f, g : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, $\mathbf{p} = (p_1, ..., p_n)$ be a real n-tuple and $\mathbf{x} = (x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $x_1 \geq ... \geq x_m \geq x_0 \geq x_{m+1} \geq ... \geq x_n$ $(m \in \{0, 1, 2, ..., n\})$, $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and (8) is satisfied. If $f, g \in C^2(I)$, then there exists $\xi \in I$ such that*

$$\frac{\Phi(\mathbf{x}; \mathbf{p}; f)}{\Phi(\mathbf{x}; \mathbf{p}; g)} = \frac{f''(\xi)}{g''(\xi)} \tag{22}$$

provided that the denominators are non-zero.

Proof. Let a function $k \in C^2(I)$ be defined as

$$k = c_1 f - c_2 g,$$

where c_1 and c_2 are defined as

$$c_1 = \Phi(\mathbf{x}; \mathbf{p}; g),$$
$$c_2 = \Phi(\mathbf{x}; \mathbf{p}; f).$$

Then, using Theorem 3.2 with $f = k$, we have

$$0 = \left(c_1 f''(\xi) - c_2 g''(\xi)\right) \Phi(\mathbf{x}; \mathbf{p}; \phi_2). \tag{23}$$

As $\Phi(\mathbf{x}; \mathbf{p}; \phi_2)$ is positive, so we have

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}.$$

After putting values, we get (22). □

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