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ON SOME CONVERSIONS OF THE JENSEN-STEFFENSEN INEQUALITY

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Abstract. Some conversions of the Jensen-Steffensen inequality for convex functions are considered. Applying *exp-convex method* improvements and reverses of the Slater-Pečarić inequality are obtained. Related Cauchy's type means are defined and some basic properties are given.

1. Introduction

A function $\varphi : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ is convex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

holds for all $x, y \in (a,b)$ and $\lambda \in [0,1]$.

It is well known that for a convex function $\varphi : (a,b) \to \mathbb{R}$, any monotonic n-tuple $\boldsymbol{x} = (x_1, ..., x_n) \in (a,b)^n$ and a real n-tuple $\boldsymbol{a} = (a_1, ..., a_n)$ that satisfies

$$0 \leq A_j = \sum_{i=1}^{j} a_i \leq A_n, \quad j = 1, ..., n, \quad A_n > 0, \tag{1.1}$$

the Jensen-Steffensen inequality

$$\varphi\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right) \leq \frac{1}{A_n}\sum_{i=1}^{n} a_i \varphi(x_i) \tag{1.2}$$

holds (see [9]).

The next integral variant of the Jensen-Steffensen inequality is proved by R. P. Boas [5].

Theorem 1. *Let $\varphi : (a,b) \to \mathbb{R}$ be a convex function. Let $f : [\alpha, \beta] \to (a,b)$ be continuous and monotonic, where $-\infty \leq a < b \leq +\infty$ and*

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$-\infty < \alpha < \beta < +\infty$, *and* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *be either continuous or of bounded variation satisfying*

$$\lambda(\alpha) \leq \lambda(x) \leq \lambda(\beta) \quad \text{for all} \quad x \in [\alpha,\beta], \quad \lambda(\beta)-\lambda(\alpha) > 0. \tag{1.3}$$

Then

$$\varphi\left(\frac{\int_\alpha^\beta f(t)\mathrm{d}\lambda(t)}{\int_\alpha^\beta \mathrm{d}\lambda(t)}\right) \leq \frac{\int_\alpha^\beta \varphi\left(f(t)\right)\mathrm{d}\lambda(t)}{\int_\alpha^\beta \mathrm{d}\lambda(t)}. \tag{1.4}$$

One important property of a convex function $\varphi : (a,b) \to \mathbb{R}$ is existence of the left and the right derivatives on (a,b), i.e for each $x \in (a,b)$ there exists $\varphi'_-(x)$ and $\varphi'_+(x)$ and it holds $\varphi'_-(x) \leq \varphi'_+(x)$. In following we denote with $\varphi'(x)$ any value in the interval $[\varphi'_-(x), \varphi'_+(x)]$. If a function φ is differentiable then $\varphi'(x) = \varphi'_-(x) = \varphi'_+(x)$.

J. Pečarić [8] proved the following companion inequality to the Jensen-Steffensen inequality which we refer to as the Slater-Pečarić inequality. Some refinements of the Slater-Pečarić inequality (1.5) are proved in [1].

Theorem 2. *Let* $\varphi : (a,b) \to \mathbb{R}$ *be a convex function. Let* $\boldsymbol{x} = (x_1,...,x_n)$ *be a monotonic* n*-tuple in* $(a,b)^n$ *and* $\boldsymbol{a} = (a_1,...,a_n)$ *be a real* n*-tuple satisfying* (1.1). *If*

$$\sum_{i=1}^n a_i\varphi'_+(x_i) \neq 0 \qquad \text{and} \qquad \frac{\sum_{i=1}^n a_i x_i \varphi'_+(x_i)}{\sum_{i=1}^n a_i\varphi'_+(x_i)} \in (a,b),$$

then

$$\frac{1}{A_n}\sum_{i=1}^n a_i\varphi(x_i) \leq \varphi\left(\frac{\sum_{i=1}^n a_i x_i \varphi'_+(x_i)}{\sum_{i=1}^n a_i\varphi'_+(x_i)}\right). \tag{1.5}$$

The next companion inequality to the Jensen-Steffensen inequality is proved by N. Elezović and J. Pečarić [6].

Theorem 3. *If* $\varphi : (a,b) \to \mathbb{R}$ *is a convex function,* $x_1,...,x_n$ *monotonic sequence in* (a,b) *and* $a_1,...,a_n$ *real numbers satisfying* (1.1), *then*

$$\begin{aligned} 0 &\leq \frac{1}{A_n}\sum_{i=1}^n a_i\varphi(x_i) - \varphi\left(\frac{1}{A_n}\sum_{i=1}^n a_i x_i\right) \qquad (1.6)\\ &\leq \frac{1}{A_n}\sum_{i=1}^n a_i x_i\varphi'(x_i) - \left(\frac{1}{A_n}\sum_{i=1}^n a_i x_i\right)\left(\frac{1}{A_n}\sum_{i=1}^n a_i\varphi'(x_i)\right). \end{aligned}$$

In following we denote $\bar{x} = \frac{1}{A_n}\sum_{i=1}^n a_i x_i$.

The inequalities (1.2), (1.5) and (1.6) can be obtained from the next more general result as special cases.

Theorem 4. [7, Theorem 1] *Let $\varphi:(a,b)\to\mathbb{R}$ be a convex function and $a_i\in\mathbb{R}$, $i=1,...,n$ be such that* (1.1) *holds. Then for any $x_i\in(a,b)$, $i=1,...,n$, such that $x_1\le x_2\le...\le x_n$ or $x_1\ge x_2\ge...\ge x_n$, the inequalities*

$$\varphi(c)+\varphi^{'}(c)(\bar{x}-c)\le\frac{1}{A_n}\sum_{i=1}^{n}a_i\varphi\left(x_i\right)\le\varphi(d)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\varphi^{'}(x_i)(x_i-d) \tag{1.7}$$

hold for all $c,d\in(a,b)$.

Remark 1. *Choosing $c=\bar{x}$ the first inequality in* (1.7) *becomes* (1.2)*. Choosing $d=\bar{x}$ the second inequality in* (1.7) *becomes* (1.6)*.*
If we choose $d\in(a,b)$ such that

$$\sum_{i=1}^{n}a_i\varphi'\left(x_i\right)\left(x_i-d\right)=0, \tag{1.8}$$

the second inequality in (1.7) *becomes*

$$\frac{1}{A_n}\sum_{i=1}^{n}a_i\varphi\left(x_i\right)\le\varphi(d).$$

Under the condition $\sum_{i=1}^{n}a_i\varphi^{'}(x_i)\neq 0$, the equality (1.8) *is equivalent to*

$$d=\frac{\sum_{i=1}^{n}a_ix_i\varphi'\left(x_i\right)}{\sum_{i=1}^{n}a_i\varphi'\left(x_i\right)}.$$

Therefore, we get (1.5)*.*

Here we also quote integral version of Theorem 4.

Theorem 5. [7, Theorem 5] *Let $\varphi:(a,b)\to\mathbb{R}$ be a convex function. Let $f:[\alpha,\beta]\to(a,b)$ be a continuous and monotonic function, where $-\infty\le a<b\le+\infty$ and $-\infty<\alpha<\beta<+\infty$, and $\lambda:[\alpha,\beta]\to\mathbb{R}$ be either continuous or of bounded variation satisfying* (1.3)*. Then $\tilde{x}$ and $\tilde{y}$ given by*

$$\tilde{x}=\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}f(t)\mathrm{d}\lambda(t),$$
$$\tilde{y}=\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}\varphi\left(f(t)\right)\mathrm{d}\lambda(t)$$

are well defined and $\tilde{x} \in (a,b)$. *Furthermore, if* $\varphi'(f)$ *and* λ *have no common discontinuity points, then the inequalities*

$$\varphi(c)+\varphi^{'}(c)(\tilde{x}-c)\leq\tilde{y}\leq\varphi(d)+\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}\varphi'\left(f(t)\right)\left(f(t)-d\right)\mathrm{d}\lambda(t) \tag{1.9}$$

hold for each $c,d\in(a,b)$.

In following we denote $\bar{f}=\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}f(t)\mathrm{d}\lambda(t)$.

In Section 2 we use so called *exp-convex method* established in [3], [4], which enables us to interpret our results in form of exponential or logarithmic convexity. Because of that we quote definitions of exponential and logarithmic convexity and some related propositions (see also [9]).

Definition 1. *A function* $\varphi:(a,b)\to\mathbb{R}$ *is said to be exponentially convex if it is continuous and*

$$\sum_{i,j=1}^{m}u_iu_j\varphi(x_i+x_j)\geq 0$$

holds for all $m\in\mathbb{N}$ *and all choices* $u_i\in\mathbb{R}$, $i=1,2,...,m$ *and* $x_i\in(a,b)$ *such that* $x_i+x_j\in(a,b)$, $1\leq i,j\leq m$.

Definition 2. *A function* $\varphi:(a,b)\to\mathbb{R}_+$ *is said to be logarithmically convex or log-convex if the function* $\log\varphi$ *is convex, or equivalently, if*

$$\varphi\left((1-\lambda)x+\lambda y\right)\leq\varphi(x)^{1-\lambda}\varphi(y)^{\lambda}$$

holds for all $x,y\in(a,b),\lambda\in[0,1]$.

Proposition 1. *Let* $\varphi:(a,b)\to\mathbb{R}$ *be a function. The following propositions are equivalent:*

(i) φ *is exponentially convex.*
(ii) φ *is continuous and*

$$\sum_{i,j=1}^{m}u_iu_j\varphi\left(\frac{x_i+x_j}{2}\right)\geq 0$$

holds for all $m\in\mathbb{N}$ *and all choices* $u_i\in\mathbb{R}$ *and every* $x_i,x_j\in(a,b)$, $1\leq i,j\leq m$.

Corollary 1. *If* $\varphi:(a,b)\to\mathbb{R}_+$ *is an exponentially convex function then* φ *is also log-convex.*

2. Applications of Exp-convex method

We define a new class of functionals which we use in sequel. In following we always suppose that $0 < a < b$ and $-\infty < \alpha < \beta < +\infty$.

Let $\boldsymbol{x} \in [a, b]^n$ be a monotonic n-tuple, $\boldsymbol{a}$ be a real n-tuple satisfying (1.1) and $c, d \in [a, b]$. We define the functionals A_1 and A_2 on $C^1([a, b])$ by

$$A_1(\varphi) = \varphi(d) + \frac{1}{A_n}\sum_{i=1}^{n} a_i \left[\varphi^{'}(x_i)(x_i - d) - \varphi(x_i)\right], \tag{2.1}$$

$$A_2(\varphi) = \frac{1}{A_n}\sum_{i=1}^{n} a_i\varphi(x_i) - \varphi(c) - \varphi^{'}(c)(\bar{x} - c).$$

We also define integral case of the previous functionals as follows.

Let $f : [\alpha, \beta] \to [a, b]$ be a continuous and monotonic function, $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation satisfying (1.3) and $c, d \in [a, b]$. We define the functionals B_1 and B_2 on $C^1([a, b])$ by

$$B_1(\varphi) = \varphi(d) + \frac{1}{\lambda(\beta) - \lambda(\alpha)}\int_{\alpha}^{\beta} \left[\varphi'(f(t))(f(t) - d) - \varphi(f(t))\right] \mathrm{d}\lambda(t), \tag{2.2}$$

$$B_2(\varphi) = \frac{1}{\lambda(\beta) - \lambda(\alpha)}\int_{\alpha}^{\beta}\varphi(f(t))\,\mathrm{d}\lambda(t) - \varphi(c) - \varphi^{'}(c)(\bar{f} - c),$$

with assumption that $\varphi'(f)$ and λ have no common discontinuity points.

Remark 2. *Notice that in case when φ is convex, then by Theorem 4 it follows $A_k(\varphi) \geq 0$ and by Theorem 5 it follows $B_k(\varphi) \geq 0$ for each $k \in \{1, 2\}$.*

In sequel we frequently use the family of convex functions $\{\phi_s; s \in \mathbb{R}\}$ defined by

$$\phi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1 \\ -\log x, & s = 0 \\ x \log x, & s = 1 \end{cases}. \tag{2.3}$$

Now we state and prove the next results.

Theorem 6. *Let $\Gamma_k : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$\Gamma_k(s) = A_k(\phi_s) \tag{2.4}$$

where A_k is defined as in (2.1) and ϕ_s as in (2.3). Then

(i) *for all* $m \in \mathbb{N}$ *and all choices* $p_i \in \mathbb{R}$, $1 \leq i \leq m$, *the matrix* $\left[\Gamma_k\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^m$ *is positive semi-definite, that is*

$$\det\left[\Gamma_k\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^m \geq 0. \tag{2.5}$$

(ii) *the function* Γ_k *is exponentially convex.*

(iii) *if in addition* Γ_k *is positive, then* Γ_k *is also log-convex. Therefore, for any* $r, s, u \in \mathbb{R}$, *such that* $r < s < u$, *the following is valid*

$$\Gamma_k(s)^{u-r} \leq \Gamma_k(r)^{u-s}\,\Gamma_k(u)^{s-r}. \tag{2.6}$$

Proof. (i) Since

$$\lim_{s\to 0}\Gamma_k(s) = \lim_{s\to 0} A_k(\phi_s) = A_k(\phi_0) = \Gamma_k(0),$$
$$\lim_{s\to 1}\Gamma_k(s) = \lim_{s\to 1} A_k(\phi_s) = A_k(\phi_1) = \Gamma_k(1),$$

it follows that Γ_k is continuous function.
Let $u_i, p_i \in \mathbb{R}$, $i = 1, ..., m$, and $p_{ij} = \frac{p_i+p_j}{2}$, $1 \leq i, j \leq m$.
We consider the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(x) = \sum_{i,j=1}^m u_i u_j \phi_{p_{ij}}(x),$$

where $\phi_{p_{ij}}$ is defined as in (2.3).
Since f is convex (see proof of [3, Theorem 3]), then applying Theorem 4 to f we have that

$$\sum_{i,j=1}^m u_i u_j \Gamma_k\left(\frac{p_i+p_j}{2}\right) \geq 0$$

holds for all choices of $m \in \mathbb{N}$, $u_i, p_i \in \mathbb{R}$, $1 \leq i \leq m$. Then the matrix $[\Gamma_k(p_{ij})]_{i,j=1}^m$ is positive semi-definite, so the inequality (2.5) holds.
(ii) Since Γ_k is also continuous, then by Proposition 1 it follows that Γ_k is exponentially convex.
(iii) If Γ_k is positive, then by Corollary 1 it follows that Γ_k is log-convex. Therefore, for $r, s, u \in \mathbb{R}$, such that $r < s < u$, we have

$$(u-s)\log\Gamma_k(r) + (r-u)\log\Gamma_k(s) + (s-r)\log\Gamma_k(u) \geq 0$$

which is equivalent to (2.6). □

On a similar way we can prove the next theorem.

Theorem 7. *Let* $\Lambda_k : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$\Lambda_k(s) = B_k(\phi_s)$$

where B_k *is defined as in* (2.2) *and* ϕ_s *as in* (2.3). *Then*

(i) *for all* $m \in \mathbb{N}$ *and all choices* $p_i \in \mathbb{R}$, $1 \leq i \leq m$, *the matrix* $\left[\Lambda_k\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^{m}$ *is positive semi-definite, that is*

$$\det\left[\Lambda_k\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^{m} \geq 0. \tag{2.7}$$

(ii) *the function* Λ_k *is exponentially convex.*

(iii) *if in addition* Λ_k *is positive, then* Λ_k *is also log-convex. Therefore, for any* $r, s, u \in \mathbb{R}$, *such that* $r < s < u$, *the following is valid*

$$\Lambda_k(s)^{u-r} \leq \Lambda_k(r)^{u-s}\,\Lambda_k(u)^{s-r}. \tag{2.8}$$

3. Mean value Theorems and Cauchy's mean

In this section we prove Lagrange's and Cauchy's type of Mean value theorem, in discrete and integral form, and introduce new means of Cauchy's type.

In following we use notation e_2 for quadratic function, i.e. $e_2(t) = t^2$, $t \in [a,b]$.

Theorem 8. *Let* A_k *be the functional defined by* (2.1) *and suppose* $A_k(e_2) \neq 0$. *If* $\varphi \in C^2([a,b])$, *then there exists* $\xi_k \in [a,b]$ *such that*

$$A_k(\varphi) = \frac{\varphi''(\xi_k)}{2} A_k(e_2). \tag{3.1}$$

Proof. Since $\varphi \in C^2([a,b])$, then there exist $m = \min_{x\in[a,b]} \varphi''(x)$ and $M = \max_{x\in[a,b]} \varphi''(x)$ such that $m \leq \varphi''(x) \leq M$ for each $x \in [a,b]$.

We define the functions $g_1 = \frac{M}{2}e_2 - \varphi$ and $g_2 = \varphi - \frac{m}{2}e_2$.

Since $g_1''(x), g_2''(x) \geq 0$, the functions g_1, g_2 are convex and applying (1.7) we obtain

$$\frac{M}{2}A_k(e_2) - A_k(\varphi) \geq 0 \quad \text{and} \quad 0 \leq A_k(\varphi) - \frac{m}{2}A_k(e_2).$$

Combining the last two inequalities we have

$$m \leq \frac{2A_k(\varphi)}{A_k(e_2)} \leq M.$$

Now we conclude that there exists $\xi_k \in [a,b]$ such that (3.1) holds. □

Theorem 9. *Let* A_k *be the functional defined by* (2.1) *and suppose* $A_k(e_2) \neq 0$. *If* $\varphi, \psi \in C^2([a,b])$, *then there exists* $\xi_k \in [a,b]$ *such that*

$$A_k(\psi)\varphi''(\xi_k) = A_k(\varphi)\psi''(\xi_k). \tag{3.2}$$

Rad Hrvat. akad. znan. umjet. 515 Matematičke znanosti 17 (2013), str. 107-122

Proof. Applying (3.1) on the function $h_k = A_k(\psi)\varphi - A_k(\varphi)\psi$ we get required result. $\square$

Theorem 9 enables us to define new means. If we set $a = \min_{1\le k\le n}\{x_k\}$ and $b = \max_{1\le k\le n}\{x_k\}$ and if we choose $\varphi = \phi_u$ and $\psi = \phi_v$, where $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0, 1$, providing that $A_k(\phi_u), A_k(\phi_v) \neq 0$, then from (3.2) we obtain

$$A_k(\phi_v)\xi_k^{u-2} = A_k(\phi_u)\xi_k^{v-2},$$

i.e.

$$\xi_k = \left(\frac{A_k(\phi_v)}{A_k(\phi_u)}\right)^{\frac{1}{v-u}}.$$

Since $a \le \xi_k \le b$, this presents a new mean on segment $[a, b]$. We use notation

$$M_{u,v}^k(\boldsymbol{x};\boldsymbol{a}) = \left(\frac{A_k(\phi_v)}{A_k(\phi_u)}\right)^{\frac{1}{v-u}}. \tag{3.3}$$

We can extend these means to the excluded cases. For $k \in \{1, 2\}$ and $u, v \in \mathbb{R}$ we define:

$$M_{u,v}^1(\boldsymbol{x};\boldsymbol{a}) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(d^v + \frac{1}{A_n}\sum_{i=1}^n a_i\left(vx_i^{v-1}(x_i-d)-x_i^v\right)\right)}{\frac{1}{u(u-1)}\left(d^u + \frac{1}{A_n}\sum_{i=1}^n a_i\left(ux_i^{u-1}(x_i-d)-x_i^u\right)\right)}\right)^{\frac{1}{v-u}}, & u \neq v;\ u, v \neq 0, 1 \\ \exp\left(\dfrac{d^u\log d + \frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^{u-1}(x_i-d)(1+u\log x_i)-x_i^u\log x_i\right)}{d^u + \frac{1}{A_n}\sum_{i=1}^n a_i\left(ux_i^{u-1}(x_i-d)-x_i^u\right)} - \dfrac{2u-1}{u(u-1)}\right), \\ u = v \neq 0, 1 \\ \exp\left(\dfrac{\log^2 d + \frac{1}{A_n}\sum_{i=1}^n a_i\left(2x_i^{-1}\log x_i(x_i-d)-\log^2 x_i\right)}{2\left(\log d + \frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^{-1}(x_i-d)-\log x_i\right)\right)} + 1\right), & u = v = 0 \\ \exp\left(\dfrac{d\log^2 d + \frac{1}{A_n}\sum_{i=1}^n a_i\left(\log x_i(x_i-d)(2+\log x_i)-x_i\log^2 x_i\right)}{2\left(d\log d + \frac{1}{A_n}\sum_{i=1}^n a_i((x_i-d)(1+\log x_i)-x_i\log x_i)\right)} - 1\right), \\ u = v = 1. \end{cases}$$

$$M^2_{u,v}(\boldsymbol{x};\boldsymbol{a}) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i {x_i}^v - c^v - vc^{v-1}(\bar{x}-c)\right)}{\frac{1}{u(u-1)}\left(\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i x_i^u - c^u - uc^{u-1}(\bar{x}-c)\right)} \right)^{\frac{1}{v-u}}, & u \neq v;\ u,v \neq 0,1 \\ \exp\left(\dfrac{\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i {x_i}^u \log x_i - c^u \log c - c^{u-1}(\bar{x}-c)(1+u\log c)}{\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i {x_i}^u - c^u - uc^{u-1}(\bar{x}-c)} - \frac{2u-1}{u(u-1)} \right), \\ u = v \neq 0,1 \\ \exp\left(\dfrac{\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i \log^2 x_i - \log^2 c - 2c^{-1}\log c(\bar{x}-c)}{2\left(\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i \log x_i - \log c - c^{-1}(\bar{x}-c)\right)} + 1 \right), & u = v = 0 \\ \exp\left(\dfrac{\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i x_i \log^2 x_i - c\log^2 c - \log c(\bar{x}-c)(2+\log c)}{2\left(\frac{1}{A_n}\sum\limits_{i=1}^{n} a_i x_i \log x_i - c\log c - (\bar{x}-c)(1+\log c)\right)} - 1 \right), & u = v = 1. \end{cases}$$

We can easily check that these means are symmetric and the special cases are limits of the general case. Note that (3.3) can be written as

$$M^k_{u,v}(\boldsymbol{x};\boldsymbol{a}) = \left(\frac{\Gamma_k(v)}{\Gamma_k(u)}\right)^{\frac{1}{v-u}},$$

where Γ_k is the function defined as in (2.4).

Now we prove the monotonicity of these means.

Theorem 10. *Let $r, s, u, v \in \mathbb{R}$ such that $r \leq u$, $s \leq v$. Then*

$$M^k_{s,r}(\boldsymbol{x};\boldsymbol{a}) \leq M^k_{v,u}(\boldsymbol{x};\boldsymbol{a}). \tag{3.4}$$

Proof. By Theorem 6 it follows that the function Γ_k is log-convex. Therefore, for any $r, s, u, v \in \mathbb{R}$, such that $r \leq u$, $s \leq v$, $r \neq s$, $u \neq v$, we have

$$\left(\frac{\Gamma_k(s)}{\Gamma_k(r)}\right)^{\frac{1}{s-r}} \leq \left(\frac{\Gamma_k(v)}{\Gamma_k(u)}\right)^{\frac{1}{v-u}}$$

which is equivalent to (3.4). The statement of theorem follows using continuous extensions. □

In following we present integral variants of the previous results without proofs.

Theorem 11. *Let B_k be the functional defined by (2.2) and suppose that $B_k(e_2) \neq 0$. If $\varphi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$B_k(\varphi) = \frac{\varphi''(\xi_k)}{2} B_k(e_2). \tag{3.5}$$

Theorem 12. *Let B_k be the functional defined by (2.2) and suppose that $B_k(e_2) \neq 0$. If $\varphi, \psi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$B_k(\psi)\varphi''(\xi_k) = B_k(\varphi)\psi''(\xi_k). \tag{3.6}$$

If we set $\operatorname{Im} f = [a,b]$, then $a = \min_{\alpha \le t \le \beta} f(t)$ and $b = \max_{\alpha \le t \le \beta} f(t)$ and if we choose $\varphi = \phi_u$ and $\psi = \phi_v$, where $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0, 1$, providing that $B_k(\phi_u), B_k(\phi_v) \neq 0$, then from (3.6) it follows

$$B_k(\phi_v)\xi_k^{u-2} = B_k(\phi_u)\xi_k^{v-2},$$

i.e.

$$\xi_k = \left(\frac{B_k(\phi_v)}{B_k(\phi_u)}\right)^{\frac{1}{v-u}},$$

what presents a new mean on segment $[a,b]$. We use notation

$$M_{u,v}^k(f;\lambda) = \left(\frac{B_k(\phi_v)}{B_k(\phi_u)}\right)^{\frac{1}{v-u}}. \tag{3.7}$$

We can extend these means to the excluded cases. For $k \in \{1,2\}$ and $u, v \in \mathbb{R}$ we define:

$$M_{u,v}^1(f;\lambda) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(d^v + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left[vf^{v-1}(t)(f(t)-d) - f^v(t)\right]\mathrm{d}\lambda(t)\right)}{\frac{1}{u(u-1)}\left(d^u + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta [uf^{u-1}(t)(f(t)-d) - f^u(t)]\mathrm{d}\lambda(t)\right)}\right)^{\frac{1}{v-u}}, & u \neq v;\ u, v \neq 0, 1 \\ \exp\left(\dfrac{d^u \log d + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left[f^{u-1}(t)(f(t)-d)(1+u\log f(t)) - f^u(t)\log f(t)\right]\mathrm{d}\lambda(t)}{d^u + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta [uf^{u-1}(t)(f(t)-d) - f^u(t)]\mathrm{d}\lambda(t)} - \dfrac{2u-1}{u(u-1)}\right), & \\ u = v \neq 0, 1 & \\ \exp\left(\dfrac{\log^2 d + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left[2f^{-1}(t)\log f(t)(f(t)-d) - \log^2 f(t)\right]\mathrm{d}\lambda(t)}{2\left(\log d + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta [f^{-1}(t)(f(t)-d) - \log f(t)]\mathrm{d}\lambda(t)\right)} + 1\right), & u = v = 0 \\ \exp\left(\dfrac{d\log^2 d + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left[\log f(t)(f(t)-d)(2+\log f(t)) - f(t)\log^2 f(t)\right]\mathrm{d}\lambda(t)}{2\left(d\log d + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta [(f(t)-d)(1+\log f(t)) - f(t)\log f(t)]\mathrm{d}\lambda(t)\right)} - 1\right), & \\ u = v = 1. & \end{cases}$$

$$M^2_{u,v}(f;\lambda)=\begin{cases}\left(\dfrac{\frac{1}{v(v-1)}\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^v(t)\mathrm{d}\lambda(t)-c^v-vc^{v-1}(\bar{f}-c)\right)}{\frac{1}{u(u-1)}\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)-c^u-uc^{u-1}(\bar{f}-c)\right)}\right)^{\frac{1}{v-u}}, & u\neq v;\ u,v\neq 0,1\\ \exp\left(\dfrac{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\log f(t)\mathrm{d}\lambda(t)-c^u\log c-c^{u-1}(\bar{f}-c)(1+u\log c)}{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)-c^u-uc^{u-1}(\bar{f}-c)}-\dfrac{2u-1}{u(u-1)}\right), & \\ u=v\neq 0,1 & \\ \exp\left(\dfrac{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log^2 f(t)\mathrm{d}\lambda(t)-\log^2 c-2c^{-1}\log c(\bar{f}-c)}{2\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log f(t)\mathrm{d}\lambda(t)-\log c-c^{-1}(\bar{f}-c)\right)}+1\right), & u=v=0\\ \exp\left(\dfrac{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(t)\log^2 f(t)\mathrm{d}\lambda(t)-c\log^2 c-\log c(\bar{f}-c)(2+\log c)}{2\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(t)\log f(t)\mathrm{d}\lambda(t)-c\log c-(\bar{f}-c)(1+\log c)\right)}-1\right), & u=v=1.\end{cases}$$

These means are also symmetric and the special cases are limits of the general case. On a similar way we can prove the monotonicity of these means.

Theorem 13. *Let $r,s,u,v\in\mathbb{R}$ such that $r\leq u$, $s\leq v$. Then*

$$M^k_{s,r}(f;\lambda)\leq M^k_{v,u}(f;\lambda).$$

4. Improvement and reverse of the Slater-Pečarić inequality

Let $\boldsymbol{x}\in[a,b]^n$ be a monotonic n-tuple and $\boldsymbol{a}$ be a real n-tuple with $a_i\neq 0$ that satisfies (1.1). With $M_u(\boldsymbol{x};\boldsymbol{a})$ we denote u-mean with quasi-weights $\boldsymbol{a}$ defined by

$$M_u(\boldsymbol{x};\boldsymbol{a})=\begin{cases}\left(\dfrac{1}{A_n}\displaystyle\sum_{i=1}^n a_i x_i^u\right)^{\frac{1}{u}}, & u\in\mathbb{R}\setminus\{0\},\\ \left(\displaystyle\prod_{i=1}^n x_i^{a_i}\right)^{\frac{1}{A_n}}, & u=0.\end{cases}$$

For any $u \in \mathbb{R}$ we have

$$\min\{x_1, ..., x_n\} \leq M_u(\boldsymbol{x}; \boldsymbol{a}) \leq \max\{x_1, ..., x_n\}.$$

Also, for any $r, s \in \mathbb{R}$, such that $r < s$, we have

$$M_r(\boldsymbol{x}; \boldsymbol{a}) \leq M_s(\boldsymbol{x}; \boldsymbol{a}).$$

For more details see [2].

With d_u we denote expression defined by

$$d_u = \frac{\sum_{i=1}^{n} a_i x_i \phi_u'(x_i)}{\sum_{i=1}^{n} a_i \phi_u'(x_i)} = \begin{cases} \frac{M_u^u(\boldsymbol{x};\boldsymbol{a})}{M_{u-1}^{u-1}(\boldsymbol{x};\boldsymbol{a})}, & u \in \mathbb{R} \setminus \{0, 1\}, \\ M_{-1}(\boldsymbol{x}; \boldsymbol{a}), & u = 0, \\ \frac{A_n \bar{x} + \sum_{i=1}^{n} a_i x_i \log x_i}{A_n(1 + \log M_0(\boldsymbol{x};\boldsymbol{a}))}, & u = 1. \end{cases} \tag{4.1}$$

We define the functionals F_u and G_u ($u \in \mathbb{R}$) on $C^1([a, b])$ by

$$\begin{aligned} F_u(\varphi) &= \varphi(d_u) - \frac{1}{A_n} \sum_{i=1}^{n} a_i \varphi(x_i), \\ G_u(\varphi) &= \varphi(d_u) + \frac{1}{A_n} \sum_{i=1}^{n} a_i \left[\varphi'(x_i)(x_i - d_u) - \varphi(x_i) \right], \end{aligned} \tag{4.2}$$

where d_u is defined as in (4.1).

Now we state and prove improvement and reverse of the Slater-Pečarić inequality.

Theorem 14. *Let F_u and G_u ($u \in \mathbb{R}$) be the functionals defined as in (4.2) and ϕ_s ($s \in \mathbb{R}$) the function defined as in (2.3). Then*

(i) *for $r, s, u \in \mathbb{R}$, such that $r < s < u$ or $u < r < s$, the following is valid*

$$F_u(\phi_u) \geq G_u(\phi_s)^{(u-r)/(s-r)} G_u(\phi_r)^{(s-u)/(s-r)}; \tag{4.3}$$

(ii) *for $r, s, u \in \mathbb{R}$, such that $r < u < s$, the following is valid*

$$F_u(\phi_u) \leq G_u(\phi_s)^{(u-r)/(s-r)} G_u(\phi_r)^{(s-u)/(s-r)}. \tag{4.4}$$

Proof. (i) Let $r, s, u \in \mathbb{R}$ such that $r < s < u$ (when $u < r < s$ the proof is analogous).

From (2.6) for $k=1$ and choosing $d=d_u$ it follows

$$\left(\phi_s(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_s^{'}(x_i)(x_i-d_u)-\phi_s\left(x_i\right)\right]\right)^{u-r}$$
$$\leq\left(\phi_r(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_r^{'}(x_i)(x_i-d_u)-\phi_r\left(x_i\right)\right]\right)^{u-s}$$
$$\times\left(\phi_u(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_u^{'}(x_i)(x_i-d_u)-\phi_u\left(x_i\right)\right]\right)^{s-r}. \tag{4.5}$$

Since

$$\phi_u(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_u^{'}(x_i)(x_i-d_u)-\phi_u\left(x_i\right)\right]$$
$$=\phi_u(d_u)-\frac{1}{A_n}\sum_{i=1}^{n}a_i\phi_u\left(x_i\right),$$

then from (4.5) it follows

$$G_u(\phi_s)^{u-r}\leq G_u(\phi_r)^{u-s}F_u(\phi_u)^{s-r},$$

which is equivalent to (4.3).
(ii) Let $r,s,u\in\mathbb{R}$, such that $r<u<s$. From (2.6) for $k=1$ and choosing $d=d_u$ it follows

$$\left(\phi_u(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_u^{'}(x_i)(x_i-d_u)-\phi_u\left(x_i\right)\right]\right)^{s-r}$$
$$\leq\left(\phi_r(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_r^{'}(x_i)(x_i-d_u)-\phi_r\left(x_i\right)\right]\right)^{s-u}$$
$$\times\left(\phi_s(d_u)+\frac{1}{A_n}\sum_{i=1}^{n}a_i\left[\phi_s^{'}(x_i)(x_i-d_u)-\phi_s\left(x_i\right)\right]\right)^{u-r},$$

i.e.

$$F_u(\phi_u)^{s-r}\leq G_u(\phi_r)^{s-u}G_u(\phi_s)^{u-r}$$

which is equivalent to (4.4). □

Theorem 15. *Let G_u $(u\in\mathbb{R})$ be the functional defined as in (4.2) and ϕ_s $(s\in\mathbb{R})$ the function defined as in (2.3). Then for all $m\in\mathbb{N}$ and all choices $p_i\in\mathbb{R}$, $1\leq i\leq m$, the following is valid*

$$\det\left[G_{p_1}\left(\phi_{p_{ij}}\right)\right]_{i,j=1}^{m}\geq 0, \tag{4.6}$$
$$\det\left[G_{p_{12}}\left(\phi_{p_{ij}}\right)\right]_{i,j=1}^{m}\geq 0, \tag{4.7}$$

where $p_{ij}=\frac{p_i+p_j}{2}$, $1\leq i,j\leq m$.

Proof. From (2.5) for $k=1$ we have

$$\det\left[\Gamma_1\left(p_{ij}\right)\right]_{i,j=1}^m=\det\left[A_1\left(\phi_{p_{ij}}\right)\right]_{i,j=1}^m\geq 0. \tag{4.8}$$

Choosing $d=d_{p_1}$ from (4.8) it follows (4.6).
Similar, if we choose $d=d_{p_{12}}$, then from (4.8) it follows (4.7). □

Now we present integral versions of the previous results.

Let $f:[\alpha,\beta]\to[a,b]$ be a continuous and monotonic function and $\lambda:[\alpha,\beta]\to\mathbb{R}$ be either continuous or of bounded variation satisfying (1.3). With $M_u(f;\lambda)$ we denote integral u-mean defined by

$$M_u(f;\lambda)=\begin{cases}\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int\limits_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)\right)^{\frac{1}{u}}, & u\in\mathbb{R}\setminus\{0\}\,,\\ \exp\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int\limits_\alpha^\beta \log f(t)\mathrm{d}\lambda(t)\right), & u=0.\end{cases}$$

With $\tilde{d}_u$ we denote expression defined by

$$\tilde{d}_u=\frac{\int\limits_\alpha^\beta \phi_u'(f(t))f(t)\mathrm{d}\lambda(t)}{\int\limits_\alpha^\beta \phi_u'(f(t))\mathrm{d}\lambda(t)}=\begin{cases}\frac{M_u^u(f;\lambda)}{M_{u-1}^{u-1}(f;\lambda)}, & u\in\mathbb{R}\setminus\{0,1\}\,,\\ M_{-1}(f;\lambda), & u=0,\\ \frac{(\lambda(\beta)-\lambda(\alpha))\bar{f}+\int_\alpha^\beta f(t)\log f(t)\mathrm{d}\lambda(t)}{(\lambda(\beta)-\lambda(\alpha))(1+\log M_0(f;\lambda))}, & u=1.\end{cases} \tag{4.9}$$

Now we define the functionals H_u and K_u $(u\in\mathbb{R})$ on $C^1([a,b])$ by

$$H_u(\varphi)=\varphi(\tilde{d}_u)-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int\limits_\alpha^\beta\varphi\left(f(t)\right)\mathrm{d}\lambda(t), \tag{4.10}$$

$$K_u(\varphi)=\varphi(\tilde{d}_u)+\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int\limits_\alpha^\beta\left[\varphi'(f(t))(f(t)-\tilde{d}_u)-\varphi\left(f(t)\right)\right]\mathrm{d}\lambda(t),$$

where $\tilde{d}_u$ is defined as in (4.9).

We state improvement and reverse of the integral Slater-Pečarić inequality without proofs.

Theorem 16. *Let H_u and K_u $(u\in\mathbb{R})$ be the functionals defined as in (4.10) and ϕ_s $(s\in\mathbb{R})$ the function defined as in (2.3). Then*

(i) *for $r,s,u\in\mathbb{R}$, such that $r<s<u$ or $u<r<s$, the following is valid*

$$H_u(\phi_u)\geq K_u(\phi_s)^{(u-r)/(s-r)}K_u(\phi_r)^{(s-u)/(s-r)};$$

(ii) *for $r,s,u\in\mathbb{R}$, such that $r<u<s$, the following is valid*

$$H_u(\phi_u)\leq K_u(\phi_s)^{(u-r)/(s-r)}K_u(\phi_r)^{(s-u)/(s-r)}.$$

Theorem 17. *Let K_u $(u \in \mathbb{R})$ be the functional defined as in* (4.10) *and ϕ_s $(s \in \mathbb{R})$ the function defined as in* (2.3)*. Then for all $m \in \mathbb{N}$ and all choices $p_i \in \mathbb{R}$, $1 \leq i \leq m$, the following is valid*

$$\det\left[K_{p_1}\left(\phi_{p_{ij}}\right)\right]_{i,j=1}^{m} \geq 0,$$
$$\det\left[K_{p_{12}}\left(\phi_{p_{ij}}\right)\right]_{i,j=1}^{m} \geq 0,$$

where $p_{ij} = \frac{p_i+p_j}{2}$, $1 \leq i,j \leq m$.

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