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# A NEW CLASS OF GENERAL REFINED HARDY-TYPE INEQUALITIES WITH KERNELS

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Abstract. Let $\mu_1$ and $\mu_2$ be positive $\sigma$-finite measures on $\Omega_1$ and $\Omega_2$ respectively, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function, and

$$K(x) = \int_{\Omega_2} k(x, y)\, d\mu_2(y),\ x \in \Omega_1.$$

We state and prove a new class of refined general Hardy-type inequalities related to the weighted Lebesgue spaces $L^p$ and $L^q$, where $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, convex functions and the integral operators $A_k$ of the form

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x, y) f(y)\, d\mu_2(y).$$

We also provide a class of new sufficient conditions for a weighted modular inequality involving operator $A_k$ to hold. As special cases of our results, we obtain refinements of the classical one-dimensional Hardy's, Pólya–Knopp's and Hardy–Hilbert's inequality and of related dual inequalities, as well as a generalization and refinement of the classical Godunova's inequality. Finally, we show that our results may be seen as generalizations of some recent results related to Riemann-Liouville's and Weyl's operator.

## 1. Introduction

To start with, we recall some well-known integral inequalities. The first of them is the classical Hardy's inequality,

$$\int_0^\infty \left( \frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \le \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx, \tag{1.1}$$

where $1 < p < \infty$, $\mathbb{R}_+ = (0, \infty)$, and $f \in L^p(\mathbb{R}_+)$ is a non-negative function. By rewriting (1.1) with the function $f$ replaced with $f^{\frac{1}{p}}$ and then by letting $p \to \infty$, we obtain the limiting case of Hardy's inequality,

$$\int_0^\infty \exp\left( \frac{1}{x} \int_0^x \ln f(t)\, dt \right) dx < e \int_0^\infty f(x)\, dx\,, \tag{1.2}$$

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which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. That inequality is referred to as Pólya–Knopp's inequality. Another two important classical inequalities, closely related to (1.1), are Hardy–Hilbert's inequality,

$$\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y}\, dx \right)^p dy \le \left( \frac{\pi}{\sin \frac{\pi}{p}} \right)^p \int_0^\infty f^p(x)\, dx, \tag{1.3}$$

and Hardy–Littlewood–Pólya's inequality

$$\int_0^\infty \left( \int_0^\infty \frac{f(y)}{\max\{x,y\}} \right)^p dy \le (pp')^p \int_0^\infty f^p(y)\, dy, \tag{1.4}$$

which hold for $1 < p < \infty$ and non-negative functions $f \in L^p(\mathbb{R}_+)$. Notice that the constants $\left(\frac{p}{p-1}\right)^p$, $e$, $\left(\frac{\pi}{\sin \frac{\pi}{p}}\right)^p$ and $(pp')^p$, respectively appearing on the right-hand sides of (1.1) – (1.4), are the best possible, that is, neither of them can be replaced with any smaller constant.

Since Hardy, Hilbert, and Pólya established inequalities (1.1), (1.2), and (1.3), they have been investigated and generalized in several directions. Further information and remarks concerning the rich history of the integral inequalities mentioned above can be found e.g. in the monographs [13,21,25,26] and expository papers [2,3,6–9,15,16,20,28] and the references given therein. Here we mention only results that to some extent have guided us in our research.

In particular, S. Kaijser et al. [17] (see also [16, 24]) pointed out that (1.1), (1.2) and (1.3) are special cases of a more general inequality of Hardy–Knopp's type with a kernel,

$$\int_0^\infty u(x)\Phi(A_k f(x))\, \frac{dx}{x} \le \int_0^\infty v(x)\Phi(f(x))\, \frac{dx}{x}, \tag{1.5}$$

where $0 < b \le \infty$, $k : (0,b) \times (0,b) \to \mathbb{R}$ and $u : (0,b) \to \mathbb{R}$ are non-negative functions, such that

$$K(x) = \int_0^x k(x,y)\, dy > 0,\ x \in (0,b), \tag{1.6}$$

and

$$v(y) = y \int_y^b u(x)\, \frac{k(x,y)}{K(x)}\, \frac{dx}{x} < \infty,\ y \in (0,b),$$

$\Phi$ is a convex function on an interval $I \subseteq \mathbb{R}$, $f : (0,b) \to \mathbb{R}$ is a function with values in $I$, and

$$A_k f(x) = \frac{1}{K(x)} \int_0^x k(x,y) f(y)\, dy,\ x \in (0,b). \tag{1.7}$$

Notice that (1.5) follows directly by only a standard application of Jensen's inequality and Fubini's theorem.

On the other hand, Godunova [11] (see also [26, Chapter VIII, p. 233]) proved that the inequality

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \Phi\left(\frac{1}{x_1\cdots x_n}\int_{\mathbb{R}_+^n} l\left(\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}\right) f(\mathbf{y})\,d\mathbf{y}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&\qquad \leq \int_{\mathbb{R}_+^n}\frac{\Phi(f(\mathbf{x}))}{x_1\cdots x_n}\,d\mathbf{x}
\end{aligned} \tag{1.8}$$

holds for all non-negative functions $l:\mathbb{R}_+^n\to\mathbb{R}_+$, such that $\int_{\mathbb{R}_+^n} l(\mathbf{x})\,d\mathbf{x}=1$, convex functions $\Phi:[0,\infty)\to[0,\infty)$, and non-negative functions $f$ on $\mathbb{R}_+^n$, such that the function $\mathbf{x}\mapsto\frac{\Phi(f(\mathbf{x}))}{x_1\cdots x_n}$ is integrable on $\mathbb{R}_+^n$.

Recently, Krulić et al. [19] unified all the above results by studying the measure spaces $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$, and the general integral operator $A_k$ defined by

$$A_kf(x)=\frac{1}{K(x)}\int_{\Omega_2}k(x,y)f(y)\,d\mu_2(y),\ x\in\Omega_1, \tag{1.9}$$

where $f:\Omega_2\longrightarrow\mathbb{R}$ is a measurable function, $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is a measurable and non-negative function, and

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y)>0,\ x\in\Omega_1. \tag{1.10}$$

They proved, again by using Jensen's inequality and Fubini's theorem, that the weighted inequality

$$\int_{\Omega_1}u(x)\Phi(A_kf(x))\,d\mu_1(x)\leq\int_{\Omega_2}v(y)\Phi(f(y))\,d\mu_2(y) \tag{1.11}$$

holds for all non-negative measurable functions $u:\Omega_1\to\mathbb{R}$, such that

$$v(y)=\int_{\Omega_1}u(x)\,\frac{k(x,y)}{K(x)}\,d\mu_1(x)<\infty,\ y\in\Omega_2, \tag{1.12}$$

convex functions $\Phi$ on an interval $I\subseteq\mathbb{R}$, and functions $f:\Omega_2\to\mathbb{R}$ with values in $I$. In the same paper they also proved a generalization of inequality (1.11). Namely, if $0<p\leq q<\infty$, $v$ is now defined with

$$v(y)=\left(\int_{\Omega_1}u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}d\mu_1(x)\right)^{\frac{p}{q}}<\infty,\, y\in\Omega_2, \tag{1.13}$$

and $\Phi$ is a non-negative convex function on an interval $I\subseteq\mathbb{R}$, then the inequality

$$\left(\int_{\Omega_1}u(x)\Phi^{\frac{q}{p}}(A_kf(x))\,d\mu_1(x)\right)^{\frac{1}{q}}\leq\left(\int_{\Omega_2}v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{1.14}$$

holds for all functions $f : \Omega_2 \to \mathbb{R}$, such that $f(\Omega_2) \subseteq I$.

In addition to proving direct inequalities of the form (1.11), there are many classical and recent results concerning the mapping properties of integral operators such as (1.7) and (1.9), that is, necessary and sufficient conditions of the Muckenhoupt type on weight functions and a kernel for boundedness of the operator $A_k$ between two function spaces. Some important and useful modular inequalities related to (1.11) can be found e.g. in [14, 18, 23]. Without stating them, here we emphasize just a class of sufficient conditions on $u$, $v$, and $k$, related to the operator (1.7), obtained in [17].

Motivated by all the results mentioned, in this paper we provide a new two-parametric class of sufficient conditions for a weighted modular inequality involving operator (1.9) to hold. Further, we state and prove a new refined general weighted Hardy–type inequality with a non-negative kernel, related to an arbitrary convex (or concave) function, and point out that our result refines relation (1.14). Applying the obtained general relation to some important particular kernels and concrete measure spaces, we derive new refinements of the classical one-dimensional Hardy's, Pólya–Knopp's and Hardy–Hilbert's inequality and related dual inequalities, as well as a generalization and a refinement of the classical Godunova's inequality. Finally, we show that our results may be seen as generalizations of some recent results related to Riemann-Liouville's and Weyl's operator.

The paper is organized in the following way. In Section 2 we establish and discuss a new class of sufficient conditions for a weighted modular inequality involving operator $A_k$ defined by (1.9) to hold, while in Section 3 we state, prove and discuss a general refined weighted Hardy–type inequality with a non-negative kernel and an arbitrary convex function. In the same section, we discuss some particular cases of the obtained general inequality, related to power and exponential functions, and to the simplest possible kernel – the one with separate variables. In the following two sections, our general results are applied to various one-dimensional settings and the Lebesgue measure. Namely, in Section 4 we obtain a new refinement of the classical one-dimensional Hardy's, Pólya–Knopp's, and related dual inequalities and point out that our results generalize some recent results related to Riemann-Liouville's and Weyl's operator. In Section 5, we obtain new generalized Hardy–Hilbert's and Hardy–Littlewood–Pólya's inequality. The paper concludes with Section 6, where a new refinement of the classical Godunova's inequality is given.

**Conventions.** Throughout this paper, all measures are assumed to be positive, all functions are assumed to be measurable, and expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{a}{\infty}$ ($a \in \mathbb{R}$), and $\frac{\infty}{\infty}$ are taken to be equal to zero. For a real parameter $0 \neq p \neq 1$, by $p'$ we denote its conjugate exponent $p' = \frac{p}{p-1}$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. By $|\Omega|_\mu$ we denote the measure of a measurable set

$\Omega$ with respect to the measure $\mu$. In particular, we use the symbol $|\ |_1$ as an abbreviation for $\|\ \|_{L^1(\Omega_1,\mu_1)}$. Also, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set. An interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while by $\operatorname{Int} I$ we denote its interior. $B(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ denotes the incomplete Beta function, defined by

$$B(x;a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt,\ \ x\in[0,1],\ \ a,b>0.$$

As usual, $B(a,b)=B(1;a,b)$ stands for the standard Beta function. Finally, inequalities like (1.11) are interpreted to mean if the right-hand side is finite, so is the left-hand side and the inequality holds.

## 2. A new class of general Hardy–type inequalities with kernels

To begin with, in this section we provide a new class of sufficient conditions on weight functions $u$ and $w$, and on a kernel $k$, for a modular inequality involving the Hardy–type operator $A_k$, defined by (1.9), to hold. The first result in that direction is given in the following theorem.

**Theorem 2.1.** *Let $0<p\le q<\infty$. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive $\sigma$-finite measures, $u$ be a weight function on $\Omega_1$, $w$ be a $\mu_2$–a.e. positive function on $\Omega_2$, $k$ be a non-negative measurable function on $\Omega_1\times\Omega_2$, and $K$ be defined on $\Omega_1$ by (1.10). Suppose that $K(x)>0$ for all $x\in\Omega_1$ and that the function $x\mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on $\Omega_1$ for each fixed $y\in\Omega_2$. Let $\Phi$ be a non-negative convex function on an interval $I\subseteq\mathbb{R}$. If*

$$A=\sup_{y\in\Omega_2} w^{-\frac{1}{p}}(y)\left(\int_{\Omega_1}u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}d\mu_1(x)\right)^{\frac{1}{q}}<\infty, \tag{2.1}$$

*then there exists a positive real constant $C$, such that the inequality*

$$\left(\int_{\Omega_1}u(x)\Phi^{\frac{q}{p}}(A_kf(x))\,d\mu_1(x)\right)^{\frac{1}{q}}\le C\left(\int_{\Omega_2}w(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{2.2}$$

*holds for all measurable functions $f:\Omega_2\to\mathbb{R}$ with values in $I$ and $A_kf$ defined on $\Omega_1$ by (1.9). Moreover, if $C$ is the smallest constant for (2.2) to hold, then $C\le A$.*

*Proof.* By using Jensen's inequality, monotonicity of the power functions $\alpha\mapsto\alpha^t$ for a positive exponent $t$, and then Minkowski's inequality, we find that

$$\begin{aligned}
&\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}\left(A_k f(x)\right)\,d\mu_1(x)\right)^{\frac{1}{q}}\\
&=\left(\int_{\Omega_1} u(x)\left[\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right)\right]^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{1}{q}}\\
&\le\left(\int_{\Omega_1} u(x)\left[\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right]^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{1}{q}}\\
&\le\left(\int_{\Omega_2}\left(w^{-\frac{q}{p}}(y)\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{p}{q}} w(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}}\\
&\le A\left(\int_{\Omega_2} w(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}}. \qquad (2.3)
\end{aligned}$$

Hence, (2.2) holds with $C = A$, so the proof is complete. ■

Following the same lines as in the proof of Theorem 2.1, we get the next corollary.

**Corollary 2.1.** *Let $-\infty < q \le p < 0$ and let the assumptions of Theorem 2.1 be satisfied with a positive convex function $\Phi$. If*

$$B = \inf_{y\in\Omega_2} w^{-\frac{1}{p}}(y)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{1}{q}} < \infty,$$

*then there exists a positive real constant $C$, such that the inequality*

$$\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}} \ge C\left(\int_{\Omega_2} w(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \qquad (2.4)$$

*holds for all measurable functions $f:\Omega_2\to\mathbb{R}$ with values in $\Omega_2$. Moreover, if $C$ is the largest constant for (2.4) to hold, then $C \ge B$.*

In order to apply Theorem 2.1 to $n$-dimensional cells in $\mathbb{R}^n_+$, we need to introduce some further notation. For $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_+$, $\mathbf{u}=(u_1,u_2,\dots,u_n)$, $\mathbf{v}=(v_1,v_n,\dots,v_n)$, we denote

$$\frac{\mathbf{u}}{\mathbf{v}}=\left(\frac{u_1}{v_1},\frac{u_2}{v_2},\dots,\frac{u_n}{v_n}\right)\ \text{and}\ \mathbf{u}^{\mathbf{v}}=u_1^{v_1}u_2^{v_2}\cdots u_n^{v_n}.$$

Especially, $\mathbf{u}^{\mathbf{1}}=\prod_{i=1}^n u_i$ and $\mathbf{u}^{\mathbf{-1}}=\left(\prod_{i=1}^n u_i\right)^{-1}$, where $\mathbf{1}=(1,1,\dots,1)$. We write $\mathbf{u}<\mathbf{v}$ if componentwise $u_i<v_i$, $i=1,\dots,n$. Relations $\le$, $>$, and $\ge$ are defined analogously. Finally,

$$(\mathbf{0},\mathbf{b})=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{0}<\mathbf{x}<\mathbf{b}\},\quad\text{and}\quad(\mathbf{b},\infty)=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{b}<\mathbf{x}<\infty\}.$$

In this setting, Theorem 2.1 reads as follows.

**Corollary 2.2.** *Let $0 < p \leq q < \infty$ and $\mathbf{0} < \mathbf{b} \leq \infty$. Let $u$ be a non-negative and $v$ be a positive function on $(\mathbf{0}, \mathbf{b})$ and let $\Phi$ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{\mathbf{y} \in (\mathbf{0}, \mathbf{b})} \left( \frac{\mathbf{y}}{v(\mathbf{y})} \right)^{\frac{1}{p}} \left( \int_{(\mathbf{y}, \mathbf{b})} u(\mathbf{x})\, \mathbf{x}^{-\frac{q}{p}-1} d\mathbf{x} \right)^{\frac{1}{q}} < \infty,$$

*then there exists a positive real constant $C$, such that the inequality*

$$\left( \int_{(\mathbf{0}, \mathbf{b})} u(\mathbf{x}) \Phi^{\frac{q}{p}} (Hf(\mathbf{x})) \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \right)^{\frac{1}{q}} \leq C \left( \int_{(\mathbf{0}, \mathbf{b})} v(\mathbf{y}) \Phi(f(\mathbf{y})) \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} \right)^{\frac{1}{p}} \tag{2.5}$$

*holds for all measurable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ with values in $I$ and*

$$Hf(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{(\mathbf{0}, \mathbf{x})} f(\mathbf{y})\, d\mathbf{y}, \ \mathbf{x} \in (\mathbf{0}, \mathbf{b}).$$

*Moreover, if $A$ is the smallest constant for (2.5) to hold, then $C \leq A$.*

*Proof.* Let $S_n = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n : \mathbf{0} < \mathbf{y} \leq \mathbf{x} < \mathbf{b}\}$ and $\Omega_1 = \Omega_2 = (\mathbf{0}, \mathbf{b})$. The proof follows directly from Theorem 2.1, applied with $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, $k = \chi_{S_n}$, and with $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$ instead of $u(\mathbf{x})$, $\mathbf{x} \in (\mathbf{0}, \mathbf{b})$. Observe that $w(\mathbf{y}) = \mathbf{y}^{-\mathbf{1}} v(\mathbf{y})$, $\mathbf{y} \in (\mathbf{0}, \mathbf{b})$. ■

**Remark 2.1.** The result given in Corollary 2.2 was published in [17, Theorem 3.1], so we see that Theorem 3.1 from [17] is just a special case of our Theorem 2.1. ■

Our analysis continues by providing a new two-parametric class of sufficient conditions for a weighted modular inequality involving the operator $A_k$ to hold. The conditions obtained depend on a real parameter $s$ and a positive function $V$ on $\Omega_2$. That result is given in the following theorem.

**Theorem 2.2.** *Let $1 < p \leq q < \infty$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma$-finite measures, $u$ be a weight function on $\Omega_1$, $v$ be a measurable $\mu_2$–a.e. positive function on $\Omega_2$, $k$ be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and $K$ be defined on $\Omega_1$ by (1.10). Let $K(x) > 0$ for all $x \in \Omega_1$ and let the function $x \mapsto u(x) \left( \frac{k(x,y)}{K(x)} \right)^q$ be integrable on $\Omega_1$ for each fixed $y \in \Omega_2$. Suppose that $\Phi : I \to [0, \infty)$ is a bijective convex function on an interval $I \subseteq \mathbb{R}$. If there exist a real parameter $s \in (1, p)$ and a positive measurable function $V : \Omega_2 \to \mathbb{R}$ such that*

$$A(s, V) = F(V, v) \sup_{y \in \Omega_2} V^{\frac{s-1}{p}}(y) \left[ \int_{\Omega_1} u(x) \left( \frac{k(x,y)}{K(x)} \right)^q d\mu_1(x) \right]^{\frac{1}{q}} < \infty,$$

*where*

$$F(V, v) = \left( \int_{\Omega_2} V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y)\, d\mu_2(y) \right)^{\frac{1}{p'}},$$

*then there is a positive real constant $C$ such that the inequality*

$$\left(\int_{\Omega_1} u(x)\Phi^q(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}} \le C\left(\int_{\Omega_2} v(y)\Phi^p(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{2.6}$$

*holds for all measurable functions $f:\Omega_2\to\mathbb{R}$ with values in $I$ and $A_k f$ defined on $\Omega_1$ by* (1.9)*. Moreover, if $C$ is the best possible constant in* (2.6)*, then*

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V). \tag{2.7}$$

*Proof.* Let $f:\Omega_2\to\mathbb{R}$ be an arbitrary measurable function with values in $I$. Applying Jensen's inequality to the left-hand side of (2.6) we get

$$\begin{aligned}&\left(\int_{\Omega_1} u(x)\Phi^q(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}}\\ &\le \left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}.\end{aligned}$$

Hence, to prove inequality (2.6) it suffices to prove that there is a real constant $C>0$, independent on $f$, such that

$$\begin{aligned}&\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}\\ &\le C\left(\int_{\Omega_2} v(y)\Phi^p(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}}.\end{aligned} \tag{2.8}$$

Taking into account properties of the function $\Phi$, let $g:\Omega_2\to\mathbb{R}$ be defined by $\Phi(g(y)) = v(y)\Phi^p(f(y))$. Then $g(\Omega_2)\subseteq I$ holds and (2.8) is equivalent to

$$\begin{aligned}&\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi^{\frac{1}{p}}(g(y))v^{-\frac{1}{p}}(y)\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}\\ &\le C\left(\int_{\Omega_2} \Phi(g(y))\,d\mu_2(y)\right)^{\frac{1}{p}}.\end{aligned} \tag{2.9}$$

Therefore, instead of proving (2.8), we prove that (2.9) holds for all measurable functions $g:\Omega_2\to\mathbb{R}$ with values in $I$. Applying Hölder's inequality, monotonicity of the power functions $\alpha\mapsto\alpha^t$ for positive exponents $t$, Minkowski's inequality, and the definitions of $F(V,v)$ and $A(s,V)$, we get the following sequence of inequalities involving an arbitrary positive measurable function $V : \Omega_2 \to \mathbb{R}$:

$$\left\{\int_{\Omega_1} u(x) \left[\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi^{\frac{1}{p}}(g(y))v^{-\frac{1}{p}}(y)\, d\mu_2(y)\right]^q d\mu_1(x)\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\Omega_1} \frac{u(x)}{K^q(x)} \left[\int_{\Omega_2}\Big(k(x,y)\Phi^{\frac{1}{p}}(g(y))V^{\frac{s-1}{p}}(y)\Big)\Big(V^{\frac{1-s}{p}}(y)v^{-\frac{1}{p}}(y)\Big)\, d\mu_2(y)\right]^q d\mu_1(x)\right\}^{\frac{1}{q}}$$

$$\leq \left\{\int_{\Omega_1} \frac{u(x)}{K^q(x)} \left(\int_{\Omega_2} k^p(x,y)\Phi(g(y))V^{s-1}(y)\, d\mu_2(y)\right)^{\frac{q}{p}} \times\right.$$

$$\left. \times \left(\int_{\Omega_2} V^{-\frac{p'(s-1)}{p}}(y)v^{1-p'}(y)\, d\mu_2(y)\right)^{\frac{q}{p'}} d\mu_1(x)\right\}^{\frac{1}{q}}$$

$$= F(V,v)\left\{\int_{\Omega_1} \frac{u(x)}{K^q(x)} \left(\int_{\Omega_2} k^p(x,y)\Phi(g(y))V^{s-1}(y)\, d\mu_2(y)\right)^{\frac{q}{p}} d\mu_1(x)\right\}^{\frac{1}{q}}$$

$$\leq F(V,v)\left\{\int_{\Omega_2} \Phi(g(y))V^{s-1}(y) \left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x)\right]^{\frac{p}{q}} d\mu_2(y)\right\}^{\frac{1}{p}}$$

(2.10)

$$\leq A(s,V)\left(\int_{\Omega_2} \Phi(g(y))\, d\mu_2(y)\right)^{\frac{1}{p}}.$$

Thus, inequalities (2.9) and (2.8) hold. Relation (2.6) follows by considering (2.7), so the proof is complete. ■

By modifying Theorem 2.2 for the setting from relations (1.6) and (1.7), we obtain the following result.

**Theorem 2.3.** *Let $1 < p \leq q < \infty$, $1 < s < p$, and $0 < b \leq \infty$. Let $u$ be a weight function on $(0,b)$, $w$ be an a.e. positive measurable function on $(0,b)$, and $k$ be a non-negative measurable function on $(0,b)\times(0,b)$ satisfying* (1.6). *Let $I$ be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If*

$$V(y) = \int_0^y w^{1-p'}(x)x^{p'-1}\, dx < \infty \tag{2.11}$$

*holds almost everywhere in $(0,b)$ and*

$$A(s) = \sup_{0<y<b} \left(\int_y^b u(x)\left(\frac{k(x,y)}{K(x)}\right)^q V^{\frac{q(p-s)}{p}}(x)\,\frac{dx}{x}\right)^{\frac{1}{q}} V^{\frac{s-1}{p}}(y) < \infty, \tag{2.12}$$

*then there exists a positive real constant $C$ such that*

$$\left(\int_0^b u(x)\Phi^q(A_kf(x))\,\frac{dx}{x}\right)^{\frac{1}{q}} \leq C\left(\int_0^b w(x)\Phi^p(f(x))\,\frac{dx}{x}\right)^{\frac{1}{p}} \tag{2.13}$$

*holds for all measurable functions* $f : (0,b) \to \mathbb{R}$ *with values in* $I$ *and the Hardy-type operator* $A_k$ *defined by* (1.7). *Moreover, if* $C$ *is the best possible constant in* (2.13), *then*

$$C \leq \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s).$$

*Proof.* Denote $S_1 = \{(x,y) \in \mathbb{R}^2 : 0 < y \leq x < b\}$ and set $\Omega_1 = \Omega_2 = (0,b)$. In Theorem 2.2, replace $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, $v(y)$, and $k$ respectively with $dx$, $dy$, $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_1}$. In this setting, inequality (2.6) reduces to (2.13). Moreover, following the lines of the proof of Theorem 2.2, the first inequality in (2.10) becomes

$$\begin{aligned}
&\left\{\int_0^b u(x)\left[\frac{1}{K(x)}\int_0^x k(x,y)\Phi^{\frac{1}{p}}(g(y))\left(\frac{y}{v(y)}\right)^{\frac{1}{p}}(y)\,dy\right]^q \frac{dx}{x}\right\}^{\frac{1}{q}} \\
&\leq \left\{\int_0^b \frac{u(x)}{K^q(x)}\left(\int_0^x k^p(x,y)\Phi(g(y))V^{s-1}(y)\,dy\right)^{\frac{q}{p}} \times\right. \\
&\qquad \left. \times \left(\int_0^x V^{-\frac{p'(s-1)}{p}}(y)v^{1-p'}(y)y^{p'-1}\,dy\right)^{\frac{q}{p'}}\frac{dx}{x}\right\}^{\frac{1}{q}}. \qquad (2.14)
\end{aligned}$$

Since definition (2.11) yields

$$\int_0^x V^{-\frac{p'(s-1)}{p}}(y)v^{1-p'}(y)y^{p'-1}\,dy = \frac{p-1}{p-s}\,V^{\frac{p-s}{p-1}}(x),\ x \in (0,b),$$

the right-hand side of (2.14) is further equal to

$$\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left\{\int_0^b \frac{u(x)}{K^q(x)}\,V^{\frac{q(p-s)}{p}}(x)\left(\int_0^x k^p(x,y)\Phi(g(y))V^{s-1}(y)\,dy\right)^{\frac{q}{p}}\frac{dx}{x}\right\}^{\frac{1}{q}}.$$

As in (2.10), the rest of the proof follows by applying Minkowski's inequality and definition (2.12) of $A(s)$. ∎

**Remark 2.2.** The result of Theorem 2.3 is given in [17, Theorem 4.4]. Hence, Theorem 4.4 in [17] can be seen a special case of Theorem 2.2. ∎

## 3. Refined Hardy-type inequalities with kernels

The rest of the paper is dedicated to new refined inequalities related to the general Hardy-type operator $A_k$ with a non-negative kernel, defined by (1.9). First, we introduce some necessary notation and recall basic facts regarding convex and concave functions. Suppose $I$ is an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ is a convex function. By $\partial\Phi(r)$ we denote the subdifferential of $\Phi$ at $r \in I$, that is, the set $\partial\Phi(r) = \{\alpha \in \mathbb{R} : \Phi(s) - \Phi(r) - \alpha(s-r) \geq 0,\, s \in I\}$. It is well-known that $\partial\Phi(r) \neq \emptyset$ for all $r \in \operatorname{Int} I$. More precisely, at each point $r \in \operatorname{Int} I$ we have $-\infty < \Phi'_-(r) \leq \Phi'_+(r) < \infty$ and $\partial\Phi(r) = [\Phi'_-(r),\, \Phi'_+(r)]$, while the set on which $\Phi$ is not differentiable is at most countable. Moreover, every function $\varphi : I \longrightarrow \mathbb{R}$, for which $\varphi(r) \in \partial\Phi(r)$ whenever $r \in \operatorname{Int} I$, is increasing on $\operatorname{Int} I$. Notice that for any such function $\varphi$ and arbitrary $r \in \operatorname{Int} I$, $s \in I$ we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geq 0$$

and further

$$\begin{aligned} \Phi(s) - \Phi(r) - \varphi(r)(s-r) &= |\Phi(s) - \Phi(r) - \varphi(r)(s-r)| \\ &\geq \left|\, |\Phi(s) - \Phi(r)| - |\varphi(r)| \cdot |s-r| \,\right|. \end{aligned} \tag{3.1}$$

On the other hand, if $\Phi : I \to \mathbb{R}$ is a concave function, that is, $-\Phi$ is convex, then $\partial\Phi(r) = \{\alpha \in \mathbb{R} : \Phi(r) - \Phi(s) - \alpha(r-s) \geq 0,\, s \in I\}$ denotes the superdifferential of $\Phi$ at the point $r \in I$. For all $r \in \operatorname{Int} I$, in this setting we have $-\infty < \Phi'_+(r) \leq \Phi'_-(r) < \infty$ and $\partial\Phi(r) = [\Phi'_+(r),\, \Phi'_-(r)] \neq \emptyset$. Hence, the inequality

$$\Phi(r) - \Phi(s) - \varphi(r)(r-s) \geq 0$$

holds for all $r \in \operatorname{Int} I$, $s \in I$, and all real functions $\varphi$ on $I$, such that $\varphi(t) \in \partial\Phi(t)$, $t \in \operatorname{Int} I$. Therefrom, we also get

$$\begin{aligned} \Phi(r) - \Phi(s) - \varphi(r)(r-s) &= |\Phi(r) - \Phi(s) - \varphi(r)(r-s)| \\ &\geq \left|\, |\Phi(s) - \Phi(r)| - |\varphi(r)| \cdot |s-r| \,\right|. \end{aligned} \tag{3.2}$$

Observe that, although the symbol $\partial\Phi(r)$ has two different notions, it will be clear from the context whether it applies to a convex or to a concave function $\Phi$. Many further information on convex and concave functions can be found e.g. in the monographs [25] and [26] and in references cited therein.

Now, we are ready to state and prove the central result of this section, that is, a new general refined weighted Hardy-type inequality with a non-negative kernel, related to an arbitrary non-negative convex function. It is given in the following theorem.

**Theorem 3.1.** *Let $t \in \mathbb{R}^+$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma$-finite measures, $u$ be a weight function on $\Omega_1$, $k$ a non-negative measurable function on $\Omega_1 \times \Omega_2$, and $K$ be defined on $\Omega_1$ by* (1.10). *Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^t$ is integrable on $\Omega_1$ for each fixed $y \in \Omega_2$, and that $v$ is defined on $\Omega_1$ by*

$$v(y) = \left( \int_{\Omega_1} u(x) \left( \frac{k(x,y)}{K(x)} \right)^t d\mu_1(x) \right)^{\frac{1}{t}}. \tag{3.3}$$

*If $\Phi$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned} &\left( \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y) \right)^t - \int_{\Omega_1} u(x)\Phi^t(A_k f(x))\, d\mu_1(x) \\ &\geq\ t \int_{\Omega_1} \frac{u(x)}{K(x)}\, \Phi^{t-1}(A_k f(x)) \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\, d\mu_1(x) \end{aligned} \tag{3.4}$$

*holds for all* $t \geq 1$ *and all measurable functions* $f : \Omega_2 \to \mathbb{R}$ *with values in* $I$*, where* $A_k f$ *is defined on* $\Omega_1$ *by* (1.9) *and the function* $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is defined by*

$$r(x,y) = \left||\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)|\right|. \tag{3.5}$$

*If* $t \in (0,1]$ *and the function* $\Phi : I \to \mathbb{R}$ *is positive and concave, then the order of terms on the left-hand side of* (3.4) *is reversed, that is, the inequality*

$$\begin{aligned}&\int_{\Omega_1} u(x)\Phi^t(A_k f(x))\,d\mu_1(x) - \left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^t\\ &\geq\ t\int_{\Omega_1} \frac{u(x)}{K(x)}\,\Phi^{t-1}(A_k f(x)) \int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{3.6}$$

*holds.*

*Proof.* First, fix an arbitrary $x \in \Omega_1$. It is not hard to see that $A_k f(x) \in I$. Moreover, for the function $h_x : \Omega_2 \to \mathbb{R}$ defined by $h_x(y) = f(y) - A_k f(x)$ we have

$$\int_{\Omega_2} k(x,y)h_x(y)\,d\mu_2(y) = 0,\ x \in \Omega_1. \tag{3.7}$$

Now, suppose that $\Phi$ is a convex function. If $A_k f(x) \in \operatorname{Int} I$, then for all $y \in \Omega_2$ by substituting $r = A_k f(x)$, $s = f(y)$ in (3.1) and multiplying the inequality obtained by $\frac{k(x,y)}{K(x)} \geq 0$, we get

$$\frac{k(x,y)}{K(x)}\,[\Phi(f(y)) - \Phi(A_k f(x)) - \varphi(A_k f(x))h_x(y)] \geq \frac{k(x,y)}{K(x)}\,r(x,y). \tag{3.8}$$

Relation (3.8) holds even if $A_k f(x)$ is an endpoint of $I$. In that case, the function $h_x$ is either non-negative or non-positive on $\Omega_2$, so (3.7) and non-negativity of the kernel $k$ imply that $k(x,y)h_x(y) = 0$ for $\mu_2$–a.e. $y \in \Omega_2$. Therefore, the identity $h_x(y) = 0$, that is, $f(y) = A_k f(x)$ holds whenever $k(x,y) > 0$ and we conclude that the both-hand sides of inequality (3.8) are equal to 0 for $\mu_2$–a.e. $y \in \Omega_2$. Since $K(x) > 0$, notice that the set of all $y \in \Omega_2$ such that $k(x,y) > 0$ is of a positive $\mu_2$ measure.

Integrating (3.8) over $\Omega_2$ we obtain

$$\begin{aligned}&\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y) - \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(A_k f(x))\,d\mu_2(y)\\ &\quad - \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\varphi(A_k f(x))h_x(y)\,d\mu_2(y)\\ &\geq \frac{1}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y).\end{aligned} \tag{3.9}$$

Observe that the second integral on the left-side of (3.9) is equal to

$$\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(A_k f(x))\,d\mu_2(y) = \Phi(A_k f(x)),$$

while applying (3.7) we get

$$\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\varphi(A_k f(x))h_x(y)\,d\mu_2(y) = 0.$$

Hence, (3.9) reduces to

$$\Phi(A_k f(x)) + \frac{1}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y) \le \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y).$$

Let $t \ge 1$. Since the functions $\Phi$, $k$, and $r$ are non-negative and the power functions with positive exponents are strictly increasing on $[0,\infty)$, we further have

$$\begin{aligned}
&\Phi^t(A_k f(x)) + t\,\frac{\Phi^{t-1}(A_k f(x))}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y) \\
&\le \left(\Phi(A_k f(x)) + \frac{1}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\right)^t \\
&\le \left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^t, \qquad (3.10)
\end{aligned}$$

where the first inequality in (3.10) is a consequence of Bernoulli's inequality. Multiplying (3.10) by $u(x)$, integrating the inequalities obtained over $\Omega_1$ and then applying Minkowski's inequality to the right-hand side of the second inequality, we get the following sequence of inequalities:

$$\begin{aligned}
&\int_{\Omega_1} u(x)\Phi^t(A_k f(x))\,d\mu_1(x) \\
&+ t\int_{\Omega_1} \frac{u(x)}{K(x)}\,\Phi^{t-1}(A_k f(x))\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x) \\
&\le \int_{\Omega_1} u(x)\left(\Phi(A_k f(x)) + \frac{1}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\right)^t d\mu_1(x) \\
&\le \int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^t d\mu_1(x) \\
&= \left\{\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^t d\mu_1(x)\right]^{\frac{1}{t}}\right\}^t \\
&\le \left\{\int_{\Omega_2} \Phi(f(y))\left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^t d\mu_1(x)\right]^{\frac{1}{t}} d\mu_2(y)\right\}^t \\
&= \left(\int_{\Omega_2} \Phi(f(y))v(y)\,d\mu_2(y)\right)^t, \qquad (3.11)
\end{aligned}$$

so (3.4) holds. The proof for a concave function $\Phi$ and $t \in (0,1]$ is similar. Namely, by the same arguments as for convex functions, from (3.2) we first obtain

$$\frac{k(x,y)}{K(x)}\left[\Phi(A_kf(x))-\Phi(f(y))+\varphi(A_kf(x))h_x(y)\right]\geq\frac{k(x,y)}{K(x)}\,r(x,y),\ x\in\Omega_1,y\in\Omega_2,$$

then

$$\begin{aligned}
&\Phi^t(A_kf(x))-t\,\frac{\Phi^{t-1}(A_kf(x))}{K(x)}\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\\
&\geq\left(\Phi(A_kf(x))-\frac{1}{K(x)}\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\right)^t\\
&\geq\left(\frac{1}{K(x)}\int_{\Omega_2}k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^t,
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_{\Omega_1}u(x)\Phi^t(A_kf(x))\,d\mu_1(x)\\
&-t\int_{\Omega_1}\frac{u(x)}{K(x)}\,\Phi^{t-1}(A_kf(x))\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)\\
&\geq\int_{\Omega_1}u(x)\left(\Phi(A_kf(x))-\frac{1}{K(x)}\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\right)^t d\mu_1(x)\\
&\geq\left(\int_{\Omega_2}\Phi(f(y))v(y)\,d\mu_2(y)\right)^t
\end{aligned}$$

that is, we get (3.6). ■

**Remark 3.1.** The discrete version of Theorem 3.1 for sequences of real numbers is given in [4, Theorem 2.1]. ■

**Remark 3.2.** In particular, for $t=1$ inequality (3.4) reduces to

$$\begin{aligned}
&\int_{\Omega_2}v(y)\Phi(f(y))\,d\mu_2(y)-\int_{\Omega_1}u(x)\Phi(A_kf(x))\,d\mu_1(x)\\
&\geq\int_{\Omega_1}\frac{u(x)}{K(x)}\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)
\end{aligned}\tag{3.12}$$

where in this setting $v$ is defined as in (1.12). Moreover, by analyzing the proof of Theorem 3.1, we see that (3.12) holds for all convex functions $\Phi: I\to\mathbb{R}$, that is, $\Phi$ does not need to be non-negative. Similarly, if $\Phi$ is any real concave function on $I$ (not necessarily positive), then (3.12) holds with the reversed order of terms on its left-hand side. This result was already proved in [5, Theorem 2.1]. ■

**Remark 3.3.** Rewriting (3.4) with $t=\frac{q}{p}\geq 1$, that is, with $0<p\leq q<\infty$ or $-\infty<q\leq p<0$, and with an arbitrary non-negative convex function $\Phi$, we obtain

$$\begin{aligned}
&\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_kf(x))\,d\mu_1(x)\\
&\geq \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}\,\Phi^{\frac{q}{p}-1}(A_kf(x))\int_{\Omega_2}k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)\geq 0, \qquad (3.13)
\end{aligned}$$

where $v$ is defined by (1.13). Therefore, we get (1.14) as an immediate consequence of Theorem 3.1 and our inequality (3.4) is a refinement of (1.14). Especially, if $p \geq 1$ or $p < 0$ (in that case, $\Phi$ should be positive), then the function $\Phi^p$ is convex as well, so by replacing $\Phi$ with $\Phi^p$ relation (3.13) becomes

$$\begin{aligned}
&\|\Phi f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|\Phi(A_kf)\|^q_{L^q_u(\Omega_1,\mu_1)}\\
&\geq \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}\,\Phi^{q-p}(A_kf(x))\int_{\Omega_2}k(x,y)r_p(x,y)\,d\mu_2(y)\,d\mu_1(x), \qquad (3.14)
\end{aligned}$$

where for $x \in \Omega_1$, $y \in \Omega_2$ we set

$$\begin{aligned}
r_p(x,y) = \big|\,&|\Phi^p(f(y)) - \Phi^p(A_kf(x))|\\
&- |p|\,\Phi^{p-1}(A_kf(x))\,|\varphi(A_kf(x))|\cdot|f(y)-A_kf(x)|\,\big|.
\end{aligned}$$

On the other hand, if $\Phi$ is a positive concave function and $t = \frac{q}{p} \in (0,1]$, that is, $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, then (3.13) holds with the reversed order of terms on its left-hand side. Moreover, if $p \in (0,1]$, then the function $\Phi^p$ is concave, so the order of terms on the left-hand side of (3.14) is reversed. ■

Now, we consider some particularly interesting convex (or concave) functions in (3.4), namely, power and exponential functions. We start with the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. For $p \geq 1$ and $p < 0$, this function is convex, while it is concave for $p \in (0,1]$. In both cases we have $\varphi(x) = px^{p-1}$, $x \in \mathbb{R}_+$. In this setting, we obtain the following direct consequence of Theorem 3.1 and Remark 3.3.

**Corollary 3.1.** *Suppose that $p, q \in \mathbb{R}$, $\frac{q}{p} > 0$, that $\Omega_1$, $\Omega_2$, $\mu_1$, $\mu_2$, $u$, $k$, and $K$ are as in Theorem 3.1, that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on $\Omega_1$ for each fixed $y \in \Omega_2$, and that the function $v$ is defined on $\Omega_1$ by (1.13). Further, suppose that $f : \Omega_2 \to \mathbb{R}$ is a non-negative measurable function (positive in the case when $p < 0$), that $A_kf$ is defined on $\Omega_1$ by (1.9), and*

$$r_{p,k}(x,y) = \big|\,|f^p(y) - (A_kf(x))^p| - |p|\cdot(A_kf(x))^{p-1}\,|f(y)-A_kf(x)|\,\big|,$$

*for* $x \in \Omega_1$, $y \in \Omega_2$. *If* $1 \leq p \leq q < \infty$ *or* $-\infty < q \leq p < 0$, *then the inequality*

$$\begin{aligned} &\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|A_k f\|^q_{L^q_u(\Omega_1,\mu_1)} \\ (3.15) \qquad &\geq \frac{q}{p}\int_{\Omega_1} \frac{u(x)}{K(x)}\,(A_k f(x))^{q-p} \int_{\Omega_2} k(x,y) r_{p,k}(x,y)\, d\mu_2(y)\, d\mu_1(x) \end{aligned}$$

*holds, while for* $0 < q \leq p < 1$ *relation* (3.15) *holds with the reversed order of terms on its left-hand side.*

**Remark 3.4.** For $p = q$ in Corollary 3.1, we obtain Corollary 2.1 in [5]. Moreover, for $p = q = 1$, relation (3.15) is trivial since its both-hand sides are equal to 0. ■

Our analysis continues by considering the convex function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$. Then $\varphi = \Phi' = \Phi$ and we obtain the following new general refined weighted Pólya–Knopp–type inequality with a kernel, which is a generalization of a result from [5, Corollary 2.2].

**Corollary 3.2.** *Let* $p, q \in \mathbb{R}$ *be such that* $0 < p \leq q < \infty$ *or* $-\infty < q \leq p < 0$. *Let* $\Omega_1$, $\Omega_2$, $\mu_1$, $\mu_2$, $u$, $k$, *and* $K$ *be as in Theorem* 3.1, *the function* $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ *be integrable on* $\Omega_1$ *for each fixed* $y \in \Omega_2$, *and the function* $v$ *be defined on* $\Omega_1$ *by* (1.13). *Then the inequality*

$$\begin{aligned} &\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|G_k f\|^q_{L^q_u(\Omega_1,\mu_1)} \\ (3.16) \qquad &\geq \frac{q}{p}\int_{\Omega_1} \frac{u(x)}{K(x)}\,(G_k f(x))^{q-p} \int_{\Omega_2} k(x,y) s_{p,k}(x,y)\, d\mu_2(y)\, d\mu_1(x) \end{aligned}$$

*holds for all positive measurable functions* $f$ *on* $\Omega_2$, *where* $G_k f(x)$ *and* $s_{p,k}(x,y)$ *are for* $x \in \Omega_1$ *and* $y \in \Omega_2$ *respectively defined by*

$$G_k f(x) = \exp\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y) \ln f(y)\, d\mu_2(y)\right)$$

*and*

$$s_{p,k}(x,y) = \left| \left|f^p(y) - (G_k f(x))^p\right| - |p|\,(G_k f(x))^p \left|\ln \frac{f(y)}{G_k f(x)}\right| \right|.$$

*Proof.* Follows by applying (3.13) with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, and replacing the function $f$ with $p \ln f$. ■

**Remark 3.5.** In particular, for $p = q$ our Corollary 3.2 reduces to Corollary 2.2 from [5]. ■

We conclude this section by considering the simplest kernels $k$, that is, those with separate variables.

**Corollary 3.3.** *Let* $p, q \in \mathbb{R}$, $\frac{q}{p} > 0$. *Let* $(\Omega, \Sigma, \mu)$ *be a measure space with a positive* $\sigma$*-finite measure* $\mu$, *let* $m \in L^1(\Omega, \mu)$ *be a non-negative function such that* $|m|_1 > 0$, $\Phi$ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$, *and $\varphi : I \to \mathbb{R}$ be any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Let $f : \Omega \to \mathbb{R}$ be a measurable function with values in $I$ and*

$$A_m f = \frac{1}{|m|_1} \int_\Omega m(y) f(y)\, d\mu(y).$$

*If $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, then the inequality*

$$[A_m(\Phi \circ f)]^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(A_m f) \ge \frac{q}{p}\, \Phi^{\frac{q}{p}-1}(A_m f) \cdot A_m r \tag{3.17}$$

*holds, where $r(y) = \big|\, |\Phi(f(y)) - \Phi(A_m f)| - |\varphi(A_m f)| \cdot |f(y) - A_m f| \,\big|$, $y \in \Omega$. If $\Phi$ is a positive concave function and $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, then* (3.17) *holds with the reversed order of terms on its left-hand side.*

*Proof.* Suppose that in Theorem 3.1 and in relation (3.13) we have $\Omega_2 = \Omega$, $\mu_2 = \mu$, $u \in L^1(\Omega_1, \mu_1)$ such that $|u|_1 > 0$, and $k$ of the form $k(x,y) = l(x)m(y)$, for some positive measurable function $l : \Omega_1 \to \mathbb{R}$. Then $K(x) = |m|_1 l(x)$ and $A_k f(x) = A_m f \in I$, $x \in \Omega_1$, while $v(y) = \frac{|u|_1^{\frac{p}{q}}}{|m|_1} m(y)$, $y \in \Omega$. Thus, (3.13) reduces to (3.17) and it does not depend on $\Omega_1$, $l$, and $u$. ■

**Remark 3.6.** Observe that for $0 < |\Omega|_\mu < \infty$ and $m(y) \equiv 1$ on $\Omega$ we have $|m|_1 = |\Omega|_\mu$, so (3.17) becomes the generalized refined Jensen's inequality

$$[A(\Phi \circ f)]^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(Af) \ge \frac{q}{p}\, \Phi^{\frac{q}{p}-1}(Af) \cdot Ar$$

where

$$Af = \frac{1}{|\Omega|_\mu} \int_\Omega f(y)\, d\mu(y)$$

and $r(y) = \big|\, |\Phi(f(y)) - \Phi(Af)| - |\varphi(Af)| \cdot |f(y) - Af| \,\big|$, $y \in \Omega$. Notice that, for $p = q$ we obtain the classical refined Jensen's inequality that was recently obtained in [5, Remark 2.4]. ■

## 4. Generalized one-dimensional Hardy's and Pólya–Knopp's inequality

In the following three sections, general results from Section 3 are applied to some usual measure spaces, convex functions, weights and kernels and new refinements and generalizations of the inequalities mentioned in the Introduction are derived. We start with the standard one-dimensional setting, that is, by considering intervals in $\mathbb{R}$ and the Lebesgue measure, and obtain generalized refined Hardy and Pólya–Knopp–type inequalities, as well as related dual inequalities. In the following theorem we generalize and refine inequality (1.5).

**Theorem 4.1.** *Let $0 < b \le \infty$ and $k : (0,b) \times (0,b) \to \mathbb{R}$, $u : (0,b) \to \mathbb{R}$ be non-negative measurable functions satisfying* (1.6) *and*

$$w(y) = y \left( \int_y^b u(x) \left( \frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} \frac{dx}{x} \right)^{\frac{p}{q}} < \infty,\ y \in (0,b).$$

*If $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, $\Phi$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\left(\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}(A_k f(x))\,\frac{dx}{x}$$

$$\ge \frac{q}{p}\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_k f(x))\int_0^x k(x,y)r(x,y)\,dy\,\frac{dx}{x} \tag{4.1}$$

*holds for all measurable functions $f : (0,b) \to \mathbb{R}$ with values in $I$, where $A_k f$ and $r$ are respectively defined by (1.7) and (3.5). If $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, and $\Phi$ is a non-negative concave function, then (4.1) holds with the reversed order of integrals on its left-hand side.*

*Proof.* Let $S_1$, $\Omega_1$, and $\Omega_2$ be as in the proof of Theorem 2.3. Relation (4.1) follows from (3.13) by replacing $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, $v(y)$, and $k$ respectively with $dx$, $dy$, $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_1}$. ■

In the following theorem we formulate a result dual to Theorem 4.1.

**Theorem 4.2.** *For $0 \le b < \infty$, let $k : (b,\infty)\times(b,\infty) \to \mathbb{R}$ and $u : (b,\infty) \to \mathbb{R}$ be non-negative measurable functions satisfying*

$$\tilde{K}(x) = \int_x^\infty k(x,y)\;dy > 0,\;\; x \in (b,\infty), \tag{4.2}$$

*and*

$$\tilde{w}(y) = y\left(\int_b^y u(x)\left(\frac{k(x,y)}{\tilde{K}(x)}\right)^{\frac{q}{p}}\frac{dx}{x}\right)^{\frac{p}{q}} < \infty,\;\; y \in (b,\infty).$$

*If $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, $\Phi$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\left(\int_b^\infty \tilde{w}(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x)\Phi^{\frac{q}{p}}(\tilde{A}_k f(x))\,\frac{dx}{x}$$

$$\ge \frac{q}{p}\int_b^\infty \frac{u(x)}{\tilde{K}(x)}\Phi^{\frac{q}{p}-1}(\tilde{A}_k f(x))\int_x^\infty k(x,y)\tilde{r}(x,y)\,dy\,\frac{dx}{x} \tag{4.3}$$

*holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$ with values in $I$ and for $\tilde{A}_k f(x)$ and $\tilde{r}(x,y)$ respectively defined by*

$$\tilde{A}_k f(x) = \frac{1}{\tilde{K}(x)}\int_x^\infty k(x,y)f(y)\;dy$$

*and*

$$\tilde{r}(x,y) = \left|\,|\Phi(f(y)) - \Phi(\tilde{A}_k f(x))| - |\varphi(\tilde{A}_k f(x))|\cdot|f(y) - \tilde{A}_k f(x)|\,\right|,$$

*where $x, y \in (b,\infty)$. If $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, and $\Phi$ is a non-negative concave function, the order of integrals on the left-hand side of (4.3) is reversed.*

*Proof.* Let $S_2 = \{(x, y) \in \mathbb{R}^2 : b < x \leq y < \infty\}$. Inequality (4.3) follows directly from (3.13), rewritten with $\Omega_1 = \Omega_2 = (b, \infty)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, and with $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_2}$ instead of $u(x)$, $v(y)$, and $k$. ■

**Remark 4.1.** For $p = q$ Theorem 4.1 and Theorem 4.2 respectively reduce to [17, Theorem 3.1] and [17, Theorem 4.3]. In particular, (4.1) refines (1.5). Of course, in that case, the function $\Phi$ does not need to be non-negative. ■

The rest of this section is dedicated to generalizations and refinements of the well-known Hardy's and Pólya–Knopp's inequality (1.1) and (1.2) and of their dual inequalities. Since being direct consequences of the above results, we state them as examples.

**Example 4.1.** Let $0 < b \leq \infty$, $\gamma \in \mathbb{R}_+$, $p, q \in \mathbb{R}$ be such that $\frac{q}{p} > 0$, and let $S_1$ be as in the proofs of Theorem 2.3 and Theorem 4.1. Let the kernel $k : (0, b) \times (0, b) \to \mathbb{R}$ and the weight function $u : (0, b) \to \mathbb{R}$ be defined by $k(x, y) = \frac{\gamma}{x^\gamma}(x - y)^{\gamma-1}\chi_{S_1}$ and $u(x) \equiv 1$. If $\frac{q}{p} \geq 1$, $\gamma > 1 - \frac{p}{q}$, $\Phi$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $f : (0, b) \to \mathbb{R}$ is a function with values in $I$, then (4.1) reads

$$\begin{aligned}
&\left(\int_0^b w_\gamma(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b \Phi^{\frac{q}{p}}(R_\gamma f(x))\,\frac{dx}{x} \\
&\geq \gamma\,\frac{q}{p}\int_0^b \Phi^{\frac{q}{p}-1}(R_\gamma f(x))\int_0^x (x - y)^{\gamma-1} r_\gamma(x, y)\,dy\,\frac{dx}{x^{\gamma+1}},
\end{aligned} \tag{4.4}$$

where $R_\gamma$ is the Riemann-Liouville operator given by

$$R_\gamma f(x) = \frac{\gamma}{x^\gamma}\int_0^x (x - y)^{\gamma-1} f(y)\,dy,\ \ x \in (0, b),$$

while for $x, y \in (0, b)$ we set

$$w_\gamma(y) = \gamma\left(\int_0^{1-\frac{y}{b}} t^{(\gamma-1)\frac{q}{p}}(1 - t)^{\frac{q}{p}-1}dt\right)^{\frac{p}{q}} = \gamma B^{\frac{p}{q}}\left(1 - \frac{y}{b}; (\gamma - 1)\frac{q}{p} + 1, \frac{q}{p}\right)$$

and

$$r_\gamma(x, y) = \left|\left|\Phi(f(y)) - \Phi(R_\gamma f(x))\right| - \left|\varphi(R_\gamma f(x))\right| \cdot \left|f(y) - R_\gamma f(x)\right|\right|.$$

Observe that $B(\,\cdot\,; \cdot\,, \cdot\,)$ denotes the incomplete Beta function defined in the Introduction. In the case when $\frac{q}{p} \in (0, 1]$ and $\Phi$ is non-negative and concave, the order of terms on the left-hand side of (4.4) is reversed and the inequality obtained holds for any $\gamma > 0$.

Rewriting (4.4) with some suitable parameters and with $\Phi$ being a power function, we get a new refined Hardy's inequality. Namely, let $\Phi(x) = x^p$, $k \in \mathbb{R}$ be such that $\frac{k-1}{p} > 0$,

$$w_{\gamma,k}(y) = B^{\frac{p}{q}}\left(1 - \left(\frac{y}{b}\right)^{\frac{k-1}{p}}; (\gamma - 1)\frac{q}{p} + 1, \frac{q}{p}\right) y^{p-k},\ \ y \in (0, b),$$

$f$ be a non-negative function on $(0,b)$ (positive, if $p<0$) and

$$Rf(x)=\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}f(y)\,dy,\ \ x\in(0,b).$$

For $1\leq p\leq q<\infty$ or $-\infty<q\leq p<0$, replace $b$ and $f(y)$ in (4.4) respectively with $b^{\frac{k-1}{p}}$ and $f\left(y^{\frac{p}{k-1}}\right)y^{\frac{p}{k-1}-1}$. After a sequence of suitable variable changes, we get the inequality

$$\begin{aligned}
&\gamma\left(\frac{p}{\gamma(k-1)}\right)^{q+1-\frac{q}{p}}\left(\int_0^b w_{\gamma,k}(y)f^p(y)\,dy\right)^{\frac{q}{p}}-\int_0^b x^{\frac{q}{p}(1-k)-1}\left(Rf(x)\right)^q dx\\
&\geq\frac{q}{p}\left|\left(\frac{p}{\gamma(k-1)}\right)^{p-1}\int_0^b x^{\frac{k-1}{p}(p-q-1)-1}(Rf(x))^{q-p}\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}\times\right.\\
&\qquad\times y^{\frac{k-1}{p}-1}\left|y^{p-k+1}f^p(y)-\left(\frac{\gamma(k-1)}{p}\right)^p x^{1-k}(Rf(x))^p\right|dy\,dx\\
&\quad-|p|\int_0^b x^{\frac{1-k}{p}q-1}(Rf(x))^{q-1}\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}\times\\
&\qquad\left.\times\left|f(y)-\frac{\gamma(k-1)}{py}\left(\frac{y}{x}\right)^{\frac{k-1}{p}}Rf(x)\right|dy\,dx\right|.
\end{aligned}\tag{4.5}$$

For $0<q\leq p<1$, the order of terms on the left-hand side of relation (4.5) is reversed. Notice that for $b=\infty$, $p=q=k>1$ and $\gamma=1$ inequality (4.5) reduces to a refinement of the classical Hardy's inequality (1.1). It can be seen that our result generalizes refined and strengthened Hardy–type inequalities from [3] and [5].

On the other hand, rewriting (4.4) with $\Phi(x)=e^x$ and $\gamma=1$, as well as with the function $y\mapsto\ln(yf(y))$ instead of a positive function $f:(0,b)\to\mathbb{R}$, we derive the following new refined strengthened Pólya–Knopp–type inequality:

$$\begin{aligned}
&\frac{p}{q}e^{\frac{q}{p}}\left(\int_0^b\left[1-\left(\frac{y}{b}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}f(y)\,dy\right)^{\frac{q}{p}}-\int_0^b x^{\frac{q}{p}-1}(Gf(x))^{\frac{q}{p}}dx\\
&\geq\frac{q}{p}\left|\int_0^b x^{\frac{q}{p}-3}(Gf(x))^{\frac{q}{p}-1}\int_0^x\left|eyf(y)-xGf(x)\right|dy\,dx\right.\\
&\quad\left.-\int_0^b x^{\frac{q}{p}-2}(Gf(x))^{\frac{q}{p}}\int_0^x\left|\ln\left(\frac{eyf(y)}{xGf(x)}\right)\right|dy\,dx\right|,
\end{aligned}\tag{4.6}$$

where $\frac{q}{p}\geq 1$ and

$$Gf(x)=\exp\left(\frac{1}{x}\int_0^x\ln f(y)\,dy\right),\ \ x\in(0,b).$$

For $p = q$ relation (4.6) reduces to a refined strengthened Pólya–Knopp's inequality from [3] and [5]. Moreover, for $b = \infty$ we obtained a refinement of the classical Pólya–Knopp's inequality (1.2). ∎

The following example provides results dual to those from Example 4.1.

**Example 4.2.** Suppose $0 \le b < \infty$, $\gamma \in \mathbb{R}_+$, $p, q \in \mathbb{R}$ are such that $\frac{q}{p} > 0$, and $S_2$ is as in the proof of Theorem 4.2. Define the kernel $k : (b,\infty) \times (b,\infty) \to \mathbb{R}$ and the weight function $u : (b,\infty) \to \mathbb{R}$ as $k(x,y) = \gamma \frac{x}{y^{\gamma+1}}(y-x)^{\gamma-1}\chi_{S_2}(x,y)$ and $u(x) \equiv 1$. For $\frac{q}{p} \ge 1$, $\gamma > 1 - \frac{p}{q}$, a non-negative convex function $\Phi$ on an interval $I \subseteq \mathbb{R}$ and a function $f : (b,\infty) \to \mathbb{R}$ with values in $I$, inequality (4.3) becomes

$$\begin{aligned}&\left(\int_b^\infty \tilde{w}_\gamma(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty \Phi^{\frac{q}{p}}(W_\gamma f(x))\,\frac{dx}{x}\\ &\ge \gamma\,\frac{q}{p}\int_b^\infty \Phi^{\frac{q}{p}-1}(W_\gamma f(x))\int_x^\infty (y-x)^{\gamma-1}\tilde{r}_\gamma(x,y)\,\frac{dy}{y^{\gamma+1}}\,dx,\end{aligned} \tag{4.7}$$

where $W_\gamma$ denotes the Weyl's operator $W_\gamma$,

$$W_\gamma f(x) = \gamma x\int_x^\infty (y-x)^{\gamma-1} f(y)\,\frac{dy}{y^{\gamma+1}},\ x \in (0,b),$$

and for $x, y \in (b,\infty)$ we define $\tilde{w}_\gamma(y) = \gamma B^{\frac{p}{q}}\left(1 - \frac{b}{y}; (\gamma-1)\frac{q}{p} + 1, \frac{q}{p}\right)$ and $\tilde{r}_\gamma(x,y) = ||\Phi(f(y)) - \Phi(W_\gamma f(x))| - |\varphi(W_\gamma f(x))| \cdot |f(y) - W_\gamma f(x)||$. If $\frac{q}{p} \in (0,1]$ and $\Phi$ is non-negative and concave, (4.7) holds for all $\gamma > 0$ and with the reversed order of terms on its left-hand side.

As in Example 4.1, to get a new refined dual Hardy's inequality, we rewrite (4.7) with $\Phi(x) = x^p$. More precisely, let $k \in \mathbb{R}$ be such that $\frac{p}{1-k} > 0$,

$$\tilde{w}_{\gamma,k}(y) = B^{\frac{p}{q}}\left(1 - \left(\frac{b}{y}\right)^{\frac{1-k}{p}}; (\gamma-1)\frac{q}{p} + 1, \frac{q}{p}\right) y^{p-k},\ y \in (b,\infty),$$

$f$ be a non-negative function on $(b,\infty)$ (positive, if $p < 0$) and

$$Wf(x) = \int_x^\infty \left[1 - \left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)\,dy,\ x \in (b,\infty).$$

For $1 \le p \le q < \infty$ or $-\infty < q \le p < 0$, substitute $b^{\frac{1-k}{p}}$ and $f\left(y^{\frac{p}{1-k}}\right) y^{\frac{p}{1-k}+1}$ in (4.7) respectively for $b$ and $f(y)$. After some computations, we obtain the inequality

$$
\begin{aligned}
&\gamma\left(\frac{p}{\gamma(1-k)}\right)^{q+1-\frac{q}{p}}\left(\int_b^\infty \tilde{w}_{\gamma,k}(y)f^p(y)\,dy\right)^{\frac{q}{p}} - \int_b^\infty x^{\frac{q}{p}(1-k)-1}\,(Wf(x))^q\,dx \\
&\geq \frac{q}{p}\left|\left(\frac{p}{\gamma(1-k)}\right)^{p-1}\int_b^\infty x^{\frac{1-k}{p}(q-p+1)-1}(Wf(x))^{q-p}\int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1}\times\right. \\
&\qquad \times y^{\frac{k-1}{p}-1}\left|y^{p-k+1}f^p(y)-\left(\frac{\gamma(1-k)}{p}\right)^p x^{1-k}(Wf(x))^p\right|dy\,dx \\
&\quad -|p|\int_b^\infty x^{\frac{1-k}{p}q-1}(Wf(x))^{q-1}\int_x^\infty\left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1}\times \\
&\qquad \left.\times\left|f(y)-\frac{\gamma(1-k)}{py}\left(\frac{x}{y}\right)^{\frac{1-k}{p}}Wf(x)\right|dy\,dx\right|. \qquad (4.8)
\end{aligned}
$$

For $0 < q \leq p < 1$, relation (4.8) holds with the reversed order of terms on its left-hand side. In the case with $p = q$, (4.8) becomes a refined and strengthened dual Hardy's inequality from Example 3.2 in [5].

Finally, for $\frac{q}{p} \geq 1$, $\gamma = 1$, $\Phi(x) = e^x$ and $y \mapsto \ln(yf(y))$ instead of a positive function $f : (b,\infty) \to \mathbb{R}$, inequality (4.7) becomes

$$
\begin{aligned}
&\frac{p}{q}e^{-\frac{q}{p}}\left(\int_b^\infty\left[1-\left(\frac{b}{y}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}f(y)\,dy\right)^{\frac{q}{p}} - \int_b^\infty x^{\frac{q}{p}-1}(\tilde{G}f(x))^{\frac{q}{p}}\,dx \\
&\geq \frac{q}{p}\left|\int_b^\infty x^{\frac{q}{p}-1}(\tilde{G}f(x))^{\frac{q}{p}-1}\int_x^\infty\left|e^{-1}yf(y)-x\tilde{G}f(x)\right|\frac{dy}{y^2}\,dx\right. \\
&\qquad \left. -\int_b^\infty x^{\frac{q}{p}}(\tilde{G}f(x))^{\frac{q}{p}}\int_x^\infty\left|\ln\frac{yf(y)}{ex\tilde{G}f(x)}\right|\frac{dy}{y^2}\,dx\right|,
\end{aligned}
$$

where

$$
\tilde{G}f(x) = \exp\left(x\int_x^\infty \ln f(y)\,\frac{dy}{y^2}\right),\ y \in (b,\infty).
$$

Thus, we proved a new refined strengthened dual Pólya–Knopp's inequality. Its special case $p = q$ was already considered in [3] and [5]. ■

## 5. Generalized one-dimensional Hardy–Hilbert's inequality

In this section, we consider Theorem 3.1, that is, inequality (3.13), with some important kernels related to $\Omega_1 = \Omega_2 = \mathbb{R}_+$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. We also assume that $d\mu_1(x) = dx$ and $d\mu_2(y) = dy$.

In the first example, we generalize and refine the classical Hardy–Hilbert's inequality (1.3).

**Example 5.1.** Let $p, q, s \in \mathbb{R}$ be such that $\frac{q}{p} > 0$ and $\frac{s-2}{p}, \frac{s-2}{p'} > -1$, and let $\alpha \in \left(-\frac{q}{p}\left(\frac{s-2}{p'}+1\right), \frac{q}{p}\left(\frac{s-2}{p}+1\right)\right)$. Denote

$$C_1 = B\left(\frac{q}{p}\left(\frac{s-2}{p}+1\right)-\alpha, \frac{q}{p}\left(\frac{s-2}{p'}+1\right)+\alpha\right)$$

and

$$C_2 = B\left(\frac{s-2}{p}+1, \frac{s-2}{p'}+1\right),$$

where $B(\,\cdot\,,\,\cdot\,)$ is the usual Beta function, and define $k : \mathbb{R}_+^2 \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$ respectively by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}}(x+y)^{-s}$ and $u(x) = x^{\alpha-1}$. Finally, let $f$ be a non-negative function on $\mathbb{R}_+$ (positive, if $p < 0$) and $Sf$ its generalized Stieltjes transform,

$$Sf(x) = \int_0^\infty \frac{f(y)}{(x+y)^s}\,dy,\ x \in \mathbb{R}_+$$

(see [1] and [27] for further information). Rewriting (3.15) with the above parameters and with $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, for $1 \le p \le q < \infty$ or $-\infty < q \le p < 0$ we obtain the inequality

$$\begin{aligned}
&C_1 C_2^{\frac{q}{p'}}\left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1} f^p(y)\,dy\right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}}(Sf(x))^q\,dx \\
&\ge \frac{q}{p}\Bigg| C_2^{p-1}\int_0^\infty x^{\alpha+q-p+\frac{s-2}{p'}(q-p+1)}(Sf(x))^{q-p}\times \\
&\qquad\times\int_0^\infty \frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left|f^p(y)y^{2-s} - C_2^{-p}x^{(p-1)(s-1)+1}(Sf(x))^p\right|\,dy\,dx \\
&\qquad -|p|\int_0^\infty x^{\alpha+q+\frac{s-2}{p'}q-1}(Sf(x))^{q-1}\times \\
&\qquad\times\int_0^\infty (x+y)^{-s}\left|f(y) - C_2^{-1}x^{\frac{s-2}{p'}+1}y^{\frac{s-2}{p}}Sf(x)\right|\,dy\,dx\Bigg|,
\end{aligned}\tag{5.1}$$

while for $0 < q \le p < 1$ the order of terms on the left-hand side of (5.1) is reversed. The case $p = q$ was already studied in [5, Example 4.1]. In particular, for $p = q > 1$, $\alpha = 0$ and $s = 1$ we have $C_1 = C_2 = B\left(\frac{1}{p}, \frac{1}{p'}\right) = \frac{\pi}{\sin\frac{\pi}{p}}$, so (5.1) provides a new generalization and refinement of the classical Hardy–Hilbert's inequality (1.3). ■

Similarly, in the next example we generalize and refine the classical Hardy–Littlewood–Pólya's inequality (1.4).

**Example 5.2.** Let the parameters $p$, $q$, $s$, $\alpha$ and the functions $u$ and $f$ be as in Example 5.1. Define $k : \mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}}\max\{x,y\}^{-s}$ and the transform $Lf$ as

$$Lf(x) = \int_0^\infty \frac{f(y)}{\max\{x,y\}^s}\,dy,\ x \in \mathbb{R}_+.$$

Finally, set

$$D_1 = \frac{p^2 p' q s}{(\alpha p p' + p'q + qs - 2q)(pq + qs - \alpha p^2 - 2q)}$$

and

$$D_2 = \frac{pp's}{(p+s-2)(p'+s-2)}.$$

Considering $1 \le p \le q < \infty$, or $-\infty < q \le p < 0$, and $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, relation (3.15) becomes

$$\begin{aligned} & D_1 D_2^{\frac{q}{p'}} \left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1} f^p(y)\,dy\right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}} (Lf(x))^q\,dx \\ & \geq \frac{q}{p}\Bigg| D_2^{p-1} \int_0^\infty x^{\alpha+q-p+\frac{s-2}{p'}(q-p+1)} (Lf(x))^{q-p} \times \\ & \qquad \times \int_0^\infty \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \left| f^p(y) y^{2-s} - D_2^{-p} x^{(p-1)(s-1)+1} (Lf(x))^p \right| dy\,dx \\ & \quad -|p| \int_0^\infty x^{\alpha+q+\frac{s-2}{p'}q-1} (Lf(x))^{q-1} \times \\ & \qquad \times \int_0^\infty \max\{x,y\}^{-s} \left| f(y) - D_2^{-1} x^{\frac{s-2}{p'}+1} y^{\frac{s-2}{p}} Lf(x) \right| dy\,dx \Bigg|. \end{aligned} \tag{5.2}$$

If $0 < q \le p < 1$, the order of terms on the left-hand side of (5.2) is reversed. For $p = q$, (5.2) reduces to [5, Example 4.2]. Moreover, since for $p = q > 1$, $\alpha = 0$ and $s = 1$ we have $D_1 = D_2 = pp'$, our result generalizes and refines (1.4). ■

We complete this section with another refined Hardy–Hilbert–type inequality, making use of the well-known reflection formula for the Digamma function $\psi$,

$$\int_0^\infty \frac{\ln t}{t-1} t^{-\alpha}\,dt = \psi'(1-\alpha) + \psi'(\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha},\ \alpha \in (0,1),$$

and of the fact that

$$Z(a,b) = \int_0^\infty t^b e^{-at} \left(1 - e^{-t}\right)^b\,dt < \infty,\ a \in \mathbb{R}_+,\ b \ge 1.$$

More precisely, $Z(a,b) = \Gamma(b+1)\phi_b^*(1, b+1, a)$, where $\phi_\mu^*$ is the so-called unified Riemann-Zeta function,

$$\phi_\mu^*(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \left(1 - ze^{-t}\right)^{-\mu}\,dt,$$

where $\mu \geq 1$, $\operatorname{Re} a > 0$ and either $|z| \leq 1$, $z \neq 1$ and $\operatorname{Re} s > 0$, or $z = 1$ and $\operatorname{Re} s > \mu$ (for more information regarding the unified Riemann-Zeta function, see e.g. [12]).

**Example 5.3.** Suppose that $\alpha \in (0,1)$ and $p, q, \beta \in \mathbb{R}$ are such that $\frac{q}{p} \geq 1$ and $\alpha\frac{q}{p} + \beta \in \left(-1, \frac{q}{p} - 1\right)$. Define the kernel $k : \mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \frac{\ln y - \ln x}{y - x}\left(\frac{x}{y}\right)^\alpha$ and the weight function $u : \mathbb{R}_+ \to \mathbb{R}$ by $u(x) = x^\beta$. Finally, denote

$$Mf(x) = \int_0^\infty \frac{\ln y - \ln x}{y - x} f(y)\, dy,\ x \in \mathbb{R}_+,$$

where $f$ is a non-negative function on $\mathbb{R}_+$ (positive, if $p < 0$),

$$E_1 = \int_0^\infty \left(\frac{\ln t}{t-1}\right)^{\frac{q}{p}} t^{\alpha\frac{q}{p}+\beta}\, dt = Z\left(\alpha\frac{q}{p} + \beta + 1, \frac{q}{p}\right) + Z\left(\frac{q}{p} - \alpha\frac{q}{p} - \beta - 1, \frac{q}{p}\right)$$

and

$$E_2 = \int_0^\infty \frac{\ln t}{t-1}\, t^{-\alpha}\, dt = \frac{\pi^2}{\sin^2 \pi\alpha}.$$

Applying (3.15) to the above parameters and to $f(y)$ replaced with $f(y)y^\alpha$, we get the inequality

$$\begin{aligned}
&E_1 E_2^{\frac{q}{p'}} \left(\int_0^\infty y^{\alpha p + (\beta+1)\frac{p}{q} - 1} f^p(y)\, dy\right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha q + \beta} (Mf(x))^q\, dx \\
&\geq \frac{q}{p} \left| E_2^{p-1} \int_0^\infty x^{\alpha(q-p+1)+\beta} (Mf(x))^{q-p} \times \right. \\
&\qquad \times \int_0^\infty y^{-\alpha} \frac{\ln y - \ln x}{y - x} \left| f^p(y) y^{\alpha p} - E_2^{-p} x^{\alpha p} (Mf(x))^p \right| dy\, dx \\
&\qquad - |p| \int_0^\infty x^{\alpha q + \beta} (Mf(x))^{q-1} \times \\
&\qquad \left. \times \int_0^\infty \frac{\ln y - \ln x}{y - x} \left| f(y) - E_2^{-1} \left(\frac{x}{y}\right)^\alpha Mf(x) \right| dy\, dx \right|.
\end{aligned} \tag{5.3}$$

Notice that for $p = q$ we have

$$E_1 = \int_0^\infty \frac{\ln t}{t-1} t^{\alpha+\beta}\, dt = \frac{\pi^2}{\sin^2 \pi(\alpha+\beta)}$$

and (5.3) reduces to the Hardy–Hilbert–type inequality obtained in [5, Example 4.3]. Therefore our result can be seen as its generalization. ■

## 6. General Godunova–type inequalities

We conclude the paper with a multidimensional result related to Godunova's inequality (1.8). Namely, let $\Omega_1 = \Omega_2 = \mathbb{R}_+^n$, $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, let $\frac{\mathbf{y}}{\mathbf{x}}$ and $\mathbf{x}^{\mathbf{y}}$ be as in Section 2, and let the kernel $k$ :

$\mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be of the form $k(\mathbf{x},\mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function.

Applying Theorem 3.1 to this setting, we get the following generalization and refinement of Godunova's inequality (1.8) and a generalization of [5, Theorem 5.1].

**Theorem 6.1.** *Let $0 < p \le q < \infty$ or $-\infty < q \le p < 0$. Let $l$ and $u$ be non-negative measurable functions on $\mathbb{R}^n_+$, such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}^n_+} l(\mathbf{y})\, d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^n_+$, and that the function $\mathbf{x} \mapsto u(\mathbf{x}) \left(\frac{l(\frac{\mathbf{y}}{\mathbf{x}})}{L(\mathbf{x})}\right)^{\frac{q}{p}}$ is integrable on $\mathbb{R}^n_+$ for each fixed $\mathbf{y} \in \mathbb{R}^n_+$. Let the function $v$ be defined on $\mathbb{R}^n_+$ by*

$$v(\mathbf{y}) = \left(\int_{\mathbb{R}^n_+} u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\right)^{\frac{q}{p}} d\mathbf{x}\right)^{\frac{p}{q}}.$$

*If $\Phi$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned} &\left(\int_{\mathbb{R}^n_+} v(\mathbf{y})\Phi(f(\mathbf{y}))\, d\mathbf{y}\right)^{\frac{q}{p}} - \int_{\mathbb{R}^n_+} u(\mathbf{x})\Phi^{\frac{q}{p}}(A_l f(\mathbf{x}))\, d\mathbf{x} \\ &\ge \frac{q}{p}\int_{\mathbb{R}^n_+} \frac{u(\mathbf{x})}{L(\mathbf{x})}\Phi^{\frac{q}{p}-1}(A_l f(\mathbf{x})) \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) r(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \end{aligned} \tag{6.1}$$

*holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ with values in $I$, where $A_l f(\mathbf{x})$ and $r(\mathbf{x},\mathbf{y})$ are for $\mathbf{x},\mathbf{y} \in \mathbb{R}^n_+$ respectively defined by*

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})}\int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\, d\mathbf{y}$$

*and*

$$r(\mathbf{x},\mathbf{y}) = \left|\,|\Phi(f(\mathbf{y})) - \Phi(A_l f(\mathbf{x}))| - |\varphi(A_l f(\mathbf{x}))| \cdot |f(\mathbf{y}) - A_l f(\mathbf{x})|\,\right|.$$

*If $\Phi$ is a positive concave function and $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, then (6.1) holds with the reversed order of terms on its left-hand side.*

**Remark 6.1.** Observe that for $p = q$ inequality (6.1) reduces to [5, Theorem 5.1]. If, additionally, $\int_{\mathbb{R}^n_+} l(\mathbf{y})\, d\mathbf{y} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}}$, we get a refinement of (1.8). ■

The above results can be rewritten with particular convex (or concave) functions, for example, with power and exponential functions. This leads to multidimensional analogues of corollaries and examples from Sections 4 and 5. Due to the lack of space, we omit them here.

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