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COSYMMEDIAN TRIANGLES IN AN ISOTROPIC PLANE

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Abstract. In this paper the concept of cosymmedian triangles in an isotropic plane is defined. A number of statements about some important properties of these triangles will be proved. Some analogies with the Euclidean case will also be considered.

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, absolute line ω, and one point on that line, the absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points. Therefore, an isotropic plane is in fact the affine plane with the pointed direction of isotropic lines and where the principle of duality is valid.

The triangle in an isotropic plane is called allowable if any two of its vertices are not parallel. Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so called *standard position*, i.e. that its circumscribed circle $\mathcal{K}$ has the equation

$$y = x^2, \tag{1}$$

and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that the equalities $q = bc - a^2$, $b^2 + bc + c^2 = -q$, $2q - 3bc = (c - a)(a - b)$, $a^2 + b^2 + c^2 = -2q$ are valid.

Key words and phrases: isotropic plane, standard triangle, cosymmedian triangles, symmedian center

Let $T_1 = (x_1, {x_1}^2)$ be the point on the circle $\mathcal{K}$, then the tangent $\mathcal{T}_1$ at T_1 to the circle $\mathcal{K}$ has the equation

$$y = 2x_1x - {x_1}^2 \tag{2}$$

since from (1) and (2) the equation $x^2 - 2x_1x + {x_1}^2 = 0$, with the double solution $x = x_1$, follows. Tangent $\mathcal{T}_1$ and the tangent $\mathcal{T}_2$ to the circle $\mathcal{K}$ at the point $T_2 = (x_2, {x_2}^2)$ with the equation $y = 2x_2x - {x_2}^2$ meet at the point $T_{12} = \left(\frac{1}{2}(x_1 + x_2),\, x_1x_2\right)$ because of

$$2x_1 \cdot \frac{1}{2}(x_1 + x_2) - {x_1}^2 = x_1x_2.$$

Tangents $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to a circle $\mathcal{K}$ at the points A, B, C determine the triangle $A_tB_tC_t$ with the vertices $A_t = \mathcal{B} \cap \mathcal{C}$, $B_t = \mathcal{C} \cap \mathcal{A}$, $C_t = \mathcal{A} \cap \mathcal{B}$, the so called *tangential triangle* of the triangle ABC. Due to $a + b + c = 0$ we have now

$$A_t = \left(-\frac{a}{2},\, bc\right), \quad B_t = \left(-\frac{b}{2},\, ca\right), \quad C_t = \left(-\frac{c}{2},\, ab\right).$$

The lines AA_t, BB_t, CC_t are the *symmedians* of the triangle ABC. In [2] it is shown that the symmedians of a triangle meet at the point K, *symmedian center* of the triangle ABC. In the case of the triangle ABC in a standard position the point K is given by

$$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right), \tag{3}$$

and, for example, the equation of the symmedian AK is

$$y = -\frac{2q}{3a}x + bc - \frac{q}{3}. \tag{4}$$

Let T be an arbitrary point. If the lines AT, BT, CT meet the circumscribed circle of the triangle ABC again at the points L, M, N (except A, B, C) then the triangle LMN is called *circum–Ceva's triangle* of the point T with respect to the triangle ABC.

Theorem 1. *The circum–Ceva's triangle of the symmedian center K with respect to the triangle ABC is the triangle DEF with the vertices $D = (d, d^2)$, $E = (e, e^2)$, $F = (f, f^2)$, where*

$$d = -a - \frac{2q}{3a}, \quad e = -b - \frac{2q}{3b}, \quad f = -c - \frac{2q}{3c}. \tag{5}$$

Proof. It is enough to prove that, for example the point D lies on the symmedian AK with the equation (4). We get

$$-\frac{2q}{3a}\,d+bc-\frac{q}{3}=\frac{2q}{3a}\left(a+\frac{2q}{3a}\right)+q+a^2-\frac{q}{3}$$
$$=a^2+\frac{4}{3}\,q+\frac{4q^2}{9a^2}=\left(a+\frac{2q}{3a}\right)^2=d^2.$$

□

Theorem 2. *The triangles ABC and DEF from Theorem 1 have common symmedians and symmmedian center.*

Proof. It is enough to prove, for example, that the intersection

$$D_t=\left(\frac{1}{2}(e+f),\,ef\right)$$

of the tangents $\mathcal{E}$ and $\mathcal{F}$ to the circle $\mathcal{K}$ at the points E and F lies on the symmedian AK with the equation (4). By using (5) we obtain

$$ef+\frac{2q}{3a}\cdot\frac{1}{2}(e+f)-bc+\frac{q}{3}=\left(b+\frac{2q}{3b}\right)\left(c+\frac{2q}{3c}\right)-\frac{q}{3a}\left(b+c+\frac{2q}{3b}+\frac{2q}{3c}\right)$$
$$-bc+\frac{q}{3}=\frac{2q}{3bc}(b^2+c^2)+\frac{4q^2}{9bc}+\frac{q}{3}-\frac{2q^2}{9abc}(b+c)+\frac{q}{3}$$
$$=\frac{2q}{3bc}(-q-bc)+\frac{4q^2}{9bc}+\frac{2q}{3}+\frac{2q^2}{9bc}=0.$$

□

The triangles ABC and DEF from Theorem 1 and 2 will be called *cosymmedian triangles*. Its relationship is symmetric.

In [3] it is shown that *Brocard angle* of the triangle ABC with the area $\triangle$ is defined by the formula

$$\omega=\frac{4\triangle}{BC^2+CA^2+AB^2}\,. \tag{6}$$

It is also proved that for the triangle ABC in the standard position the equality

$$\omega=-\frac{1}{3q}(b-c)(c-a)(a-b) \tag{7}$$

is valid.

Theorem 3. *If the triangle DEF is cosymmedian to the triangle ABC given in the standard position, then its area $\triangle'$ is given by*

$$\triangle' = -\frac{q^3}{2p^2}\,\omega^3, \tag{8}$$

and the lengths of its sides are given by the formulae

$$EF = a\,\frac{q}{p}\,\omega, \quad FD = b\,\frac{q}{p}\,\omega, \quad DE = c\,\frac{q}{p}\,\omega, \tag{9}$$

where ω is the Brocard angle of the triangle ABC.

Proof. Owing to (5) and (7) we get, for example

$$\begin{aligned} e-f &= \frac{2q}{3c}-\frac{2q}{3b}+c-b=\left(\frac{2q}{3bc}-1\right)(b-c)=\frac{b-c}{3bc}(2q-3bc) \\ &= \frac{a}{3p}(b-c)(c-a)(a-b)=\frac{a}{3p}(-3q\omega)=-a\cdot\frac{q\omega}{p} \end{aligned}$$

and analogously

$$f-d=-b\cdot\frac{q\omega}{p}, \quad d-e=-c\cdot\frac{q\omega}{p},$$

wherefrom

$$(e-f)(f-d)(d-e)=-\frac{q^3}{p^2}\,\omega^3. \tag{10}$$

Now

$$2\triangle' = \begin{vmatrix} d & d^2 & 1 \\ e & e^2 & 1 \\ f & f^2 & 1 \end{vmatrix} = (e-f)(f-d)(d-e) = -\frac{q^3}{p^2}\,\omega^3$$

and, for example

$$EF=-(e-f)=a\cdot\frac{q}{p}\,\omega.$$

□

According to (9) it follows

$$EF^2+FD^2+DE^2 = (a^2+b^2+c^2)\,\frac{q^2}{p^2}\,\omega^2 = -\frac{2q^3}{p^2}\,\omega^2,$$

wherefrom, in accordance with the formula (8), by the analogy to (6) the formula

$$\omega' = \frac{4\triangle'}{EF^2+FD^2+DE^2}=\omega$$

holds. This consideration gives us next theorem.

Theorem 4. *The Brocard angles of the cosymmedian triangles are equal.*

According to Delens ([5]) in the Euclidean geometry cosymmedian triangles are equibrocardial i.e., they have equal Brocard angles.

For the triangle ABC in a standard position the abscissa of the midpoint A_m of the side BC is $-\frac{a}{2}$. Therefore the length $AA_m = -\frac{a}{2} - a = -\frac{3}{2}a$. It means that the lengths of the medians of the triangle ABC are proportional to the numbers a, b and c. Therefore, according to (9), the lengths of the sides of the triangle DEF are proportional to the lengths of the medians of the triangle ABC. In [5] Delens has the same statement.
The following interesting statements for the cosymmedian triangles in an isotropic plane are valid.

Theorem 5. *The axis of homology of the cosymmedian triangles is the common Lemoine line of these triangles.*

Proof. The Lemoine line of the triangle ABC is, according to [2], the axis of homology of this triangle and its tangential triangle and also the polar line of the symmedian center of the triangle ABC with respect to its circumscribed circle. Cosymmedian triangles have the common circumscribed circles and common symmedian center and therefore have also common Lemoine line. By virtue of Theorem 2 in [2] the line BC meet the Lemoine line at the point $\left(-\frac{q}{3a}, \frac{q}{3} - bc\right)$. It is enough to prove that this point lies on the line through the points $E = (e, e^2)$ and $F = (f, f^2)$, which has the equation $y = (e+f)x - ef$. Owing to (5) it follows

$$\frac{q}{3} - bc - (e+f)\left(-\frac{q}{3a}\right) + ef = \frac{q}{3} - bc - \frac{q}{3a}\left(b + c + \frac{2q}{3b} + \frac{2q}{3c}\right)$$
$$+ \left(b + \frac{2q}{3b}\right)\left(c + \frac{2q}{3c}\right) = \frac{q}{3} - \frac{q}{3} \cdot \frac{b+c}{a} - \frac{2q^2}{9abc}(b+c) + \frac{2q}{3bc}(b^2+c^2) + \frac{4q^2}{9bc}$$
$$= \frac{2q}{3} + \frac{2q^2}{9bc} + \frac{2q}{3bc}(-q - bc) + \frac{4q^2}{9bc} = 0.$$

□

In the Euclidean geometry the statement of Theorem 5 (without a proof) can be found in [6].

Theorem 6. *If the triangles ABC and DEF are cosymmedian triangles with the common symmedian center K, then the distances of the point K to the sides of the hexagon $AFBDCE$ are proportional to the lengths of these sides, so we have the equalities*

$$\frac{area\,BDK}{BD^2} = \frac{area\,DCK}{DC^2} = \frac{area\,CEK}{CE^2}$$

$$= \frac{area\, EAK}{EA^2} = \frac{area\, AFK}{AF^2} = \frac{area\, FBK}{FB^2}$$

(In the Euclidean geometry Casey in [7] has this statement without a proof.)

Proof. Owing to (5) we obtain

$$BD = d - b = -a - b - \frac{2q}{3a} = c - \frac{2q}{3a} = -\frac{1}{3a}(2q - 3ac).$$

The line with the equation $y = (b+d)x - bd$ obviously passes through the points $B = (b, b^2)$ and $D = (d, d^2)$. The distance of the point $K = (x, y)$, given by (3), to this line is equal to

$$y - (b+d)x + bd = -\frac{q}{3} - \left(b - a - \frac{2q}{3a}\right)\frac{3p}{2q} - b\left(a + \frac{2q}{3a}\right)$$
$$= \frac{1}{6aq}[-2aq^2 + 9a(a-b)p + 6pq - 6a^2bq - 4bq^2]$$
$$= \frac{1}{6aq}[-2aq^2 - 9a^2c(a+c)(2a+c) + 6abq(c-a) + 4(a+c)q^2]$$
$$= \frac{1}{6aq}[4(c-a)q^2 + 6aq^2 - 9a^2c(2a^2 + 2ac + 2c^2 + ac - c^2) + 6ab(c-a)q]$$
$$= \frac{1}{6aq}[4(c-a)q^2 + 6ab(c-a)q + 9a^2c^2(c-a) + 6aq^2 + 18a^2cq]$$
$$= \frac{1}{6aq}[(c-a)(4q^2 + 6abq + 9a^2c^2) + 6aq(q + 3ca)]$$
$$= \frac{1}{6aq}[(c-a)(4q^2 + 6abq + 9a^2c^2) - 6aq(c-a)^2]$$
$$= \frac{c-a}{6aq}[4q^2 - 6a(a+c)q + 9a^2c^2 - 6a(c-a)q]$$
$$= \frac{c-a}{6aq}(4q^2 - 12acq + 9a^2c^2) = \frac{c-a}{6aq}(2q - 3ca)^2$$
$$= -\frac{c-a}{2q}(2q - 3ca) \cdot BD = -\frac{1}{2q}(c-a)(a-b)(b-c) \cdot BD = \frac{3}{2}\,\omega \cdot BD.$$

Therefore we obtain, for example

$$2\, area\, BDK = \frac{3}{2}\,\omega \cdot BD \cdot BD, \quad \text{i.e.} \quad \frac{area\, BDK}{BD^2} = \frac{3}{4}\,\omega.$$

□

Now, we will consider some interesting properties of the points D, E, F from Theorems 1 – 6. So we have the following theorem.

Theorem 7. *Let L, M, N be the feet of the perpendiculars from T to the lines BC, CA, AB. Then the point L is the midpoint of the points M and N, if and only if the point T is parallel to the second intersection D of the symmedian AK of the triangle ABC with its circumscribed circle.*

Proof. In [1] the following equations of the lines BC, CA, AB are derived

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab,$$

so if x is the abscissa of the point T then the ordinate of the points L, M, N are equal to $-ax-bc$, $-bx-ca$, $-cx-ab$. The point L is the midpoint of the points M and N provided that $2(ax + bc) = (b + c)x + ca + ab$, i.e., $3ax = q - 3bc$ or further $3ax = -3a^2 - 2q$. Finally, according to $(5)_1$ we get $x = d$. □

In the Euclidean geometry Cristescu [8] states that L is the midpoint of M and N, if and only if, $T = D$.

If $T_i = (x_i, x_i{}^2)$ $(i = 1, 2)$ are the points on the circle (1), then because $(x_1 + x_2)x_i - x_1x_2 = x_i{}^2$ the line T_1T_2 has the equation $y = (x_1 + x_2)x - x_1x_2$. Referring to [4] the Steiner point of the standard triangle ABC is of the form $S = (s, s^2)$, where $s = -\frac{3p}{q}$. The line DS has the equation $y = (d + s)x - ds$, i.e. the equation

$$y = -\left(a + \frac{2q}{3a} + \frac{3p}{q}\right)x - \frac{3p}{q}\left(a + \frac{2q}{3a}\right),$$

and because

$$a\left(a + \frac{2q}{3a} + \frac{3p}{q}\right) - \frac{3p}{q}\left(a + \frac{2q}{3a}\right) = bc - q + \frac{2q}{3} - \frac{2p}{a} = -\frac{q}{3} - bc$$

it passes through the point $A' = \left(-a, -\frac{q}{3} - bc\right)$. Owing to

$$a^2 - bc - \frac{q}{3} = -q - \frac{q}{3} = -\frac{4}{3}\,q$$

the points $A = (a, a^2)$ and A' have the midpoint $G = \left(0, -\frac{2}{3}\,q\right)$, and it is according to [1] the centroid of the triangle ABC. We have proved:

Theorem 8. *If DEF is the circum–Ceva's triangle of the symmedian center of an allowable triangle ABC and if A', B', C' are the points symmetrical to the points A, B, C with respect to the centroid G of the* *triangle ABC, then the lines DA', EB', FC' pass through the Steiner point S of that triangle.*

In the Euclidean case Mineur in [9] and Leemans in [10] have this statetment.

Theorem 9. *If DEF is the circum–Ceva's triangle of the symmedian center K of an allowable triangle ABC and if D', E', F' are the points symmetrical to the points D, E, F with respect to the lines BC, CA, AB, then the segments $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ can be seen from the points D', E', F' under the angles $-A$, $-C$, $-B$; $-C$, $-B$, $-A$; $-B$, $-A$, $-C$. The points D', E', F' are the intersections of the medians AG, BG, CG of the triangle ABC with the circles symmetrical to the circumscribed circle of the triangle ABC with respect to the lines BC, CA, AB.*

This theorem generalizes the Euclidean result, see Stoll [11].

Proof. The ordinate of the midpoint of the point $D = (d, d^2)$, given by (5), and the point

$$D' = \left(d, -q - bc - \frac{4q^2}{9a^2}\right)$$

is equal to

$$\begin{aligned}\frac{1}{2}\left(d^2 - q - bc - \frac{4q^2}{9a^2}\right) &= \frac{1}{2}\left(a + \frac{2q}{3a}\right)^2 - \frac{q}{2} - \frac{bc}{2} - \frac{2q^2}{9a^2}\\ &= \frac{a^2}{2} + \frac{2}{3}q - \frac{1}{2}(bc - a^2) - \frac{bc}{2}\\ &= a\left(a + \frac{2q}{3a}\right) - bc = -ad - bc,\end{aligned}$$

so it lies on the line BC with the equation $y = -ax - bc$. Owing to [2] the median AG of the triangle ABC has the equation

$$y = \frac{3bc - q}{3a}x - \frac{2q}{3}.$$

It passes through the point D' because of

$$\begin{aligned}\frac{q - 3bc}{3a} \cdot \left(a + \frac{2q}{3a}\right) - \frac{2q}{3} &= -bc + \frac{q}{3} - \frac{2bcq}{3a^2} + \frac{2q^2}{9a^2} - \frac{2q}{3}\\ &= -\frac{q}{3} - bc - \frac{2q(q + a^2)}{3a^2} + \frac{2q^2}{9a^2}\\ &= -q - bc - \frac{4q^2}{9a^2}.\end{aligned}$$

The lines AD', BD', CD' have the slopes

$$\begin{aligned}\frac{a^2+q+bc+\frac{4q^2}{9a^2}}{a+a+\frac{2q}{3a}} &= \frac{2bc+\frac{4q^2}{9a^2}}{2a+\frac{2q}{3a}} = \frac{1}{3a}\cdot\frac{2q^2+9a^2(q+a^2)}{q+3a^2} \\ &= \frac{1}{3a}\cdot\frac{2q^2+9a^2q+9a^4}{q+3a^2} = \frac{1}{3a}(2q+3a^2) = \frac{2q}{3a}+a, \\ \frac{b^2+q+bc+\frac{4q^2}{9a^2}}{b+a+\frac{2q}{3a}} &= \frac{\frac{4q^2}{9a^2}-c^2}{\frac{2q}{3a}-c} \\ &= \frac{2q}{3a}+c\end{aligned}$$

and analogously $\frac{2q}{3a}+b$. So, we obtain our desired result

$$\measuredangle(BD',CD') = b-c = -A,$$
$$\measuredangle(CD',AD') = a-b = -C,$$
$$\measuredangle(AD',BD') = c-a = -B.$$

□

References

[1] R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec, J. Beban–Brkić, *Metrical relationships in a standard triangle in an isotropic plane*, Mathematical Communications **10**(2005), 149–157.

[2] Z. Kolar–Begović, R. Kolar–Šuper, J. Beban–Brkić, V. Volenec, *Symmedians and the symmedian center of the triangle in an isotropic plane*, Mathematica Pannonica **17**(2006), 287–301.

[3] Z. Kolar–Begović, R. Kolar–Šuper, V. Volenec, *Brocard angle of the triangle in an isotropic plane*, Rad HAZU **503**(2009), 55–60.

[4] V. Volenec, Z. Kolar–Begović, R. Kolar–Šuper, *Steiner's ellipses of the triangle in an isotropic plane*, Mathematica Pannonica, **21**(2010), 229-238.

[5] P. Delens, *Sur la géométrie du tétraèdre*, Mathesis, **51**(1937), 444–456.

[6] E. Valdès, *Sur la géométrie du triangle*, Nouv. Ann. Math. (3) **11**(1892), 249–250.

[7] J. Casey, *Géométrie élémentaire récente* , Mathesis, **9**(1889), 5–70.

[8] V. Cristescu, *Sur la droite de Simson*, Mathesis **36**(1922), 324–325.

[9] A. Mineur, *Sur la point de Steiner*, Mathesis **47**(1933), 360–361.

[10] J Leemans, *Sur la point de Steiner*, Mathesis **50**(1936), 379.

[11] Stoll, *Aufgabe 2019*, Zeitschr. Math. Naturwiss. Unterr. **33**(1902), 262.

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