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CAUCHY TYPE MEANS RELATED TO THE CONVERSE JENSEN-STEFFENSEN INEQUALITY

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Abstract. In this paper we apply so called exp-convex method to the converse Jensen-Steffensen inequality in order to interpret it in the form of exponentially convex functions. The outcome is a new class of Cauchy type means and some new interesting inequalities related to them.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $\varphi : I \to \mathbb{R}$ a convex function on I. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple in I^n and $\boldsymbol{p} = (p_1, \ldots, p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$, then the well known Jensen's inequality

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) \tag{1.1}$$

holds (see for example [9, p. 43]). If φ is strictly convex then (1.1) is strict unless $x_i = c$ for all $i \in \{j : p_j > 0\}$.

It is well known that the assumption "$\boldsymbol{p}$ is a nonnegative n-tuple" can be relaxed at the expense of more restrictions on the n-tuple $\boldsymbol{x}$. Namely, if $\boldsymbol{p}$ is a real n-tuple such that

$$0 \leq P_j \leq P_n\ ,\ j = 1, \ldots, n\ ; \quad P_n > 0\ , \tag{1.2}$$

where $P_j = \sum_{i=1}^{j} p_i$, then for any monotonic n-tuple $\boldsymbol{x}$ (increasing or decreasing) in I^n we have

$$\bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in \begin{cases} [x_1, x_n]\,, & \boldsymbol{x} \text{ increasing} \\ [x_n, x_1]\,, & \boldsymbol{x} \text{ decreasing} \end{cases}\ , \tag{1.3}$$

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and for any function φ convex on I, (1.1) still holds. Inequality (1.1) considered under conditions (1.2) is known as the Jensen-Steffensen inequality (see for example [9, p. 57]) for convex functions. The equality case for strictly convex functions is not so simple as in the case of Jensen's inequality and it was thoroughly investigated in [1].

It is known that the Jensen-Steffensen inequality can be stated in a more general integral form. It is given in the following theorem (see for example [9, p. 58]).

Theorem 1. *Let $f : [\alpha, \beta] \to (a, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $-\infty \leq a < b \leq +\infty$. Let $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation satisfying*

$$\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta) \quad \text{for all } t \in [\alpha, \beta], \quad \lambda(\beta) - \lambda(\alpha) > 0.$$

Then for any convex function $\varphi : (a, b) \to \mathbb{R}$ the inequality

$$\varphi\left(\tfrac{1}{\lambda(\beta)-\lambda(\alpha)} \int_\alpha^\beta f(t) \mathrm{d}\lambda(t)\right) \leq \tfrac{1}{\lambda(\beta)-\lambda(\alpha)} \int_\alpha^\beta \varphi(f(t)) \mathrm{d}\lambda(t)$$

holds.

Another well known inequality related to Jensen's inequality is converse Jensen's inequality (see for example [9, p. 98])

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(x_i) \leq \frac{b - \bar{x}}{b - a} \varphi(a) + \frac{\bar{x} - a}{b - a} \varphi(b), \tag{1.4}$$

which holds when $\varphi : I \to \mathbb{R}$ is a convex function on I, $[a, b] \subset I$, $-\infty < a < b < +\infty$ and $\boldsymbol{p}, \boldsymbol{x}$ are as in (1.1).

Since Jensen's inequality remains valid under Steffensen's conditions for n-tuples $\boldsymbol{x}$ and $\boldsymbol{p}$ it was reasonable to think that (1.4) would be valid too, but this was not the case (one can rather easily find a counterexample). A converse Jensen-Steffensen inequality as well as two inequalities complementary to the Jensen-Steffensen inequality have been recently established in [6] and they are stated in the following two theorems.

Theorem 2. *Let I be an open interval in $\mathbb{R}$ and $[a, b] \subseteq I$, $-\infty < a < b < +\infty$. For $n > 2$ let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a monotonic n-tuple in $[a, b]^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ a real n-tuple satisfying*

$$\begin{aligned} &w_i \neq 0, \quad i = 1, \ldots, n, \\ &0 \leq W_j \leq W_n = 1, \quad j = 1, \ldots, n, \end{aligned} \tag{1.5}$$

where $W_n = \sum_{i=1}^{n} w_i$ *and* $W_j = \sum_{i=1}^{j} w_i$. *If* $\varphi : I \to \mathbb{R}$ *is a convex function on* I, *then*

$$\sum_{i=1}^{n} w_i \varphi(x_i) \leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) + \varphi(\bar{x}). \tag{1.6}$$

If φ *is strictly convex, the equality holds in* (1.6) *if and only if there exists* $l \in \{2, \ldots, n-1\}$ *such that* $x_l = \bar{x} = (a+b)/2$ *and*

$$\begin{cases} (x_1 = a \wedge x_n = b) \vee (x_1 = b \wedge x_n = a), \\ (\forall j \in \{2, \ldots, l\}) \left(\bar{W}_j = 0 \vee x_{j-1} = x_j\right), \\ (\forall j \in \{l, \ldots, n-1\}) (W_j = 0 \vee x_j = x_{j+1}), \end{cases}$$

where $\bar{W}_j = \sum_{i=j}^{n} w_i$.

Theorem 3. *Let* I *be an open interval in* $\mathbb{R}$ *and* $[a,b] \subseteq I$, $-\infty < a < b < +\infty$. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *and* $\boldsymbol{w} = (w_1, \ldots, w_n)$ *be real* n*-tuples satisfying the conditions of Theorem 2. If* $\varphi : I \to \mathbb{R}$ *is a convex function on* I, *then*

$$\begin{aligned} &\sum_{i=1}^{n} w_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leq \varphi(a) + \varphi(b) - \varphi(\bar{x}) - \varphi(a+b-\bar{x}) \\ &\leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right). \end{aligned} \tag{1.7}$$

If φ *is strictly convex, the equality holds in the first inequality in* (1.7) *if and only if one of the following two cases occurs:*

(i) *either* $\bar{x} = a$ *or* $\bar{x} = b$,
(ii) *there exists* $l \in \{2, \ldots, n-1\}$ *such that* $\bar{x} = a + b - x_l$ *and*

$$\begin{cases} (x_1 = a \wedge x_n = b) \vee (x_1 = b \wedge x_n = a), \\ (\forall j \in \{2, \ldots, l\}) \left(\bar{W}_j = \sum_{i=j}^{n} w_i = 0 \vee x_{j-1} = x_j\right), \\ (\forall j \in \{l, \ldots, n-1\}) (W_j = 0 \vee x_j = x_{j+1}). \end{cases}$$

If φ *is strictly convex the second inequality in* (1.7) *becomes equality if and only if* $\bar{x} = (a+b)/2$.

Remark 1. *If we denote* (1.7) *as* $(1) \leq (2) \leq (3)$ *and the equality conditions for* $(1) = (3)$ *as* (EQ_1), *for* $(1) = (2)$ *as* (EQ_2) *and for* $(2) = (3)$ *as* (EQ_3) *we see that* $(EQ_1) \iff (EQ_2) \wedge (EQ_3)$ *which is to be expected.*

In the same paper integral versions of Theorem 2 and Theorem 3 has been established. For the sake of brevity throughout the rest of the paper we denote

$$\overline{f}=\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}f(t)\mathrm{d}\lambda(t)\in[a,b]\,.$$

Theorem 4. *Let $f:[\alpha,\beta]\to[a,b]$ be a continuous and monotonic function, where $-\infty<\alpha<\beta<+\infty$ and $-\infty<a<b<+\infty$. Let $\lambda:[\alpha,\beta]\to\mathbb{R}$ be either continuous or of bounded variation satisfying*

$$\lambda(\alpha)\leq\lambda(t)\leq\lambda(\beta)\quad\text{for all } t\in[\alpha,\beta]\,,\qquad\lambda(\beta)-\lambda(\alpha)>0. \tag{1.8}$$

Then for any continuous convex function $\varphi:[a,b]\to\mathbb{R}$ the inequality

$$\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}\varphi(f(t))\mathrm{d}\lambda(t)\leq\varphi(a)+\varphi(b)-2\varphi\left(\frac{a+b}{2}\right)+\varphi\left(\overline{f}\right) \tag{1.9}$$

holds.

Theorem 5. *Let the functions f and λ be as in Theorem 4. Then for any continuous convex function $\varphi:[a,b]\to\mathbb{R}$ the inequalities*

$$\begin{aligned}&\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}\varphi(f(t))\mathrm{d}\lambda(t)-\varphi\left(\overline{f}\right)\\&\leq\varphi(a)+\varphi(b)-\varphi\left(\overline{f}\right)-\varphi\left(a+b-\overline{f}\right)\\&\leq\varphi(a)+\varphi(b)-2\varphi\left(\frac{a+b}{2}\right)\end{aligned} \tag{1.10}$$

hold.

In Section 2 we apply so called exp-convex method to the results presented in Theorem 3 and Theorem 5. In Section 3 we introduce new means of Cauchy type and using results from Section 2 we establish comparison inequalities for them.

2. Exp-convex method

In this section we use so called *exp-convex method* established in [3] in order to interpret our results in the form of exponentially convex functions or (in the special case) log-convex functions (for the results related to the log-convex method see [2], [4], [7], [10] and [11]). As a consequence we obtain some new interesting inequalities.

Throughout this section I denotes an open interval in $\mathbb{R}$, $\mathbb{R}_+$ denotes the set $\{x\in\mathbb{R}:x>0\}$ and log denotes the natural logarithm function.

The following definitions can be found in [8] and [9].

Definition 1. *A function $\varphi : I \to \mathbb{R}$ is said to be exponentially convex if it is continuous and if*

$$\sum_{i,j=1}^{m} \xi_i \xi_j \varphi(x_i + x_j) \geq 0,$$

holds for any choice of $m \in \mathbb{N}$, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ and $\boldsymbol{x} = (x_1, \ldots, x_m) \in I^m$ such that $x_i + x_j \in I$.

Definition 2. *A function $\varphi : I \to \mathbb{R}_+$ is said to be logarithmically convex or log-convex if the function $\log \varphi$ is convex, or equivalently, if*

$$\varphi((1-\lambda)x + \lambda y) \leq \varphi(x)^{1-\lambda} \varphi(y)^{\lambda}$$

holds for all $x, y \in I, \lambda \in [0,1]$.

The following proposition is given in [3].

Proposition 1. *Let $\varphi : I \to \mathbb{R}$ be a function. The following assertions are equivalent:*

(i) *φ is exponentially convex.*
(ii) *φ is continuous and*

$$\sum_{i,j=1}^{m} \xi_i \xi_j \varphi\left(\tfrac{x_i + x_j}{2}\right) \geq 0,$$

for any choice of $m \in \mathbb{N}$, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ and $\boldsymbol{x} = (x_1, \ldots, x_m) \in I^m$.

Remark 2. *We can easily see that for positive functions exponential convexity implies log-convexity (consider Proposition 1 for $m = 2$).*

The following lemmas give two characterization inequalities for convex functions (see [9, p. 2]).

Lemma 1. *Let φ be a convex function on an interval $I \subseteq \mathbb{R}$. Then for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ the following is valid*

$$(x_3 - x_2)\varphi(x_1) + (x_1 - x_3)\varphi(x_2) + (x_2 - x_1)\varphi(x_3) \geq 0.$$

Lemma 2. *Let φ be a convex function on an interval $I \subseteq \mathbb{R}$. Then for any $x_1, x_2, y_1, y_2, \in I$ such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$ the following is valid*

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}.$$

Next we define a class of functionals which we will use in the sequel.

For real numbers a, b such that $0 < a < b < +\infty$, a monotonic n-tuple $\boldsymbol{x} \in [a, b]^n$ and a real n-tuple $\boldsymbol{w}$ satisfying (1.5) we define the functionals $F_k,\ k \in \{1, 2, 3\}$, on $C([a, b])$ by

$$F_1(\varphi) = \varphi(a) + \varphi(b) + \varphi(\bar{x}) - 2\varphi\left(\frac{a+b}{2}\right) - \sum_{i=1}^{n} w_i \varphi(x_i),$$

$$F_2(\varphi) = \varphi(a) + \varphi(b) - \varphi(a + b - \bar{x}) - \sum_{i=1}^{n} w_i \varphi(x_i), \tag{2.1}$$

$$F_3(\varphi) = \varphi(\bar{x}) + \varphi(a + b - \bar{x}) - 2\varphi\left(\frac{a+b}{2}\right).$$

Notice that when φ is convex, by Theorem 3 it follows that $F_k(\varphi) \geq 0$. Also, when φ is strictly convex and the corresponding condition (EQ_k) is not satisfied, then by Theorem 3 it follows that $F_k(\varphi) > 0$. Observe that all the functionals F_k are linear (this property will be needed later).

In the sequel we will frequently use a family of convex functions described in the following lemma.

Lemma 3. [5] *Let t be a real number. We define the function $\phi_t : \mathbb{R}_+ \to \mathbb{R}$ by*

$$\phi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0, 1 \\ -\log x, & t = 0 \\ x \log x, & t = 1 \end{cases}. \tag{2.2}$$

Then $\phi_t''(x) = x^{t-2}$, hence ϕ_t is convex on $\mathbb{R}_+$.

Now we can state and prove the next result.

Theorem 6. *The function $\Omega_k : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Omega_k(t) = F_k(\phi_t), \tag{2.3}$$

where F_k is defined as in (2.1) *and ϕ_t as in* (2.2)*, is exponentially convex.*

Proof. Since

$$\lim_{t\to 0} \Omega_k(t) = \lim_{t\to 0} F_k(\phi_t) = F_k(\phi_0) = \Omega_k(0),$$
$$\lim_{t\to 1} \Omega_k(t) = \lim_{t\to 1} F_k(\phi_t) = F_k(\phi_1) = \Omega_k(1),$$

it follows that Ω_k is continuous.

Let $u_i, p_i \in \mathbb{R}$, $i = 1, ..., m$, and $p_{ij} = \frac{p_i + p_j}{2}$, $1 \leq i, j \leq m$.

We consider the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(x) = \sum_{i,j=1}^{m} u_i u_j \phi_{p_{ij}}(x),$$

where $\phi_{p_{ij}}$ is defined as in (2.2).

Then

$$f''(x) = \sum_{i,j=1}^{m} u_i u_j x^{p_{ij}-2} = \left(\sum_{i=1}^{m} u_i x^{\frac{p_i}{2}-1}\right)^2 \geq 0,$$

hence f is convex on $\mathbb{R}_+$.

Applying Theorem 3 to f and $\boldsymbol{x}, \boldsymbol{w}$ as in (2.1) we have that

$$\sum_{i,j=1}^{m} u_i u_j \Omega_k\left(\tfrac{p_i+p_j}{2}\right) \geq 0$$

holds for all choices of $m \in \mathbb{N}$, $u_i,\ p_i \in \mathbb{R}$, $1 \leq i \leq m$.

Since Ω_k is also continuous, then by Proposition 1 it follows that Ω_k is exponentially convex. □

Theorem 7. *Let Ω_k be as in Theorem 6. If in addition the corresponding condition (EQ_k) is not satisfied, then Ω_k is log-convex. Therefore:*

(i) *for any $r, s, t \in \mathbb{R}$, such that $r < s < t$, we have*

$$\Omega_k(s)^{t-r} \leq \Omega_k(r)^{t-s}\, \Omega_k(t)^{s-r}\,; \tag{2.4}$$

(ii) *for any $s, t, u, v \in \mathbb{R}$, such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, we have*

$$\left(\frac{\Omega_k(t)}{\Omega_k(s)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Omega_k(v)}{\Omega_k(u)}\right)^{\frac{1}{v-u}}. \tag{2.5}$$

Proof. (i) By Theorem 6 it follows that the function Ω_k is exponentially convex.

Note that the function ϕ_t given by (2.2) is strictly convex so $F_k(\phi_t) > 0$.

Then Ω_k is positive and hence log-convex.

By Lemma 1, for $r, s, t \in \mathbb{R}$, such that $r < s < t$, we have

$$(t-s)\log\Omega_k(r) + (r-t)\log\Omega_k(s) + (s-r)\log\Omega_k(t) \geq 0,$$

which is equivalent to (2.4).

(ii) Since Ω_k is log-convex, by Lemma 2 it follows that for any s, t, u, $v \in \mathbb{R}$, such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, the inequality

$$\frac{\log\Omega_k(t) - \log\Omega_k(s)}{t-s} \leq \frac{\log\Omega_k(v) - \log\Omega_k(u)}{v-u} \tag{2.6}$$

holds. Then (2.5) easily follows from (2.6). □

Now we present integral versions of the previous results. We define a new class of functionals as follows.

Let $f : [\alpha, \beta] \to [a, b]$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $0 < a < b < +\infty$. Let $\lambda : [\alpha, \beta] \to \mathbb{R}$

be continuous or of bounded variation satisfying (1.8). We define the functionals $G_k,\ k \in \{1,2,3\}$, on $C([a,b])$ by

$$G_1(\varphi) = \varphi(a) + \varphi(b) + \varphi\left(\overline{f}\right) - 2\varphi\left(\frac{a+b}{2}\right) - \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \varphi(f(t))\mathrm{d}\lambda(t),$$

$$G_2(\varphi) = \varphi(a) + \varphi(b) - \varphi\left(a+b-\overline{f}\right) - \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \varphi(f(t))\mathrm{d}\lambda(t), \quad (2.7)$$

$$G_3(\varphi) = \varphi\left(\overline{f}\right) + \varphi\left(a+b-\overline{f}\right) - 2\varphi\left(\frac{a+b}{2}\right).$$

Notice that when φ is convex by Theorem 5 it follows that $G_k(\varphi) \geq 0$ and that all G_k are linear.

We can prove the next two theorems in a similar way as Theorems 6 and 7.

Theorem 8. *The function $\Delta_k : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Delta_k(t) = G_k(\phi_t), \quad (2.8)$$

where G_k is defined as in (2.7) *and ϕ_t as in* (2.2), *is exponentially convex.*

Theorem 9. *Let Δ_k be the function as in Theorem 8. If in addition Δ_k is positive, then Δ_k is log-convex. Therefore:*

(i) *for any $r,s,t \in \mathbb{R}$, such that $r<s<t$, we have*

$$\Delta_k(s)^{t-r} \leq \Delta_k(r)^{t-s}\,\Delta_k(t)^{s-r};$$

(ii) *for any $s,t,u,v \in \mathbb{R}$, such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, we have*

$$\left(\frac{\Delta_k(t)}{\Delta_k(s)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Delta_k(v)}{\Delta_k(u)}\right)^{\frac{1}{v-u}}.$$

3. Applications

In this section we prove Lagrange's and Cauchy's types of Mean value theorem. As consequences we introduce new means of Cauchy's type, in discrete and integral form, and prove the monotonicity of these means.

In the following we denote with e_2 the quadratic function, that is $\mathrm{e}_2 : [a,b] \to \mathbb{R}$, $\mathrm{e}_2(t) = t^2$.

Theorem 10. *Let F_k be the functional on $C([a,b])$ defined as in* (2.1). *Suppose that the corresponding condition (EQ_k) is not satisfied. If $\varphi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$F_k(\varphi) = \frac{\varphi''(\xi_k)}{2}F_k(\mathrm{e}_2), \quad (3.1)$$

Proof. Since $\varphi \in C^2([a,b])$ there exist $m = \min_{x\in[a,b]} \varphi''(x)$ and $M = \max_{x\in[a,b]} \varphi''(x)$ such that $m \le \varphi''(x) \le M$ for each $x \in [a,b]$.

We define the functions $g_1, g_2 : [a,b] \to \mathbb{R}$ by

$$g_1 = \tfrac{M}{2}\mathrm{e}_2 - \varphi \quad \text{and} \quad g_2 = \varphi - \tfrac{m}{2}\mathrm{e}_2.$$

Then $g_1, g_2 \in C^2([a,b])$ and

$$g_1''(x) = M - \varphi''(x) \ge 0 \quad \text{and} \quad g_2''(x) = \varphi''(x) - m \ge 0,$$

hence the functions g_1, g_2 are convex.

By Theorem 3 it follows that

$$\frac{M}{2}F_k(\mathrm{e}_2) - F_k(\varphi) \ge 0$$

and

$$0 \le F_k(\varphi) - \frac{m}{2}F_k(\mathrm{e}_2).$$

Since $F_k(\mathrm{e}_2) \ne 0$, by combining the last two inequalities we obtain

$$m \le \frac{2F_k(\varphi)}{F_k(\mathrm{e}_2)} \le M.$$

Since $\varphi \in C^2([a,b])$ there exists $\xi_k \in [a,b]$ such that

$$\varphi''(\xi_k) = \frac{2F_k(\varphi)}{F_k(\mathrm{e}_2)}.$$

□

Theorem 11. *Let F_k be the functional on $C([a,b])$ defined as in (2.1). Suppose that the corresponding condition (EQ_k) is not satisfied. If $\varphi, \psi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$F_k(\psi)\varphi''(\xi_k) = F_k(\varphi)\psi''(\xi_k). \tag{3.2}$$

Proof. We define the function $h_k : [a,b] \to \mathbb{R}$ by

$$h_k = F_k(\psi)\varphi - F_k(\varphi)\psi.$$

Then $h_k \in C^2([a,b])$. Applying Theorem 10 we get

$$F_k(h_k) = \tfrac{h_k''(\xi_k)}{2}F_k(\mathrm{e}_2). \tag{3.3}$$

Since $F_k(h_k) = 0$, it follows $h_k''(\xi_k) = 0$, that is

$$F_k(\psi)\varphi''(\xi_k) - F_k(\varphi)\psi''(\xi_k) = 0.$$

□

Theorem 11 enables us to define new means. If we set $a = \min_{1\le k\le n}\{x_k\}$ and $b = \max_{1\le k\le n}\{x_k\}$ and if we choose $\varphi = \phi_u$ and $\psi = \phi_v$, where $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0, 1$, then from (3.2) we obtain

$$F_k(\phi_v)\xi_k^{u-2} = F_k(\phi_u)\xi_k^{v-2},$$

that is

$$\xi_k = \left(\frac{F_k(\phi_v)}{F_k(\phi_u)}\right)^{\frac{1}{v-u}},$$

which represents a new family of means on the segment $[a, b]$. We use notation

$$M_{u,v}^k(\boldsymbol{x};\boldsymbol{w}) = \left(\frac{F_k(\phi_v)}{F_k(\phi_u)}\right)^{\frac{1}{v-u}}. \tag{3.4}$$

We can extend these means to the excluded cases. For $k \in \{1, 2, 3\}$ and $u, v \in \mathbb{R}$ we define:

$$M_{u,v}^1(\boldsymbol{x};\boldsymbol{w}) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(a^v+b^v-2\left(\frac{a+b}{2}\right)^v+\bar{x}^v-\sum\limits_{i=1}^n w_i x_i^v\right)}{\frac{1}{u(u-1)}\left(a^u+b^u-2\left(\frac{a+b}{2}\right)^u+\bar{x}^u-\sum\limits_{i=1}^n w_i x_i^u\right)}\right)^{\frac{1}{v-u}}, & u \neq v;\ u, v \neq 0, 1 \\ \exp\left(\dfrac{a^u\log a+b^u\log b-2\left(\frac{a+b}{2}\right)^u\log\left(\frac{a+b}{2}\right)+\bar{x}^u\log\bar{x}-\mathcal{C}}{a^u+b^u-2\left(\frac{a+b}{2}\right)^u+\bar{x}^u-\sum\limits_{i=1}^n w_i x_i^u}-\dfrac{2u-1}{u(u-1)}\right), & u = v \neq 0, 1 \\ \exp\left(\dfrac{\log^2 a+\log^2 b-2\log^2\left(\frac{a+b}{2}\right)+\log^2\bar{x}-\sum\limits_{i=1}^n w_i\log^2 x_i}{2\left(\log a+\log b-2\log\left(\frac{a+b}{2}\right)+\log\bar{x}-\sum\limits_{i=1}^n w_i\log x_i\right)}+1\right), & u = v = 0 \\ \exp\left(\dfrac{a\log^2 a+b\log^2 b-(a+b)\log^2\left(\frac{a+b}{2}\right)+\bar{x}\log^2\bar{x}-\mathcal{D}}{2\left(a\log a+b\log b-(a+b)\log\left(\frac{a+b}{2}\right)+\bar{x}\log\bar{x}-\sum\limits_{i=1}^n w_i x_i\log x_i\right)}-1\right), & u = v = 1 \end{cases}$$

$$M^2_{u,v}(\boldsymbol{x};\boldsymbol{w}) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(a^v+b^v-\widetilde{x}^v-\sum\limits_{i=1}^n w_i x_i^v\right)}{\frac{1}{u(u-1)}\left(a^u+b^u-\widetilde{x}^u-\sum\limits_{i=1}^n w_i x_i^u\right)}\right)^{\frac{1}{v-u}}, & u \neq v;\ u,v \neq 0,1 \\ \exp\left(\dfrac{a^u\log a+b^u\log b-\widetilde{x}^u\log\widetilde{x}-\mathcal{C}}{a^u+b^u-\widetilde{x}^u-\sum\limits_{i=1}^n w_i x_i^u} - \dfrac{2u-1}{u(u-1)}\right), & u = v \neq 0,1 \\ \exp\left(\dfrac{\log^2 a+\log^2 b-\log^2\widetilde{x}-\sum\limits_{i=1}^n w_i\log^2 x_i}{2\left(\log a+\log b-\log\widetilde{x}-\sum\limits_{i=1}^n w_i\log x_i\right)}+1\right), & u = v = 0 \\ \exp\left(\dfrac{a\log^2 a+b\log^2 b-\widetilde{x}\log^2\widetilde{x}-\mathcal{D}}{2\left(a\log a+b\log b-\widetilde{x}\log\widetilde{x}-\sum\limits_{i=1}^n w_i\log x_i\right)}-1\right), & u = v = 1 \end{cases}$$

$$M^3_{u,v}(\boldsymbol{x};\boldsymbol{w}) = \begin{cases} \left(\dfrac{\frac{1}{v(v-1)}\left(\bar{x}^v+\widetilde{x}^v-2\left(\frac{a+b}{2}\right)^v\right)}{\frac{1}{u(u-1)}\left(\bar{x}^u+\widetilde{x}^u-2\left(\frac{a+b}{2}\right)^u\right)}\right)^{\frac{1}{v-u}}, & u \neq v;\ u,v \neq 0,1 \\ \exp\left(\dfrac{\bar{x}^u\log\bar{x}+\widetilde{x}^u\log\widetilde{x}-2\left(\frac{a+b}{2}\right)^u\log\left(\frac{a+b}{2}\right)}{\bar{x}^u+\widetilde{x}^u-2\left(\frac{a+b}{2}\right)^u} - \dfrac{2u-1}{u(u-1)}\right), & u = v \neq 0,1 \\ \exp\left(\dfrac{\log^2\bar{x}+\log^2\widetilde{x}-2\log^2\left(\frac{a+b}{2}\right)}{2\left(\log\bar{x}+\log\widetilde{x}-2\log\left(\frac{a+b}{2}\right)\right)}+1\right), & u = v = 0 \\ \exp\left(\dfrac{\bar{x}\log^2\bar{x}+\widetilde{x}\log^2\widetilde{x}-(a+b)\log^2\left(\frac{a+b}{2}\right)}{2\left(\bar{x}\log\bar{x}+\widetilde{x}\log\widetilde{x}-(a+b)\log\left(\frac{a+b}{2}\right)\right)}-1\right), & u = v = 1 \end{cases}$$

where

$$\widetilde{x} = a+b-\bar{x},$$

$$\mathcal{C} = \textstyle\sum_{i=1}^n w_i x_i^u \log x_i \quad \text{and} \quad \mathcal{D} = \sum_{i=1}^n w_i x_i \log^2 x_i.$$

We can easily check that these means are symmetric and the special cases are limits of the general case. That is, we have

$$M^k_{u,u}(\boldsymbol{x};\boldsymbol{w}) = \lim_{v\to u} M^k_{u,v}(\boldsymbol{x};\boldsymbol{w}),$$

$$M^k_{0,0}(\boldsymbol{x};\boldsymbol{w}) = \lim_{u\to 0} M^k_{u,u}(\boldsymbol{x};\boldsymbol{w}),$$

$$M^k_{1,1}(\boldsymbol{x};\boldsymbol{w}) = \lim_{u\to 1} M^k_{u,u}(\boldsymbol{x};\boldsymbol{w}).$$

Notice that (3.4) can be rewritten as

$$M^k_{u,v}(\boldsymbol{x};\boldsymbol{w}) = \left(\frac{\Omega_k(v)}{\Omega_k(u)}\right)^{\frac{1}{v-u}},$$

where Ω_k is the function defined as in (2.3).

Now we prove the monotonicity of these means.

Theorem 12. *Let $s,t,u,v \in \mathbb{R}$ be such that $s \leq u$, $t \leq v$. Then*

$$M^k_{t,s}(\boldsymbol{x};\boldsymbol{w}) \leq M^k_{v,u}(\boldsymbol{x};\boldsymbol{w}). \tag{3.5}$$

Proof. By Theorem 7 it follows that the function Ω_k is log-convex. Therefore, for any $s, t, u, v \in \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, inequality (2.5) holds which is equivalent to (3.5). The statement of theorem follows using continuous extensions. □

In the following we present the integral versions of previous results. We can prove the next two theorems in a similar way as Theorems 10 and 11.

Theorem 13. *Let G_k be the functional on $C([a,b])$ defined as in* (2.7). *Suppose that $G_k(\mathrm{e}_2) \neq 0$. If $\varphi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$G_k(\varphi) = \frac{\varphi''(\xi_k)}{2} G_k(\mathrm{e}_2). \tag{3.6}$$

Theorem 14. *Let G_k be the functional on $C([a,b])$ defined as in* (2.7). *Suppose that $G_k(\mathrm{e}_2) \neq 0$. If $\varphi, \psi \in C^2([a,b])$, then there exists $\xi_k \in [a,b]$ such that*

$$G_k(\psi)\varphi''(\xi_k) = G_k(\varphi)\psi''(\xi_k). \tag{3.7}$$

If we set $\operatorname{Im} f = [a,b]$, that is $a = \min_{\alpha \leq t \leq \beta} f(t)$ and $b = \max_{\alpha \leq t \leq \beta} f(t)$, and if we choose $\varphi = \phi_u$ and $\psi = \phi_v$, where $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0, 1$, providing that $G_k(\varphi_u), G_k(\varphi_v) \neq 0$, then from (3.7) we obtain

$$\xi_k = \left(\frac{G_k(\phi_v)}{G_k(\phi_u)} \right)^{\frac{1}{v-u}}.$$

This represents a new class of integral means on the segment $[a,b]$. We use notation

$$M^k_{u,v}(f;\lambda) = \left(\frac{G_k(\phi_v)}{G_k(\phi_u)} \right)^{\frac{1}{v-u}}. \tag{3.8}$$

For $k \in \{1,2,3\}$ and $u, v \in \mathbb{R}$ we can extend these means to the excluded cases as follows:

$$M^1_{u,v}(f;\lambda)=\begin{cases}\left(\dfrac{\frac{1}{v(v-1)}\left(a^v+b^v-2\left(\frac{a+b}{2}\right)^v+\overline{f}^v-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^v(t)\mathrm{d}\lambda(t)\right)}{\frac{1}{u(u-1)}\left(a^u+b^u-2\left(\frac{a+b}{2}\right)^u+\overline{f}^u-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)\right)}\right)^{\frac{1}{v-u}}, \\ u\neq v;\ u,v\neq 0,1 \\ \exp\left(\dfrac{a^u\log a+b^u\log b-2\left(\frac{a+b}{2}\right)^u\log\left(\frac{a+b}{2}\right)+\overline{f}^u\log\overline{f}-\mathcal{E}}{a^u+b^u-2\left(\frac{a+b}{2}\right)^u+\overline{f}^u-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)}-\dfrac{2u-1}{u(u-1)}\right), \\ u=v\neq 0,1 \\ \exp\left(\dfrac{\log^2 a+\log^2 b-2\log^2\left(\frac{a+b}{2}\right)+\log^2\overline{f}-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log^2 f(t)\mathrm{d}\lambda(t)}{2\left(\log a+\log b-2\log\left(\frac{a+b}{2}\right)+\log\overline{f}-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log f(t)\mathrm{d}\lambda(t)\right)}+1\right), \\ u=v=0 \\ \exp\left(\dfrac{a\log^2 a+b\log^2 b-(a+b)\log^2\left(\frac{a+b}{2}\right)+\overline{f}\log^2\overline{f}-\mathcal{F}}{2\left(a\log a+b\log b-(a+b)\log\left(\frac{a+b}{2}\right)+\overline{f}\log\overline{f}-\mathcal{G}\right)}-1\right), \\ u=v=1\end{cases}$$

$$M^2_{u,v}(f;\lambda)=\begin{cases}\left(\dfrac{\frac{1}{v(v-1)}\left(a^v+b^v-\widetilde{f}^v-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^v(t)\mathrm{d}\lambda(t)\right)}{\frac{1}{u(u-1)}\left(a^u+b^u-\widetilde{f}^u-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)\right)}\right)^{\frac{1}{v-u}}, \\ u\neq v;\ u,v\neq 0,1 \\ \exp\left(\dfrac{a^u\log a+b^u\log b-\widetilde{f}^u\log\widetilde{f}-\mathcal{E}}{a^u+b^u-\widetilde{f}^u-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\mathrm{d}\lambda(t)}-\dfrac{2u-1}{u(u-1)}\right), u=v\neq 0,1 \\ =\exp\left(\dfrac{\log^2 a+\log^2 b-\log^2\widetilde{f}-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log^2 f(t)\mathrm{d}\lambda(t)}{2\left(\log a+\log b-\log\widetilde{f}-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \log f(t)\mathrm{d}\lambda(t)\right)}+1\right), \\ u=v=0 \\ =\exp\left(\dfrac{a\log^2 a+b\log^2 b-\widetilde{f}\log^2\widetilde{f}-\mathcal{F}}{2\left(a\log a+b\log b-\widetilde{f}\log\widetilde{f}-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(t)\log f(t)\mathrm{d}\lambda(t)\right)}-1\right), \\ u=v=1\end{cases}$$

$$M^3_{u,v}(f;\lambda)=\begin{cases}\left(\dfrac{\frac{1}{v(v-1)}\left(\overline{f}^v+\widetilde{f}^v-2\left(\frac{a+b}{2}\right)^v\right)}{\frac{1}{u(u-1)}\left(\overline{f}^u+\widetilde{f}^u-2\left(\frac{a+b}{2}\right)^u\right)}\right)^{\frac{1}{v-u}}, u\neq v;\ u,v\neq 0,1\\ \exp\left(\dfrac{\overline{f}^u\log\overline{f}+\widetilde{f}^u\log\widetilde{f}-2\left(\frac{a+b}{2}\right)^u\log\left(\frac{a+b}{2}\right)}{\overline{f}^u+\widetilde{f}^u-2\left(\frac{a+b}{2}\right)^u}-\dfrac{2u-1}{u(u-1)}\right), u=v\neq 0,1\\ \exp\left(\dfrac{\log^2\overline{f}+\log^2\widetilde{f}-2\log^2\left(\frac{a+b}{2}\right)}{2\left(\log\overline{f}+\log\widetilde{f}-2\log\left(\frac{a+b}{2}\right)\right)}+1\right), u=v=0\\ \exp\left(\dfrac{\overline{f}\log^2\overline{f}+\widetilde{f}\log^2\widetilde{f}-(a+b)\log^2\left(\frac{a+b}{2}\right)}{2\left(\overline{f}\log\overline{f}+\widetilde{f}\log\widetilde{f}-(a+b)\log\left(\frac{a+b}{2}\right)\right)}-1\right), u=v=1\end{cases}$$

where

$$\widetilde{f}=a+b-\overline{f},$$
$$\mathcal{E}=\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f^u(t)\log f(t)\mathrm{d}\lambda(t),$$
$$\mathcal{F}=\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(t)\log^2 f(t)\mathrm{d}\lambda(t),$$
$$\mathcal{G}=\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(t)\log f(t)\mathrm{d}\lambda(t).$$

We can easily check that these means are symmetric and the special cases are limits of the general case. That is,

$$M^k_{u,u}(f;\lambda)=\lim_{v\to u}M^k_{u,v}(f;\lambda),$$
$$M^k_{0,0}(f;\lambda)=\lim_{u\to 0}M^k_{u,u}(f;\lambda),$$
$$M^k_{1,1}(f;\lambda)=\lim_{u\to 1}M^k_{u,u}(f;\lambda).$$

Notice that (3.8) can be rewritten as

$$M^k_{u,v}(f;\lambda)=\left(\frac{\Delta_k(v)}{\Delta_k(u)}\right)^{\frac{1}{v-u}}$$

where Δ_k is the function defined as in (2.8).

Similary as in discrete case we can prove the monotonicity of these means which we express in the next theorem.

Theorem 15. *Let $s,t,u,v\in\mathbb{R}$ be such that $s\leq u$, $t\leq v$. Then*

$$M^k_{t,s}(f;\lambda)\leq M^k_{v,u}(f;\lambda).$$

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