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ON SOME GENERALIZED NORM TRIANGLE INEQUALITIES

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Abstract. In this paper we characterize equality attainedness in some recently obtained generalized norm triangle inequalities.

1. Introduction

In [3] Dragomir gave the following bounds for the norm of $\sum_{j=1}^{n} \alpha_j x_j$, where $\alpha_j \in \mathbb{C}$ and x_j, $j = 1, \dots, n$, are arbitrary elements of a normed linear space X :

$$\max_{i\in\{1,\dots,n\}} \left\{ |\alpha_i| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\alpha_j - \alpha_i| \|x_j\| \right\}$$

$$\leq \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \min_{i\in\{1,\dots,n\}} \left\{ |\alpha_i| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_i| \|x_j\| \right\}. \tag{1}$$

In the case $\alpha_j = \frac{1}{\|x_j\|}$, where x_j are non-zero elements of X, this result reduces to Theorem 2.1 proved in [13], which in its turn implies the following generalization of the triangle inequality and its reverse inequality obtained by Kato et al. in [6]:

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{j\in\{1,\dots,n\}} \|x_j\|$$

$$\leq \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{j\in\{1,\dots,n\}} \|x_j\|. \tag{2}$$

When $n = 2$ inequalities in (2) yield those established by Maligranda in [8] (see also [9]) and can be written as the estimates for the *angular*

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distance $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|$ (see [2]) between non-zero elements x and y:

$$\frac{\|x-y\|-|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}} \le \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le \frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}. \tag{3}$$

(Another proof of the first inequality in (3) was given by Mercer in [11].) The second inequality in (3) is a refinement of the Massera-Schäffer inequality [10]

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}} \quad (x,y\in X\setminus\{0\}),$$

which is sharper than the Dunkl-Williams inequality [5]

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le \frac{4\|x-y\|}{\|x\|+\|y\|} \quad (x,y\in X\setminus\{0\}).$$

One more generalization of the second inequality in (3) was recently obtained in [4], where new bounds for the norm of $\sum_{j=1}^n \alpha_j x_j$ are established. It was proved there that

$$\max_{i\in\{1,\ldots,n\}}\left\{\left|\sum_{j=1}^n \alpha_j\right|\|x_i\| - \sum_{j=1}^n |\alpha_j|\|x_j-x_i\|\right\}$$

$$\le \left\|\sum_{j=1}^n \alpha_j x_j\right\| \le \min_{i\in\{1,\ldots,n\}}\left\{\left|\sum_{j=1}^n \alpha_j\right|\|x_i\| + \sum_{j=1}^n |\alpha_j|\|x_j-x_i\|\right\}, \tag{4}$$

where $\alpha_j\in\mathbb{C}$ and x_j, $j=1,\ldots,n$, are elements of a normed linear space X.

In [3, Theorems 2 and 3] Dragomir also provided the following dual versions of inequalities from (3), that is, he obtained lower and upper bounds for $\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|$, where x and y are two non-zero elements of a normed linear space:

$$0 \le \frac{\|x-y\|}{\min\{\|x\|,\|y\|\}} - \frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}$$

$$\le \left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\| \le \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}} + \frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}} \tag{5}$$

and

$$\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}} - \frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}$$

$$\le \left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\| \le \frac{\|x+y\|}{\min\{\|x\|,\|y\|\}} + \frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}. \tag{6}$$

In this paper we give alternative proofs for the inequalities in (5) and (6). We also consider the case of equality in each of the inequalities in (1) and (4) for elements of a strictly convex normed linear space.

2. The results

As a special case of (1) we have the following dual versions of inequalities from (3).

Theorem 2.1. *Let X be a normed linear space and x, y non-zero elements of X. Then we have*

$$\begin{aligned} 0 &\le \frac{\|x-y\|}{\min\{\|x\|,\|y\|\}} - \frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}} \\ &\le \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \le \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}} + \frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}. \end{aligned}$$

Proof. If $n = 2$ then by putting $x_1 := x$, $x_2 := -y$ and $\alpha_1 := \frac{1}{\|y\|}$, $\alpha_2 := \frac{1}{\|x\|}$ in (1) we get

$$\begin{aligned} &\max\left\{ \frac{\|x-y\|}{\|y\|} - \frac{|\|x\|-\|y\||}{\|x\|}, \frac{\|x-y\|}{\|x\|} - \frac{|\|x\|-\|y\||}{\|y\|} \right\} \\ \le &\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \le \min\left\{ \frac{\|x-y\|}{\|y\|} + \frac{|\|x\|-\|y\||}{\|x\|}, \frac{\|x-y\|}{\|x\|} + \frac{|\|x\|-\|y\||}{\|y\|} \right\}. \end{aligned}$$

Clearly,

$$\begin{aligned} &\max\left\{ \frac{\|x-y\|}{\|y\|} - \frac{|\|x\|-\|y\||}{\|x\|}, \frac{\|x-y\|}{\|x\|} - \frac{|\|x\|-\|y\||}{\|y\|} \right\} \\ &= \frac{\|x-y\|}{\min\{\|x\|,\|y\|\}} - \frac{|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}. \end{aligned}$$

It remains to show that

$$\begin{aligned} &\min\left\{ \frac{\|x-y\|}{\|y\|} + \frac{|\|x\|-\|y\||}{\|x\|}, \frac{\|x-y\|}{\|x\|} + \frac{|\|x\|-\|y\||}{\|y\|} \right\} \\ &= \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}} + \frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}. \end{aligned}$$

To see this, let us suppose that $\|x\| \le \|y\|$. Since $\|x-y\| - |\|x\|-\|y\|| \ge 0$ it follows that

$$\frac{\|x-y\| - |\|x\|-\|y\||}{\|y\|} \le \frac{\|x-y\| - |\|x\|-\|y\||}{\|x\|},$$

so

$$\frac{\|x-y\|}{\|y\|} + \frac{|\|x\|-\|y\||}{\|x\|} \le \frac{\|x-y\|}{\|x\|} + \frac{|\|x\|-\|y\||}{\|y\|}.$$

Therefore,

$$\min\left\{\frac{\|x-y\|}{\|y\|}+\frac{|\|x\|-\|y\||}{\|x\|},\frac{\|x-y\|}{\|x\|}+\frac{|\|x\|-\|y\||}{\|y\|}\right\}$$
$$=\frac{\|x-y\|}{\|y\|}+\frac{|\|x\|-\|y\||}{\|x\|}=\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}+\frac{|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}$$

and the result follows. □

Theorem 2.2. *Let X be a normed linear space and x, y non-zero elements of X. Then we have*

$$\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}$$
$$\leq\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|\leq\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}+\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}.$$

Proof. If $n=2$ then by putting $x_1:=x$, $x_2:=y$ and $\alpha_1:=\frac{1}{\|y\|}$, $\alpha_2:=-\frac{1}{\|x\|}$ in (1) we get

$$\max\left\{\frac{\|x+y\|}{\|y\|}-\frac{\|x\|+\|y\|}{\|x\|},\frac{\|x+y\|}{\|x\|}-\frac{\|x\|+\|y\|}{\|y\|}\right\}$$
$$\leq\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|\leq\min\left\{\frac{\|x+y\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|x\|},\frac{\|x+y\|}{\|x\|}+\frac{\|x\|+\|y\|}{\|y\|}\right\}.$$

Clearly,

$$\max\left\{\frac{\|x+y\|}{\|y\|}-\frac{\|x\|+\|y\|}{\|x\|},\frac{\|x+y\|}{\|x\|}-\frac{\|x\|+\|y\|}{\|y\|}\right\}$$
$$=\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}-\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}.$$

Let us show that

$$\min\left\{\frac{\|x+y\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|x\|},\frac{\|x+y\|}{\|x\|}+\frac{\|x\|+\|y\|}{\|y\|}\right\}$$
$$=\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}+\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}.$$

To see this, suppose that $\|x\|\leq\|y\|$. Since $\|x\|+\|y\|-\|x+y\|\geq 0$ it follows that

$$\frac{\|x\|+\|y\|-\|x+y\|}{\|y\|}\leq\frac{\|x\|+\|y\|-\|x+y\|}{\|x\|},$$

so

$$\frac{\|x+y\|}{\|x\|}+\frac{\|x\|+\|y\|}{\|y\|}\leq\frac{\|x+y\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|x\|}.$$

Therefore,

$$\min\left\{\frac{\|x+y\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|x\|},\frac{\|x+y\|}{\|x\|}+\frac{\|x\|+\|y\|}{\|y\|}\right\}$$

$$=\frac{\|x+y\|}{\|x\|}+\frac{\|x\|+\|y\|}{\|y\|}=\frac{\|x+y\|}{\min\{\|x\|,\|y\|\}}+\frac{\|x\|+\|y\|}{\max\{\|x\|,\|y\|\}}$$

and the theorem is proved. □

The following results describe the case of equality in each of the inequalities in (1) for elements of a strictly convex normed linear space. The proofs can be obtained similarly as the proofs of Theorem 2.6 and Theorem 2.8 from [13] and hence we omit them.

Theorem 2.3. *Let X be a strictly convex normed linear space, $x_1,\ldots,x_n$ non-zero elements of X and $\alpha_1,\ldots,\alpha_n\in\mathbb{C}$. Then the following two statements are mutually equivalent.*

(i) $\displaystyle\left\|\sum_{j=1}^n\alpha_jx_j\right\|=\min_{i\in\{1,\ldots,n\}}\left\{|\alpha_i|\left\|\sum_{j=1}^n x_j\right\|+\sum_{j=1}^n|\alpha_j-\alpha_i|\|x_j\|\right\}.$

(ii) *$\alpha_1=\cdots=\alpha_n$ or there exist $i\in\{1,\ldots,n\}$ and $v\in X$ satisfying $\dfrac{\alpha_j-\alpha_i}{|\alpha_j-\alpha_i|}\dfrac{x_j}{\|x_j\|}=v$ for all $j\in\{1,\ldots,n\}$ such that $\alpha_j\neq\alpha_i$ and $\displaystyle\alpha_i\sum_{j=1}^n x_j=|\alpha_i|\left\|\sum_{j=1}^n x_j\right\|v$.*

Theorem 2.4. *Let X be a strictly convex normed linear space, $x_1,\ldots,x_n$ non-zero elements of X and $\alpha_1,\ldots,\alpha_n\in\mathbb{C}$. Then the following two statements are mutually equivalent.*

(i) $\displaystyle\left\|\sum_{j=1}^n\alpha_jx_j\right\|=\max_{i\in\{1,\ldots,n\}}\left\{|\alpha_i|\left\|\sum_{j=1}^n x_j\right\|-\sum_{j=1}^n|\alpha_j-\alpha_i|\|x_j\|\right\}.$

(ii) *$\alpha_1=\cdots=\alpha_n$ or there exist $i\in\{1,\ldots,n\}$ and $v\in X$ satisfying $\dfrac{\alpha_i-\alpha_j}{|\alpha_i-\alpha_j|}\dfrac{x_j}{\|x_j\|}=v$ for all $j\in\{1,\ldots,n\}$ such that $\alpha_j\neq\alpha_i$ and $\displaystyle\sum_{j=1}^n\alpha_jx_j=\left\|\sum_{j=1}^n\alpha_jx_j\right\|v$.*

Remark 2.5. Index i from the statement (ii) of Theorem 2.3 (resp. Theorem 2.4) is precisely the index for which $|\alpha_k|\left\|\sum_{j=1}^n x_j\right\|+\sum_{j=1}^n|\alpha_j-\alpha_k|\|x_j\|$, $k=1,\ldots,n$, attains its minimum (resp. $|\alpha_k|\left\|\sum_{j=1}^n x_j\right\|-\sum_{j=1}^n|\alpha_j-\alpha_k|\|x_j\|$, $k=1,\ldots,n$, attains its maximum).

In what follows we consider the case of equality in each of the inequalities in (4) for elements of a strictly convex normed linear space. To do this, we need the following result, the proof of which can be found in [6, Lemma 1].

Lemma 2.6. *If $x_1, \dots, x_n$ are non-zero elements of a strictly convex normed linear space X, then the following statements are mutually equivalent.*

(i) $\left\| \sum_{j=1}^{n} x_j \right\| = \sum_{j=1}^{n} \|x_j\|$.

(ii) $\frac{x_1}{\|x_1\|} = \cdots = \frac{x_n}{\|x_n\|}$.

Theorem 2.7. *Let X be a strictly convex normed linear space, $x_1, \dots, x_n$ non-zero elements of X and $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \{0\}$. Then for every $i \in \{1, \dots, n\}$ the following two statements are mutually equivalent.*

(i) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \left| \sum_{j=1}^{n} \alpha_j \right| \|x_i\| + \sum_{j=1}^{n} |\alpha_j| \|x_j - x_i\|$.

(ii) *$x_1 = \cdots = x_n$ or there exists $v \in X$ satisfying $\frac{\alpha_j}{|\alpha_j|} \frac{x_j - x_i}{\|x_j - x_i\|} = v$ for all $j \in \{1, \dots, n\}$ such that $x_j \neq x_i$ and $\sum_{j=1}^{n} \alpha_j x_i = \left| \sum_{j=1}^{n} \alpha_j \right| \|x_i\| v$.*

Proof. If $x_1 = \cdots = x_n$ we are done. So, suppose that this is not the case.

Let us denote $J = \{j \in \{1, \dots, n\} : x_j \neq x_i\}$. Note that (i) is equivalent to

$$\left\| \sum_{j=1}^{n} \alpha_j x_i + \sum_{j \in J} \alpha_j (x_j - x_i) \right\| = \left| \sum_{j=1}^{n} \alpha_j \right| \|x_i\| + \sum_{j \in J} |\alpha_j| \|x_j - x_i\|. \tag{7}$$

First, let us suppose that $\sum_{j=1}^{n} \alpha_j \neq 0$.

By Lemma 2.6, (7) holds if and only if there is $v \in X$ satisfying

$$\frac{\sum_{j=1}^{n} \alpha_j x_i}{\left| \sum_{j=1}^{n} \alpha_j \right| \|x_i\|} = \frac{\alpha_j (x_j - x_i)}{|\alpha_j| \|x_j - x_i\|} = v, \quad j \in J.$$

In the case when $\sum_{j=1}^{n}\alpha_j = 0$, (7) can be written as

$$\left\|\sum_{j\in J}\alpha_j(x_j - x_i)\right\| = \sum_{j\in J}|\alpha_j|\|x_j - x_i\|. \tag{8}$$

Again, by Lemma 2.6, we deduce that (8) holds precisely when there is $v \in X$ such that

$$\frac{\alpha_j(x_j - x_i)}{|\alpha_j|\|x_j - x_i\|} = v, \quad j \in J.$$

This proves the theorem. □

As an immediate consequence of Theorem 2.7 and the second inequality in (4) we obtain the following result.

Corollary 2.8. *Let X be a strictly convex normed linear space, $x_1, \dots, x_n$ non-zero elements of X and $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \{0\}$. Then the following two statements are mutually equivalent.*

(i) $\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \min_{i\in\{1,\dots,n\}}\left\{\left|\sum_{j=1}^{n}\alpha_j\right|\|x_i\| + \sum_{j=1}^{n}|\alpha_j|\|x_j - x_i\|\right\}.$

(ii) $x_1 = \cdots = x_n$ *or there exist* $i \in \{1, \dots, n\}$ *and* $v \in X$ *satisfying* $\frac{\alpha_j}{|\alpha_j|}\frac{x_j - x_i}{\|x_j - x_i\|} = v$ *for all* $j \in \{1, \dots, n\}$ *such that* $x_j \neq x_i$ *and* $\sum_{j=1}^{n}\alpha_j x_i = \left|\sum_{j=1}^{n}\alpha_j\right|\|x_i\|v.$

Theorem 2.9. *Let X be a strictly convex normed linear space, $x_1, \dots, x_n$ non-zero elements of X and $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \{0\}$. Then for every $i \in \{1, \dots, n\}$ the following two statements are mutually equivalent.*

(i) $\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \left|\sum_{j=1}^{n}\alpha_j\right|\|x_i\| - \sum_{j=1}^{n}|\alpha_j|\|x_j - x_i\|.$

(ii) $x_1 = \cdots = x_n$ *or there exists* $v \in X$ *satisfying* $\frac{\alpha_j}{|\alpha_j|}\frac{x_i - x_j}{\|x_i - x_j\|} = v$ *for all* $j \in \{1, \dots, n\}$ *such that* $x_j \neq x_i$ *and* $\sum_{j=1}^{n}\alpha_j x_j = \left\|\sum_{j=1}^{n}\alpha_j x_j\right\|v.$

Proof. If $x_1 = \cdots = x_n$ we are done. So, suppose that this is not the case.

Let us denote $J = \{j \in \{1, \dots, n\} : x_j \neq x_i\}$. Put $y := \sum_{j=1}^{n}\alpha_j x_i$ and $z := \sum_{j=1}^{n}\alpha_j(x_i - x_j)$.

(i)⇒(ii) Passing the proof of the first inequality in (4) (see [4, Theorem 1]) we deduce that (i) holds if and only if the following two conditions are satisfied:

$$\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \left\|\sum_{j=1}^{n}\alpha_j x_i\right\| - \left\|\sum_{j=1}^{n}\alpha_j (x_i - x_j)\right\| \tag{9}$$

and

$$\left\|\sum_{j\in J}\alpha_j (x_i - x_j)\right\| = \sum_{j\in J}|\alpha_j|\|x_i - x_j\|. \tag{10}$$

By Lemma 2.6, (10) holds if and only if there is $v \in X$ satisfying

$$\frac{\alpha_j}{|\alpha_j|}\frac{x_i - x_j}{\|x_i - x_j\|} = v, \quad j \in J. \tag{11}$$

Now we have

$$z = \sum_{j\in J}\alpha_j (x_i - x_j) = \sum_{j\in J}|\alpha_j|\|x_i - x_j\|v = \|z\|v.$$

Since $\|z\| = \sum_{j\in J}|\alpha_j|\|x_i - x_j\| \neq 0$, we get

$$\frac{z}{\|z\|} = v.$$

Note that (9) can be written as $\|y - z\| = \|y\| - \|z\|$, i.e., $\|(y - z) + z\| = \|y - z\| + \|z\|$. So, by Lemma 2.6 it follows that

$$y - z = \|y - z\|\frac{z}{\|z\|} = \|y - z\|v.$$

Thus,

$$\sum_{j=1}^{n}\alpha_j x_j = \left\|\sum_{j=1}^{n}\alpha_j x_j\right\|v.$$

(ii) ⇒ (i) To prove (i) we must show that (9) and (10) hold. Since (10) ⇔ (11) and (11) holds by the assumption, it remains to prove (9). As in the first part of the proof, (11) implies $\frac{z}{\|z\|} = v$. Also, by the assumption we have $y - z = \|y - z\|v$. Thus, $y = z + \|y - z\|v = \|z\|v + \|y - z\|v$ from which it follows that $\|y\| = \|z\| + \|y - z\|$, which is the equality (9). This completes the proof. □

As a consequence of Theorem 2.9 and the first inequality in (4) we have the following result.

Corollary 2.10. *Let X be a strictly convex normed linear space, $x_1,\ldots,x_n$ non-zero elements of X and $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}$. Then the following two statements are mutually equivalent.*

(i) $\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \max_{i\in\{1,\ldots,n\}}\left\{\left|\sum_{j=1}^{n}\alpha_j\right|\|x_i\| - \sum_{j=1}^{n}|\alpha_j|\|x_j - x_i\|\right\}$.

(ii) $x_1 = \cdots = x_n$ *or there exist* $i \in \{1,\ldots,n\}$ *and* $v \in X$ *satisfying* $\frac{\alpha_j}{|\alpha_j|}\frac{x_i - x_j}{\|x_i - x_j\|} = v$ *for all* $j \in \{1,\ldots,n\}$ *such that* $x_j \neq x_i$ *and* $\sum_{j=1}^{n}\alpha_j x_j = \left\|\sum_{j=1}^{n}\alpha_j x_j\right\| v$.

Concluding remarks

It was shown in [1] that for non-zero elements $x_1,\ldots,x_n$ of a pre-Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$ the equality $\|\sum_{j=1}^{n}x_j\| = \sum_{j=1}^{n}\|x_j\|$ holds if and only if there exist $i \in \{1,\ldots,n\}$ and a state φ on $\mathcal{A}$ such that $\varphi(\langle x_j, x_i\rangle) = \|x_j\|\|x_i\|$ for all $j \in \{1,\ldots,n\}\setminus\{i\}$, where $\langle\cdot,\cdot\rangle$ stands for an $\mathcal{A}$-valued inner product on X. (For the definition and basic results on (pre)-Hilbert C^*-modules the reader is referred to [7] or [14].) By using this result, Pečarić and Rajić [12] described the case of equality in each of the inequalities in (1), where x_j are non-zero elements of a pre-Hilbert C^*-module X, and scalars α_j are chosen to be $\frac{1}{\|x_j\|}$. In a similar way, one can obtain the characterizations of the case of equality in each of the inequalities in (1) and (4) for non-zero elements x_j of a pre-Hilbert C^*-module X and non-zero complex numbers α_j. For instance, to describe the equality attainedness in the second inequality in (4) we consider two different cases: $\sum_{j=1}^{n}\alpha_j \neq 0$ or $\sum_{j=1}^{n}\alpha_j = 0$. In the first case the equality holds precisely when $x_1 = \cdots = x_n$, or there exist $i \in \{1,\ldots,n\}$ and a state φ on $\mathcal{A}$ satisfying

$$\alpha_k\left(\sum_{j=1}^{n}\overline{\alpha}_j\right)\varphi(\langle x_i, x_k - x_i\rangle) = |\alpha_k|\left|\sum_{j=1}^{n}\alpha_j\right|\|x_i\|\,\|x_k - x_i\|$$

for all $k \in \{1,\ldots,n\}$ such that $x_k \neq x_i$.

In the second case the equality holds if and only if $x_1 = \cdots = x_n$, or there exist $i, k \in \{1,\ldots,n\}$ for which $x_i \neq x_k$, and a state φ on $\mathcal{A}$ satisfying

$$\overline{\alpha}_j\alpha_k\varphi(\langle x_j - x_i, x_k - x_i\rangle) = |\alpha_j||\alpha_k|\,\|x_j - x_i\|\,\|x_k - x_i\|$$

for all $j \in \{1,\ldots,n\}\setminus\{k\}$ such that $x_j \neq x_i$.

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