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ON GERGONNE POINT OF THE TRIANGLE IN ISOTROPIC PLANE

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Abstract. Using the standard position of the allowable triangle in the isotropic plane relationships between this triangle and its contact and tangential triangle are studied. Thereby different properties of the symmedian center, the Gergonne point, the Lemoine line and the de Longchamps line of these triangles are obtained.

It has been shown in [3] that any allowable triangle ABC in the isotropic plane I_2 can be set in the so called *standard position* by choosing an appropriate affine coordinate system and having the circumcircle equation $y = x^2$, while its vertices are of the form

$$A = (a,\ a^2), \quad B = (b,\ b^2), \quad C = (c,\ c^2),$$

with $a + b + c = 0$. Along with the abbreviations

$$p = abc, \quad q = bc + ca + ab$$

other useful relations hold too, for example: $a^2 = bc - q$ and $a^2 + b^2 + c^2 = -2q$, wherefrom it follows that $q < 0$.

In order to prove any statement on any allowable triangle it is sufficient to prove the considered statement for the triangle in the standard position (the expression *standard triangle* will further on be in use).

Following [1] the inscribed circle of the standard triangle ABC has the equation

$$\mathcal{K}_i \ \ldots \ y = \frac{1}{4}x^2 - q \tag{1}$$

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while the points of contact with the straight lines BC, CA, AB are

$$A_i = (-2a,\, bc - 2q), \quad B_i = (-2b,\, ca - 2q), \quad C_i = (-2c,\, ab - 2q). \quad (2)$$

We call the triangle $A_iB_iC_i$ *the contact triangle* of the reference triangle ABC.

Theorem 1. *Let $A_iB_iC_i$ be the contact triangle of the triangle ABC, and A_h, B_h, C_h the feet of its altitudes. Then, the pairs of line segments BC, A_hA_i; CA, B_hB_i; AB, C_hC_i have the same midpoints, and the points A, B, C lie successively on the bisectors of the line segments B_iC_i, C_iA_i, A_iB_i.*

Proof. Let us first recall from [3] that, for example, $A_h = (a, q-2bc)$, and that the vertices of the contact triangle $A_iB_iC_i$ of the standard triangle ABC are given in (2). The points A_h and A_i have the midpoint with the abscissa $\frac{1}{2}(a-2a) = -\frac{a}{2}$, being in the same time according to [3] the abscissa of the midpoint of the side BC. Since for example $\frac{1}{2}(-2b-2c) = a$, the bisector of the side B_iC_i passes through the point$A$. □

The incircles of the triangles ABA_i and ACA_i have in the Euclidean geometry the same point of contact with the straight line AA_i. In the isotropic plane we have the following:

Theorem 2. *If A_i is the point of contact of the line BC with the incircle of the triangle ABC, then the circles inscribed in the triangles ABA_i and ACA_i have points of contact with the line AA_i symmetric according to the midpoint of the line segment AA_i and at the same time being parallel with the points C and B respectively.*

Proof. Let u_1 and u_2 be the abscissae of the points of contact of the two considered circles with the line AA_i. Applying Theorem 1 we get $b + u_1 = a - 2a$ and $c + u_2 = a - 2a$, i.e. $u_1 = c$ and $u_2 = b$. □

Theorem 3. *The potential axis of the circumcircle and the incircle of a triangle is its de Longchamps line.*

Proof. Eliminating the terms involving x^2 from the circumcircle equations $y = x^2$ and (1) we get the equation $y = -\frac{4}{3}q$, that is the potential axis of the two circles. On the other hand, according to [5] the latter equation is the de Longchamps line of the triangle ABC. □

Theorem 4. *A given triangle and its contact triangle have the same Euler line, i.e. parallel centroids, and corresponding altitudes of equal length.*

Proof. The points B_i and C_i from (2) lie on the line with the equation

$$y = \frac{a}{2}x - bc - q \tag{3}$$

since for example for the first of the two points we get

$$\frac{a}{2}(-2b) - bc - q = -ab - bc - q = ca - 2q.$$

We compute the isotropic span of the point A_i given in (2) to the line (3), i.e.

$$bc - 2q - \frac{a}{2}(-2a) + bc + q = 2bc + a^2 - q = 3bc - 2q,$$

while the isotropic span of the point $A = (a, a^2)$ to the line BC with the equation $y = -ax - bc$ (see [1]) is

$$a^2 + a \cdot a + bc = 2a^2 + bc = 3bc - 2q,$$

showing that the two spans are equal. The three points (2) have the centroid $G_i = (0, -\frac{5}{3}q)$, parallel to the centroid $G = (0, -\frac{2}{3}q)$ of the triangle ABC. □

Corollary 1. *The sides B_iC_i, C_iA_i, A_iB_i of the contact triangle $A_iB_iC_i$ of the standard triangle ABC are given successively in the equations*

$$y = \frac{a}{2}x - bc - q, \quad y = \frac{b}{2}x - ca - q, \quad y = \frac{c}{2}x - ab - q. \tag{4}$$

For the slopes of the lines CA and AB are, according to [3], $-b$ and $-c$ respectively their bisector has the slope $-\frac{1}{2}(b + c) = \frac{a}{2}$, and it is parallel to the first line (4). Hence,

Corollary 2. *The sides of the contact triangle of a given triangle are parallel to the angle bisectors of that triangle.*

The equation of a polar of any point (x_0, y_0) with respect to the circle (1) is $y + y_0 = \frac{1}{2}xx_0 - 2q$. By choosing $x_0 = 0,\ y_0 = -\frac{5}{3}q$ we get $y = -\frac{q}{3}$, being according to [3] the equation of the orthic axis of the triangle ABC. Thus we have:

Corollary 3. *The orthic axis of a triangle is a polar of the centroid of its contact triangle with respect to its inscribed circle.*

Theorem 5. *Assuming that $A_iB_iC_i$ is a contact triangle of the allowable triangle ABC, the lines AA_i, BB_i, CC_i intersect in one point Γ.*

Proof. The line with the equation

$$y = \frac{q}{3a}x + bc - \frac{4}{3}q$$

passes through the point $A = (a, a^2)$ as well as through the point A_i from (2) due to

$$\frac{q}{3} + bc - \frac{4}{3}q = bc - q = a^2, \text{ and } -\frac{2}{3}q + bc - \frac{4}{3}q = bc - 2q = a^2,$$

representing therefore the line AA_i. This line passes through the point

$$\Gamma = (-\frac{3p}{q}, -\frac{4q}{3}) \tag{5}$$

as well, since the equality $-\frac{p}{a} + bc - \frac{4}{3}q = -\frac{4}{3}q$ is valid. Because of the symmetry in a, b, and c the point Γ lies on the lines BB_i, CC_i as well. □

By accordance with the Euclidean case we call the point Γ from Theorem 5 *Gergonne point* of the triangle ABC. Γ obviously lies on the de Longchamps line with the equation $y = -\frac{4}{3}q$.

Corollary 4. *The Gergonne point of the standard triangle ABC is given in (5). The Gergonne point of an allowable triangle lies on its de Longchamps line.*

Let us recall now a notion and some facts on a *tangential triangle.* The tangential triangle of the given triangle ABC is a triangle determined by the three tangents to the circumcircle of the triangle ABC in its vertices. It has been shown in [1] that the vertices of the tangential triangle in the case of a standard triangle are

$$A_t = (-\frac{a}{2}, bc), \quad B_t = (-\frac{b}{2}, ca), \quad C_t = (-\frac{c}{2}, ab) \tag{6}$$

and for example the side B_tC_t has the equation

$$y = 2ax + q - bc \tag{7}$$

since for example for the line (7) and the point B_t in (6) we get

$$2a(-\frac{b}{2}) + q - bc = b^2 + q = ca.$$

The triangle $A_tB_tC_t$ with the vertices given in (6) has the centroid $G_t = (0, \frac{q}{3})$.

In [2] it is shown that the triangles ABC and $A_tB_tC_t$ are homological. The center respectively the axis of homology are the *symmedian center* K and the *Lemoine line* $\mathcal{L}$ of the triangle ABC. In case of the standard triangle ABC we have

$$K = (\frac{3p}{2q}, -\frac{q}{3}) \tag{8}$$

and

$$\mathcal{L} \ \ldots \ y = \frac{3p}{q}x + \frac{q}{3}. \tag{9}$$

Assuming that G is the centroid of the triangle ABC, then the homothecy $(G, -2)$ maps each point P to its *anticomplementary point* P'. Thus by the equality $2P + P' = 3G$, i.e. $P' = 3G - 2P$ follows that the point $P' = (-2x, -2q-2y)$ is anticomplementary to the point $P = (x, y)$. Therefore, we have:

Theorem 6. *The Gergonne point of a triangle is anticomplementary to its symmedian center.*

Proof. Really, the point Γ from (5) is anticomplementary to the point K from (8) due to

$$-2q - 2(-\frac{q}{3}) = -\frac{4}{3}q. \qquad \square$$

Theorem 7. *The Gergonne point of a triangle is isogonal to the centroid of its tangential triangle with respect to the given triangle, this centroid being parallel to the centroid of the given triangle.*

Proof. The centroid of the triangle $A_tB_tC_t$ with the vertices given in (6) is the point

$$G_t = (0, \frac{q}{3}). \tag{10}$$

The proof indicates that the point G_t from (10) is isogonal to the point Γ from (5). Indeed, for the point $G_t = (x, y)$ we have

$$y - x^2 = \frac{q}{3}, \quad xy - qx - p = -p, \quad px - qy - y^2 = -\frac{4}{9}q^2,$$

and according to [4] for its isogonal point (x', y') we get

$$x' = \frac{xy + qx - p}{y - x^2} = -\frac{3p}{q}, \quad y' = \frac{px - qy - y^2}{y - x^2} = -\frac{4}{3}\,q. \qquad \square$$

Theorem 8. *The Gergonne point of the allowable triangle ABC is the homology center of that triangle and its contact triangle. In the case of the standard triangle ABC the homology axis of the same transformation is the line with the equation*

$$y = -\frac{3p}{2q}x - \frac{2q}{3}. \tag{11}$$

Proof. From the equations $y = -ax - bc$ and (3) for the lines BC and B_iC_i we obtain the equation $-ax = \frac{a}{2}x - q$ with the solution $x = \frac{2q}{3a}$, while inserting it in $y = -ax - bc$ we get $y = -\frac{2}{3}\,q - bc$. The obtained values for x and y satisfy (11) because of $\frac{p}{a} = bc$. $\square$

The triangle ABC is a contact triangle of its tangential triangle $A_tB_tC_t$. Therefore the symmedian center K of the triangle ABC is the Gergonne point Γ_t of $A_tB_tC_t$. Further on, the triangle ABC is a tangential triangle of its contact triangle $A_iB_iC_i$. Hence, the Gergonne point Γ of the triangle ABC is the symmedian center K_i of $A_iB_iC_i$, while the axis of homology from Theorem 8 is the Lemoine line $\mathcal{L}_i$ for that triangle. The line $\mathcal{L}_i$ with the equation (11) obviously passes through the centroid $G = (0, -\frac{2q}{3})$ of the triangle ABC. Thus, we have:

Corollary 5. *The centroid of a triangle lies on the Lemoine line of its contact triangle.*

Applying Corollary 5 to the tangential triangle $A_tB_tC_t$, it follows:

Corollary 6. *The Lemoine line of a triangle passes through the centroid of its tangential triangle.*

The claim of Corollary 6 can be deduced directly by checking the stated properties for the point G_t from (10) and the line $\mathcal{L}_i$ from (9).

The next statement follows from above as well.

Corollary 7. *The symmedian center of a triangle is the Gergonne point of its tangential triangle. The Gergonne point of a triangle is the symmedian center of its contact triangle.*

By adding the equation (9) and the equation (11) previously multiplied by 2, we get the equation $3y = -q$, i.e. according to [3] the equation of the *orthic axis* $\mathcal{H}$ of the triangle ABC. This very axis is the axis of homology of the triangle ABC and its orthic triangle $A_hB_hC_h$. Using $y = -\frac{q}{3}$ in (9) or (11) we get $x = -\frac{2q^2}{9p}$, providing that

$$T = (-\frac{2q^2}{9p}, -\frac{q}{3}) \tag{12}$$

is the common point of the three observed lines. Hence, we have:

Theorem 9. *The Lemoine line $\mathcal{L}$ and the orthic axis $\mathcal{H}$ of the allowable triangle ABC intersect in the point T which lies on the Lemoine line $\mathcal{L}_i$ of its contact triangle $A_iB_iC_i$. In case of a standard triangle ABC, T is given in (12).*

The triangle ABC is homological with both of the triangles $A_tB_tC_t$ and $A_iB_iC_i$. On the other hand, the next theorem provides that the two latter triangles are homological.

Theorem 10. *The tangential triangle $A_tB_tC_t$ and the contact triangle $A_iB_iC_i$ of the allowable triangle ABC are homological. The center respectively the axis of that homology are, in the case of the standard triangle ABC, the point*

$$S = (-\frac{3p}{4q}, \frac{2}{3}q) \tag{13}$$

and the line $\mathcal{S}$ with the equation

$$\mathcal{S} \ \ldots \ y = \frac{3p}{4q}x - \frac{5}{3}q. \tag{14}$$

Proof. The line with the equations $y = \frac{4q}{3a}x + \frac{2}{3}q + bc$ passes through the points A_t, A_i, and S given in (6), (2), and (13) respectively because

$$\frac{4q}{3a}(-\frac{a}{2}) + \frac{2}{3}q + bc = bc,$$

$$\frac{4q}{3a}(-2a) + \frac{2}{3}q + bc = bc - 2q,$$

and

$$\frac{4q}{3a}(-\frac{3p}{4q}) + \frac{2}{3}q + bc = \frac{2}{3}q.$$

This is the reason that the point S lies on the line A_tA_i and at the same time on analogous lines B_tB_i and C_tC_i. The point

$$D = (-\frac{4q}{3a}, -\frac{5}{3}q - bc) \tag{15}$$

lies on the lines B_tC_t and B_iC_i with the equations (7) and (3) as well as on the line $\mathcal{S}$ with the equation (14) because each of the three sums

$$2a(-\frac{4q}{3a}) + q - bc, \quad \frac{a}{2}(-\frac{4q}{3a}) - bc - q, \quad \frac{3p}{4q}(-\frac{4q}{3a}) - \frac{5}{3}q$$

equals $-\frac{5}{3}q - bc$. Therefore the line (14) passes through the point $B_tC_t \cap B_iC_i$ and through the analogous points $C_tA_t \cap C_iA_i$ and $A_tB_t \cap A_iB_i$ too. □

We use now the rule from the proof of Theorem 6 to deduce that, because of $-2q - 2(-\frac{q}{3}) = -\frac{4}{3}q$, the point T from (12) is anticomplementary to the point

$$T_i = (\frac{4q^2}{9p}, -\frac{4}{3}q). \tag{16}$$

This very point lies on the lines $\mathcal{L}_i$ and $\mathcal{S}$ from (11) and (14) because each of the two sums

$$-\frac{3p}{2q}(\frac{4q^2}{9p}) - \frac{2}{3}q, \quad \frac{3p}{4q}(\frac{4q^2}{9p}) - \frac{5}{3}q$$

equals $-\frac{4}{3}q$. We have thus proved the following theorem.

Theorem 11. *Let $A_tB_tC_t$ and $A_iB_iC_i$ be the tangential and the contact triangle of the allowable triangle ABC, and let $\mathcal{L}$, $\mathcal{L}_i$, and $\mathcal{S}$ be the axes of homology for pairs of triangles ABC, $A_tB_tC_t$; ABC, $A_iB_iC_i$, and $A_tB_tC_t$, $A_iB_iC_i$. Then the point $T_i = \mathcal{L}_i \cap \mathcal{S}$ is anticomplementary to the point $T = \mathcal{L} \cap \mathcal{L}_i$ regarding the triangle ABC.*

Pursuant to Theorem 6, the symmedian center of a triangle is complementary to its Gergonne point. Therefore, the symmedian center K_t of the triangle $A_tB_tC_t$ is complementary with respect to that triangle to its Gergonne point Γ_t which is the point K from (8), being the symmedian center of the triangle ABC according to Theorem 7. This means that the equality $2K_t + K = 3G_t$ is valid, with $G_t = (0, \frac{q}{3})$ being the centroid of the triangle $A_tB_tC_t$. Hence, for the point $K_t = (x, y)$ we get

$$2x = -\frac{3p}{2q}, \quad 2y = 3 \cdot \frac{q}{3} - (-\frac{q}{3}) = \frac{4}{3}q$$

showing that K_t coincides with the point S from (13). We have thus proved Theorem 12.

Theorem 12. *By the notation from the previous theorem, the center of homology of the triangles $A_tB_tC_t$ and $A_iB_iC_i$ is the symmedian center K_t of the triangle $A_tB_tC_t$. In case of standard triangle ABC we have that*

$$K_t = (-\frac{3p}{4q}, \ \frac{2}{3}q).$$

The geometrical meaning of the line $\mathcal{S}$ is given by the theorem below.

Theorem 13. *Let $A_{ii}B_{ii}C_{ii}$ be the contact triangle of the contact triangle $A_iB_iC_i$ of the allowable triangle ABC. The line $\mathcal{S}$ from Theorem 10 is the Lemoine line $\mathcal{L}_{ii}$ of the triangle $A_{ii}B_{ii}C_{ii}$.*

Proof. Let $\mathcal{K}_{ii}$ be the circle with the equation

$$y = \frac{1}{16}x^2 - 2q. \tag{17}$$

From this equation and the equation of the line B_iC_i given in (3) we get $\frac{1}{16}x^2 - 2q = \frac{a}{2}x - bc - q$, which due to $bc - q^2 = a^2$ becomes $\frac{1}{16}x^2 - \frac{a}{2}x + a^2 = 0$, i.e. an equation with double solution $x = 4a$. Therefore the line B_iC_i touches the circle $\mathcal{K}_{ii}$ in the point A_{ii} having the abscissa $4a$ and the ordinate $a^2 - 2q$, i.e. $bc - 3q$. We thus provide the first of the three analogous equalities:

$$A_{ii} = (4a, \ bc - 3q), \quad B_{ii} = (4b, \ ca - 3q), \quad C_{ii} = (4c, \ ab - 3q). \tag{18}$$

The equation of the line $B_{ii}C_{ii}$ is

$$y = -\frac{a}{4}x - bc - 2q \tag{19}$$

because for example, for the point B_{ii} from (18) we have

$$-\frac{a}{4}(4b) - bc - 2q = b^2 - 2q = ca - 3q.$$

Further on, we see at once that the the point $D = B_iC_i \cap \mathcal{S}$ lies on the line (19) owing to

$$-\frac{a}{4}(-\frac{4q}{3a}) - bc - 2q = -\frac{5}{3}q - bc,$$

wherefrom we deduce that $\mathcal{S}$ passes through the point $B_iC_i \cap B_{ii}C_{ii}$ and analogously through the points $C_iA_i \cap C_{ii}A_{ii}$ and $A_iB_i \cap A_{ii}B_{ii}$. Hence, we conclude that $\mathcal{S}$ is the axis of homology of the triangle $A_iB_iC_i$ and its contact triangle $A_{ii}B_{ii}C_{ii}$. □

Corollary 8. *The contact triangle $A_iB_iC_i$ of the standard triangle ABC has the incircle $\mathcal{K}_{ii}$ with the equation (17) and the contact triangle $A_{ii}B_{ii}C_{ii}$ with the vertices given in (18), while for example the equation of its side $B_{ii}C_{ii}$ is given in (19). The Lemoine line $\mathcal{L}_{ii}$ of the triangle $A_{ii}B_{ii}C_{ii}$ has the equation (14).*

We see now that the point $\mathcal{L}_i \cap \mathcal{L}_{ii} = \mathcal{L}_i \cap \mathcal{S} = T_i$ in the triangle $A_iB_iC_i$ has the same role as the point $T = \mathcal{L} \cap \mathcal{L}_i$ in the triangle ABC, and using Theorem 9 on the triangle $A_iB_iC_i$ it follows that its orthic axis $\mathcal{H}_i$ passes through the point T_i and has the equation $y = -\frac{4}{3}q$. We then apply Theorem 11 on the triangle $A_iB_iC_i$. It follows that the point T_{ii} of the triangle $A_{ii}B_{ii}C_{ii}$ which corresponds to the point T_i of the triangle $A_iB_iC_i$, is anticomplementary to the latter point regarding the triangle $A_iB_iC_i$, that is the equality $2T_i + T_{ii} = 3G_i$ is valid. From the equality (16) and $G_i = (0,\ -\frac{5}{3}q)$ we get

$$T_{ii} = (-\frac{8q^2}{9p}, -\frac{7}{3}q). \tag{20}$$

On the other hand, the point T_{ii} from (18) lies on the line $\mathcal{L}$ with the equation (9) due to

$$\frac{3p}{q}(-\frac{8q^2}{9p}) + \frac{q}{3} = -\frac{7}{3}q.$$

We have actually proved:

Theorem 14. *By the notation from Theorem 11 the point $T_{ii} = \mathcal{L} \cap \mathcal{L}_{ii}$ is anticomplementary to the point $T_i = \mathcal{L}_i \cap \mathcal{S}$ regarding the triangle $A_iB_iC_i$.*

Using Theorem 11 again and the fact that the point T_i is anticomplementary to the point T regarding the triangle ABC we get $2T + T_i = 3G$ that together with the equality $2T_i + T_{ii} = 3G_i$ gives the equality $4T - T_{ii} = 6G - 3G_i$. For $G = (0,\ -\frac{2}{3}q)$ being the midpoint of the points $G_t = (0,\ \frac{q}{3})$ and $G_i = (0,\ -\frac{5}{3}q)$ we have $G_t + G_i = 2G$, i.e. $6G - 3G_i = 3G_t$. Hence, it follows that $4T - T_{ii} = 3G_t$ which written in the form of $4(T - G_t) = T_{ii} - G_t$ proves that the homothecy $(G_t,\ 4)$ maps the point T into the point T_{ii}. So, we have:

Theorem 15. *If G_t is the centroid of the tangential triangle $A_tB_tC_t$ of the allowable triangle ABC, then the homothecy $(G_t, 4)$ maps the point T from Theorem 11 into the point T_{ii} from Theorem 14.*

Relationships between all the discussed elements are given in the Figure 1.

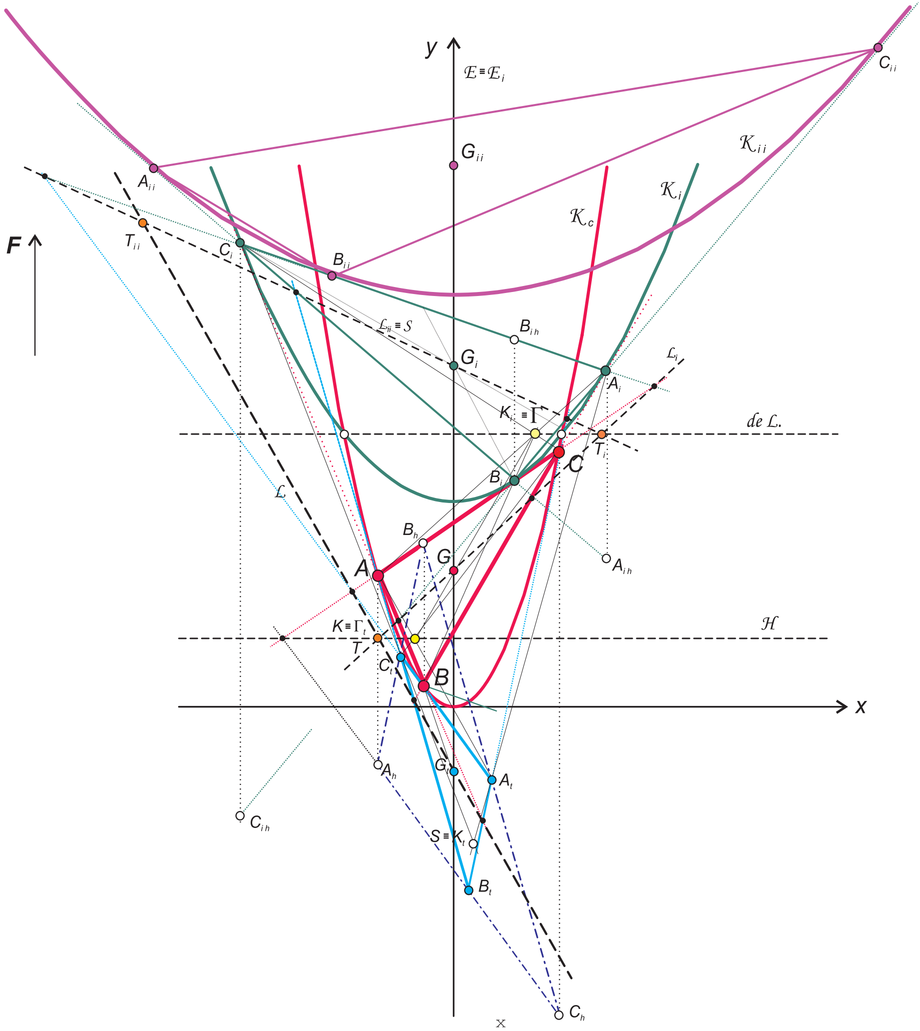

Figure 1.

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