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# CERTAIN BINARY RELATIONS AND OPERATIONS AND THEIR USE IN RESEARCH OF BICENTRIC POLYGONS

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Abstract. In the article we consider certain binary relations and operations and their use in research of bicentric $n$-gons where $n \geq 3$ is an odd integer. The considered binary relations and operations are defined on the set whose elements are integers $1, 2, \ldots, \frac{n-1}{2}$ which are relatively prime to $n$. We have found that some properties concerning bicentric $n$-gons can be a source or generator for many useful ideas and procedures in number theory and theory of groups. So using partition and ordering concerning bicentric $n$-gons, where $n$ is an odd integer we have found some interesting relations concerning number theory.

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## 1. Introduction

The article is closely connected with the articles [5] and [6]. The most part of the article deals with some kinds of binary relations and operations closely connected with bicentric $n$-gons where $n \geq 3$ is an odd integer. Some of the obtained results can be interesting not only in theory of bicentric $n$-gons but also in number theory and theory of groups.

First we state some results from [6] which will be used in the following.

Let $n \geq 3$ is an odd integer and let $\mathbb{S}$ denotes the set given by

$$\mathbb{S} = \left\{ x : x \in \left\{ 1, 2, \ldots, \frac{n-1}{2} \right\} \text{ and } \mathrm{GCD}(x, n) = 1 \right\}. \tag{1.1}$$

***Definition A.*** *Let $f : \mathbb{S} \to \mathbb{S}$ be function defined by*

$$f(x) = \begin{cases} 2x & \textit{if } 2x \in \mathbb{S} \\ n - 2x & \textit{if } 2x \notin \mathbb{S} \end{cases} \tag{1.2}$$

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**THEOREM A.** *The function $f$ is one to one mapping from $\mathbb{S}$ to $\mathbb{S}$.*

(It is easy to show that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. If $k \in \mathbb{S}$ is even then equation $2x = k$ has solution in $\mathbb{S}$, but if $k$ is odd then equation $k = n - 2x$ has solution in $\mathbb{S}$.)

**Corollary A. 1.** *The function $f$ determines a partition of the set $\mathbb{S}$.*

**Example A. 1.** *Let $n = 17$. Then partition of the set $\mathbb{S} = \{1, 2, \dots, 8\}$ has two cosets $\mathbb{C}_1 = \{1, 2, 4, 8\}$ and $\mathbb{C}_2 = \{3, 5, 6, 7\}$ since in this case*

$$\begin{aligned} &f(1) = 2, \quad f(2) = 4, \quad f(4) = 8, \quad f(8) = 1, \\ &f(3) = 6, \quad f(6) = 5, \quad f(5) = 7, \quad f(7) = 3. \end{aligned} \tag{1.3}$$

**Corollary A. 2.** *The function $f$ determines one (cyclic) ordering of elements in each coset.*

**Example A. 2.** *If $n = 17$ then instead of* (1.3) *we can write*

$$1 \to 2 \to 4 \to 8 \to 1, \tag{1.4a}$$

$$3 \to 6 \to 5 \to 7 \to 3. \tag{1.4b}$$

where, for brevity, instead of $f(x) = y$ we write $x \to y$.

**Example A. 3.** *Let $n = 31$. Then $\mathbb{S} = \{1, 2, \dots, 15\}$ and*

$$\begin{aligned} &1 \to 2 \to 4 \to 8 \to 15 \to 1, \\ &3 \to 6 \to 12 \to 7 \to 14 \to 3, \\ &5 \to 10 \to 11 \to 9 \to 13 \to 5. \end{aligned}$$

*The partition of $\mathbb{S}$ is $\{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3\}$ where*

$$\mathbb{C}_1 = \{1, 2, 4, 8, 15\}, \quad \mathbb{C}_2 = \{3, 6, 12, 7, 14\}, \quad \mathbb{C}_3 = \{5, 10, 11, 9, 13\}.$$

As will be seen the partition and ordering determined by function $f$ have very interesting and important properties concerning bicentric polygons.

In this connection let us remark that function $f$ defined by (1.2) can be also defined by

$$f\left(\frac{n - x}{2}\right) = x \text{ if } x \in \mathbb{S} \text{ is odd}, \tag{1.5a}$$

$$f\left(\frac{x}{2}\right) = x \text{ if } x \in \mathbb{S} \text{ is even}. \tag{1.5b}$$

So, if $n = 11$ then $\mathbb{S} = \{1, 2, 3, 4, 5\}$ and we have

$$\begin{aligned} f\left(\frac{11-1}{2}\right) &= 1 \quad \text{or} \quad 5 \to 1, \\ f\left(\frac{11-5}{2}\right) &= 5 \quad \text{or} \quad 3 \to 5, \\ f\left(\frac{11-3}{2}\right) &= 3 \quad \text{or} \quad 4 \to 3, \\ f(2) &= 4 \quad \text{or} \quad 2 \to 4, \\ f(1) &= 2 \quad \text{or} \quad 1 \to 2. \end{aligned}$$

Thus $1 \to 2 \to 4 \to 3 \to 5 \to 1$.

Since the function $f$ is one to one mapping from $\mathbb{S}$ to $\mathbb{S}$ there is the function $f^{-1}$ from $\mathbb{S}$ to $\mathbb{S}$ given by

$$f^{-1}(x) = \begin{cases} \frac{n-x}{2} & \text{if } x \text{ is an odd integer} \\ \frac{x}{2} & \text{if } x \text{ is an even integer} \end{cases}. \tag{1.6}$$

The ordering obtained using function $f^{-1}$ is opposite to the ordering obtained using function $f$. So, if $n = 11$, the ordering obtained using function $f^{-1}$ is $1 \to 5 \to 3 \to 4 \to 2 \to 1$.

Now about notation and some results concerning bicentric polygons which will be used in the article.

A polygon which is both chordal and tangential is shortly called bicentric polygon. The first one that was concerned with bicentric polygons is German mathematician Nicolaus Fuss (1755–1826). He found relations (conditions) for bicentric quadrilateral, pentagon, hexagon, heptagon and octagon given in [1] and [2].

Although Fuss found relations only for bicentric $n$-gons, $4 \leq n \leq 8$, it is in his honor to call such relations Fuss' relation also in the case $n > 8$.

The very remarkable theorem concerning bicentric polygons is given in [4] by French mathematician Poncelet (1788–1867), so called *Poncelet's closure theorem* for circles, can be stated as follows.

Let $C_1$ and $C_2$ be two circles, where $C_2$ is inside of $C_1$. If there is a bicentric $n$-gon $A_1 \dots A_n$ such that $C_1$ is its circumcircle and $C_2$ its incircle then for every point $P_1$ on $C_1$ there are points $P_1, \dots P_n$ on $C_1$ such that $P_1, \dots P_n$ is a bicentric $n$-gon whose circumcircle is $C_1$ and incircle $C_2$.

Although this celebrated Poncelet's closure theorem dates from nineteenth century, many mathematicians have been working on number of problems in connection with this theorem. In this article we deal with certain important properties and relations in this connection.

If $A_1 \ldots A_n$ is considered bicentric $n$-gon then it is usually to be used the following notation

$R$**:** radius of circumcircle of the $n$-gon $A_1 \ldots A_n$,
$r$**:** radius of incircle of the $n$-gon $A_1 \ldots A_n$,
$d$**:** distance between centers of circumcircle and incircle.

By

$$F_n^{(k)}(R,d,r)=0 \tag{1.7}$$

is denoted Fuss' relation for bicentric $n$-gons whose rotation number for $n$ is $k$, that is, it is valid

$$\sum_{i=1}^{n} \text{measure of } \measuredangle A_i M A_{i+1} = k\cdot 360,$$

where $M$ is the center of the incircle of $A_1 \ldots A_n$ and $n+1 = n(\mod n)$.

Of course, for rotation number $k$ hold relations

$$\begin{aligned} &1 \le k \le \frac{n-1}{2} \text{ if } n \text{ is odd,}\\ &1 \le k \le \frac{n-2}{2} \text{ if } n \text{ is even,} \end{aligned} \tag{1.8}$$

where $\mathrm{GCD}(k,n)=1$.

Here let us remark that instead of saying that $k$ is rotation number for $n$ we shall also say that $k$ is number of *outscription* or *circumscription* for bicentric $n$-gons. These numbers will play important role in the following.

Let $(R_k, d_k, r_k)$ be a solution of Fuss' relation (1.7) and let $\hat{R}_k$, $\hat{d}_k$, $\hat{r}_k$ be given by

$$\hat{R}_k = \frac{R_k^2-d_k^2}{2r_k},\quad \hat{d}_k = \frac{2R_k d_k r_k}{R_k^2-d_k^2}, \tag{1.9a}$$

$$\hat{r}_k = \sqrt{-(R_k^2+d_k^2-r_k^2)+\left(\frac{R_k^2-d_k^2}{2r_k}\right)^2+\left(\frac{2R_k d_k r_k}{R_k^2-d_k^2}\right)^2}. \tag{1.9b}$$

From (1.9a) it is clear that $\hat{R}_k > 0$ and $\hat{d}_k > 0$ since $R_k > d_k + r_k$. The proof that also $\hat{r}_k > 0$ can be written as

$$\left[-(R_k^2+d_k^2-r_k^2)+\left(\frac{R_k^2-d_k^2}{2r_k}\right)^2+\left(\frac{2R_k d_k r_k}{R_k^2-d_k^2}\right)^2\right]\cdot 4r_k^2(R_k^2-d_k^2)^2$$
$$= \left(R_k^2 - 2r_k^2R_k^2 - 2d_k^2R_k^2 - 2d_k^2r_k^2\right)^2.$$

***Notice*** 1. For brevity in the following we shall often write $\hat{k}$ instead of $f(k)$. Also let us remark that the relation given by (1.9) is in fact the relation (2) in [6].

Now, using some properties of the function $f$, we state the following conjecture which is slightly modified Conjecture 2.3 given in [6].

**Conjecture 1.** *Let $n \geq 3$ be an odd integer and let $k$ be an integer from the set $\mathbb{S}$ given by (1.1), that is, let $(R_k, d_k, r_k)$ be a solution of Fuss' relation (1.7). Then*

$$\left(\hat{R}_k, \hat{d}_k, \hat{r}_k\right) = \left(R_{\hat{k}}, d_{\hat{k}}, r_{\hat{k}}\right) \tag{1.10a}$$

*and*

$$F_n^{(\hat{k})}\left(R_{\hat{k}}, d_{\hat{k}}, r_{\hat{k}}\right) = 0, \tag{1.10b}$$

*where $\hat{R}_k$, $\hat{d}_k$, $\hat{r}_k$ are calculated using notation (1.9).*

First we have proved this conjecture for $n = 3, 5, 7, 9$. (See [5, Theorems 1,3,5].) Then we have proved this conjecture for $n = 11, 13, 15, 17$. For odd $n > 17$ we found that the capacity of usual (standard) computer is insufficient.

Now using partition of the set $\mathbb{S}$ determined by function $f$, the Conjecture 1 can be modified and stated as follows.

**Conjecture 2.** *Let $n \geq 3$ be any given odd integer and let $C_i$ be a coset of the partition of the set $\mathbb{S}$ determined by function $f$. Let $k_1, k_2, \ldots, k_v$ be all elements of the coset $C_i$ and let $k_1 \to k_2 \to \ldots \to k_v \to k_1$. Let $(R_{k_1}, d_{k_1}, r_{k_1})$ be solutions of Fuss' relations $F_n^{(k_1)}(R, d, r) = 0$. Then*

(1.11)

$$\left(\hat{R}_{k_1}, \hat{d}_{k_1}, \hat{r}_{k_1}\right) = \left(R_{k_2}, d_{k_2}, r_{k_2}\right), \quad \left(\hat{R}_{k_2}, \hat{d}_{k_2}, \hat{r}_{k_2}\right) = \left(R_{k_3}, d_{k_3}, r_{k_3}\right),$$

$$\ldots$$

$$\left(\hat{R}_{k_v}, \hat{d}_{k_v}, \hat{r}_{k_v}\right) = \left(R_{k_1}, d_{k_1}, r_{k_1}\right),$$

*where $v$ is the number of elements in the coset $\mathbb{C}_i$.*

So if $n = 5$ we have coset $\{1, 2\}$ where $1 \to 2 \to 1$. In accordance with $1 \to 2$ and $2 \to 1$ we have relations

$$\left(R_{\hat{1}}, d_{\hat{1}}, r_{\hat{1}}\right) = \left(R_2, d_2, r_2\right), \quad \left(R_{\hat{2}}, d_{\hat{2}}, r_{\hat{2}}\right) = \left(R_1, d_1, r_1\right).$$

If $n = 7$ we have coset $\{1, 2, 3\}$, where $1 \to 2 \to 3 \to 1$. In accordance with $1 \to 2$, $2 \to 3$, $3 \to 1$ we have relations

$$\begin{aligned}
\left(R_{\hat{1}}, d_{\hat{1}}, r_{\hat{1}}\right) &= (R_2, d_2, r_2)\,,\\
\left(R_{\hat{2}}, d_{\hat{2}}, r_{\hat{2}}\right) &= (R_3, d_3, r_3)\,,\\
\left(R_{\hat{3}}, d_{\hat{3}}, r_{\hat{3}}\right) &= (R_1, d_1, r_1)\,.
\end{aligned}$$

If $n = 17$ we have two cosets. See Example A.1 and Example A.2. Concerning coset $C_1$ we have

$$\begin{aligned}
\left(R_{\hat{1}}, d_{\hat{1}}, r_{\hat{1}}\right) &= (R_2, d_2, r_2)\,,\\
\left(R_{\hat{2}}, d_{\hat{2}}, r_{\hat{2}}\right) &= (R_4, d_4, r_4)\,,\\
\left(R_{\hat{4}}, d_{\hat{4}}, r_{\hat{4}}\right) &= (R_8, d_8, r_8)\,,\\
\left(R_{\hat{8}}, d_{\hat{8}}, r_{\hat{8}}\right) &= (R_1, d_1, r_1)\,.
\end{aligned}$$

Analogously holds for coset $C_2$.

***Notice*** 2. Since the elements of the set $\mathbb{S}$ are rotation numbers for a given odd $n \geq 3$, the function $f$ determines a partition of these numbers. Also the function $f$ determines a cyclic ordering of the elements in each coset. This partition and the ordering concerning function $f$ on the set $\mathbb{S}$ is compatible with partition and ordering determined by relation (1.9) on the set of the corresponding classes of bicentric $n$-gons for a given odd $n$. This compatibility can be very useful in researching of bicentric $n$-gons, where $n \geq 3$ is an odd integer. (An example is the article [6].)

In connection with Conjecture 2 the following question can be arisen.

If $m \geq 1$ is a given integer how can be found the set $\mathbb{G}_m$ of all odd integers such that for any two integers from $\mathbb{G}_m$ we get cosets with $m$ elements? Here will be shown how we can get the set $\mathbb{G}_m$ for $m = 3, 4, \ldots, 9$.

Let $m$ be an integer such that $3 \leq m \leq 9$ and let $n$ be an integer from the set $\mathbb{G}_m$ given by

$$\mathbb{G}_m = \{2^m - 1, 2^m + 1, D\},$$

where $D$ is the sequence of all divisor of $2^m - 1$ and all divisor of $2^m + 1$ such that none of them is less then $2m + 1$. Let $\mathbb{S}_n$ denotes the set

$$\left\{x : \; x \in \{1, \ldots, \frac{n-1}{2}\} \text{ and } \operatorname{GCD}(x, n) = 1\right\}.$$

Then partition of the set $\mathbb{S}_n$ determined by function $f$ has cosets with $m$ elements. For example:

$$\mathbb{G}_3 = \{7, 9\},\ \mathbb{G}_4 = \{15, 17\},\ \mathbb{G}_5 = \{31, 33, 11\},$$
$$\mathbb{G}_6 = \{63, 65, 13, 21\},\ \mathbb{G}_7 = \{127, 129, 43\},$$
$$\mathbb{G}_8 = \{255, 257, 51, 85\},\ \mathbb{G}_9 = \{511, 513, 19, 27, 57, 73, 171\}.$$

Concerning $m = 1$ and $m = 2$ we have $n = 3$ if $m = 1$ and $n = 5$ if $m = 2$.

The above examples strongly suggest that analogously holds generally for any integer $m > 9$. If the corresponding conjecture is a true one then there exists a partition of all odd integers $n \geq 3$ such that for any $m \geq 1$ we get one class. What can be implications of this partition to bicentric polygons it may be a theme of investigation.

## 2. Certain binary relations and operations and their use in research of bicentric polygons

First we prove the following theorem which is a true one for every odd $n \geq 3$ for which Conjecture 2 is also true.

**THEOREM 1.** *Let* $(R_{k_i}, d_{k_i}, r_{k_i})$ *and* $\left(R_{\hat{k}_i}, d_{\hat{k}_i}, r_{\hat{k}_i}\right)$ *be as in Conjecture 2. Then for each* $i = 1, 2, \ldots, v$ *it is valid*

(2.1a) $$R_{k_i} d_{k_i} = R_{\hat{k}_i} d_{\hat{k}_i}$$

(2.1b) $$(R_{k_i} + d_{k_i})^2 - r_{k_i}^2 = \left(R_{\hat{k}_i} + d_{\hat{k}_i}\right)^2 - r_{\hat{k}_i}^2,$$

(2.1c) $$(R_{k_i} - d_{k_i})^2 - r_{k_i}^2 = \left(R_{\hat{k}_i} - d_{\hat{k}_i}\right)^2 - r_{\hat{k}_i}^2.$$

*Proof.* It is easy to see that from (1.9) it follows

$$\hat{R}_k \hat{d}_k = R_k d_k$$
$$\left(\hat{R}_k + \hat{d}_k\right)^2 - \hat{r}_k^2 = (R_k + d_k)^2 - r_k^2,$$
$$\left(\hat{R}_k - \hat{d}_k\right)^2 - \hat{r}_k^2 = (R_k - d_k)^2 - r_k^2.$$

□

**Corollary 1.1.** *From* (2.1) *it follows*

$$R_{k_i}^2 + d_{k_i}^2 - r_{k_i}^2 = R_{\hat{k}_i}^2 + d_{\hat{k}_i}^2 - r_{\hat{k}_i}^2, \quad i = 1, \ldots, v.$$

Now we prove the following theorem which is a true one if and only if the Conjecture 2 is a true one. Without loss of generality we can take

$n = 17$ since essentially the same argument applies in all of the other cases.

**THEOREM 2.** *Let $n = 17$. Then we have cosets $\mathbb{C}_1 = \{1, 2, 4, 8\}$ and $\mathbb{C}_2 = \{3, 6, 5, 7\}$ where $1 \to 2 \to 4 \to 8 \to 1$ and $3 \to 6 \to 5 \to 7 \to 3$. Let*

$$(R_i, d_i, r_i),\ i = 1, 2, 4, 8 \quad \text{and} \quad (R_j, d_j, r_j),\ j = 3, 6, 5, 7 \tag{2.2}$$

*be solutions of Fuss' relation $F_{17}(R, d, r) = 0$ such that*

$$(\hat{R}_1, \hat{d}_1, \hat{r}_1) = (R_2, d_2, r_2), \quad (\hat{R}_2, \hat{d}_2, \hat{r}_2) = (R_4, d_4, r_4), \tag{2.3a}$$

$$(\hat{R}_4, \hat{d}_4, \hat{r}_4) = (R_8, d_8, r_8), \quad (\hat{R}_8, \hat{d}_8, \hat{r}_8) = (R_1, d_1, r_1) \tag{2.3b}$$

*and*

$$(\hat{R}_3, \hat{d}_3, \hat{r}_3) = (R_6, d_6, r_6), \quad (\hat{R}_6, \hat{d}_6, \hat{r}_6) = (R_5, d_5, r_5), \tag{2.4a}$$

$$(\hat{R}_5, \hat{d}_5, \hat{r}_5) = (R_7, d_7, r_7), \quad (\hat{R}_7, \hat{d}_7, \hat{r}_7) = (R_3, d_3, r_3). \tag{2.4b}$$

*Then the following relations holds good*

$$R_1^2 = R_2 \left( R_2 + r_2 + \sqrt{(R_2 + r_2)^2 - d_2^2} \right), \tag{2.5a}$$

$$d_1^2 = R_2 \left( R_2 + r_2 - \sqrt{(R_2 + r_2)^2 - d_2^2} \right), \tag{2.5b}$$

$$r_1^2 = (R_2 + r_2)^2 - d_2^2, \tag{2.5c}$$

$$R_2^2 = R_4 \left( R_4 + r_4 + \sqrt{(R_4 + r_4)^2 - d_4^2} \right), \tag{2.6a}$$

$$d_2^2 = R_4 \left( R_4 + r_4 - \sqrt{(R_4 + r_4)^2 - d_4^2} \right), \tag{2.6b}$$

$$r_2^2 = (R_4 + r_4)^2 - d_4^2, \tag{2.6c}$$

$$R_4^2 = R_8 \left( R_8 + r_8 + \sqrt{(R_8 + r_8)^2 - d_8^2} \right), \tag{2.7a}$$

$$d_4^2 = R_8 \left( R_8 + r_8 - \sqrt{(R_8 + r_8)^2 - d_8^2} \right), \tag{2.7b}$$

$$r_4^2 = (R_8 + r_8)^2 - d_8^2, \tag{2.7c}$$

(2.8a) $$R_8^2 = R_1\left(R_1 - r_1 + \sqrt{(R_1 - r_1)^2 - d_1^2}\right),$$

(2.8b) $$d_8^2 = R_1\left(R_1 - r_1 - \sqrt{(R_1 - r_1)^2 - d_1^2}\right),$$

(2.8c) $$r_8^2 = (R_1 - r_1)^2 - d_1^2.$$

*Analogously holds for the solutions* $(R_j, d_j, r_j),\ j = 3, 6, 5, 7.$

In this connection let us remark that the following rule needs to be used. If $k$ is an odd integer from the set $\{1, \ldots, 8\}$ then we have the expressions

(2.9a) $$R_k\left(R_k - r_k + \sqrt{(R_k - r_k)^2 - d_k^2}\right),$$

(2.9b) $$R_k\left(R_k - r_k - \sqrt{(R_k - r_k)^2 - d_k^2}\right),$$

(2.9c) $$(R_k - r_k)^2 - d_k^2.$$

But if $k$ is an even integer from the set $\{1, \ldots, 8\}$ then we have the following expressions

(2.10a) $$R_k\left(R_k + r_k + \sqrt{(R_k + r_k)^2 - d_k^2}\right),$$

(2.10b) $$R_k\left(R_k + r_k - \sqrt{(R_k + r_k)^2 - d_k^2}\right),$$

(2.10c) $$(R_k + r_k)^2 - d_k^2.$$

So, for example, for $k = 3$ and $k = 6$ we have relations

$$R_3^2 = R_6\left(R_6 + r_6 + \sqrt{(R_6 + r_6)^2 - d_6^2}\right),$$

$$R_6^2 = R_5\left(R_5 - r_5 + \sqrt{(R_5 - r_5)^2 - d_5^2}\right).$$

(More about this for some odd $n > 3$ can be seen Corollaries 1.3 and 3.2 in [5].)

*Proof.* First it is clear that (2.3) can be written as

$$(R_1, d_1, r_1) \to (R_2, d_2, r_2) \to (R_4, d_4, r_4) \to (R_8, d_8, r_8) \to (R_1, d_1, r_1),$$

where the arrow $\to$ replaces the word *implies*. Also from (2.5), (2.6), (2.7) and (2.8) it is clear that

$$(R_1, d_1, r_1) \leftarrow (R_2, d_2, r_2) \leftarrow (R_4, d_4, r_4) \leftarrow (R_8, d_8, r_8) \leftarrow (R_1, d_1, r_1),$$

where the arrow $\leftarrow$ replaces the words *follows from.*

Thus we have to prove that $(R_1, d_1, r_1), \dots, (R_8, d_8, r_8)$ given by (2.5), (2.6), (2.7) and (2.8) have the properties that holds (2.3). The proof is very easy. So, for example, the proof that

$$\text{the relations (2.5)} \implies (\hat{R}_1, \hat{d}_1, \hat{r}_1) = (R_2, d_2, r_2)$$

can be as follows. First we have

$$\begin{aligned}\frac{R_1^2 - d_1^2}{2r_1} &= \frac{R_2\left(R_2 + r_2 + \sqrt{(R_2 + r_2)^2 - d_2^2}\right) - R_2\left(R_2 + r_2 - \sqrt{(R_2 + r_2)^2 - d_2^2}\right)}{2\sqrt{(R_2 + r_2)^2 - d_2^2}} \\ &= R_2.\end{aligned}$$

In the same way can be found that $\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2} = d_2$ and

$$\frac{2(R_1^2 + d_1^2)r_1^2 - (R_1^2 - d_1^2)^2}{2\left(R_1^2 - d_1^2\right) r_1} = r_2$$

or $-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2 = r_2^2.$

This proves Theorem 2. □

**Corollary 2.1.** *It is valid*

$$\begin{aligned}&R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_4^2 + d_4^2 - r_4^2 = R_8^2 + d_8^2 - r_8^2,\\ &R_1 d_1 = R_2 d_2 = R_4 d_4 = R_8 d_8.\end{aligned}$$

*Analogously holds for the solutions* $(R_j, d_j, r_j),\ j = 3, 6, 5, 7.$

The following corollary refers to one relatively very simply way how using relations (1.9) can be obtained Fuss' relations.

**Corollary 2.2.** *Let* $(R_1, d_1, r_1) \in \mathbb{R}_+^3$ *such that* $R_1 > d_1 + r_1$ *and let* $(R_2, d_2, r_2)$ *be given by* $(R_2, d_2, r_2) = (\hat{R}_1, \hat{d}_1, \hat{r}_1)$*. Then from*

$$(\hat{R}_1, \hat{d}_1, \hat{r}_1) = (R_2, d_2, r_2)$$

*after rationalization and factorization we get the following relation*

$$\left(R_1^2 - d_1^2 - 2R_1 r_1\right) F_5(R_1, d_1, r_1) = 0, \tag{2.11}$$

*where* $R^2 - d^2 - 2Rr = 0$ *is Euler relation for triangle and* $F_5(R, d, r) = 0$ *is Fuss' relation for bicentric pentagons.*

This relations can be obtained more simply using one of the relations $\hat{R}_1 = R_2$, $\hat{d}_1 = d_2$, $\hat{r}_1 = r_2$. So from

$$\frac{R_1^2 - d_1^2}{2r_1} = R_1\left(R_1 - r_1 + \sqrt{(R_1 - r_1)^2 - d_1^2}\right)$$

we get the relation (2.11).

In the same way can be proceed and get Fuss' relations $F_7(R,d,r)=0$, $F_9(R,d,r)=0$ and so on. (Of course, computer capacity needs to be enough for chosen $n$. Cf. with the method given in [6] using relations (11) and (12).)

Now, let $n=17$. Let by $F_{17}^{<1>}(R,d,r)=0$ be denoted Fuss' relation for bicentric 17-gons whose rotation numbers are odd integers from the set $\{1,2,\ldots,8\}$ and let by $F_{17}^{<2>}(R,d,r)=0$ be denoted Fuss' relation for bicentric 17-gons whose rotation numbers are even integers from the set $\{1,2,\ldots,8\}$. (These relation can be obtained using relations (11) and (12) in [6].) Let for $R$ and $d$ in $F_{17}^{<1>}(R,d,r)=0$ be put $R=7$, $d=1$. Then the solutions of the equation $F_{17}^{<1>}(7,1,r)=0$ are

$$r_1=5.999949896\ldots,\qquad r_3=5.646332581\ldots,$$
$$r_5=4.117389221\ldots,\qquad r_7=1.883868466\ldots.$$

Also the solutions of the equation $F_{17}^{<2>}(7,1,r)=0$ are

$$r_2=5.958123110\ldots,\qquad r_4=5.001520087\ldots,$$
$$r_6=3.060512535\ldots,\qquad r_8=0.635878342\ldots.$$

Here let us remark that using relation (1.15) given in [5] for calculation tangent lengths can be found that 17-gons from the class $C_{17}^{(j)}(7,1,r_j),\ j=1,\ldots,8$, have rotation number $j$. Also let us remark that each triple $(R_1,s_1,t_1)$ where $R_1=7,\ s_1=1,\ t_1=r_j,\ j=1,\ldots,8$, determines one coset

$$\{(R_1,s_1,t_1),(R_2,s_2,t_2),(R_3,s_3,t_3),(R_4,s_4,t_4)\},$$

where

$$(R_2,s_2,t_2)=(\hat{R}_1,\hat{s}_1,\hat{t}_1),\qquad (R_3,s_3,t_3)=(\hat{R}_2,\hat{s}_2,\hat{t}_2),$$
$$(R_4,s_4,t_4)=(\hat{R}_3,\hat{s}_3,\hat{t}_3),\qquad (R_1,s_1,t_1)=(\hat{R}_4,\hat{s}_4,\hat{t}_4).$$

Thus can be obtained 8 cosets. Using these cosets can be verified the all relations in Theorem 1 and Theorem 2. In this connection let us remark that between triples $(7,1,r_j),\ j=1,\ldots,8$, there is no two such that $(\hat{R},\hat{d},\hat{r}_j)=(7,1,r_i)$, where $R=7$, $d=1$. Thus the obtained cosets contain 32 solutions of Fuss' relation $F_{17}(R,d,r)=0$. The half of those refer to $F_{17}^{<i>}(R,d,r)=0,\ i=1,2$.

Now will be in short about some interesting facts concerning solutions of Fuss' relation $F_n(R,d,r)=0$, where $n\geq 3$ is an odd integer, and the solutions of fuss relation $F_{2n}(R,d,r)=0$.

**i**$_1$) Let $n=3$ and let $(R_1,d_1,r_1)\in\mathbb{R}^3_+$ be a solution of Euler's relation for triangles

$$R^2-d^2-2Rr=0. \tag{2.12}$$

Let $\mathcal{R}_0$, $\delta_0$, $\rho_0$ and $\mathcal{R}_1$, $\delta_1$, $\rho_1$ be given by

$$\mathcal{R}_0^2 = R_1\left(R_1 - r_1 + \sqrt{(R_1-r_1)^2 - d_1^2}\right),$$

$$\delta_0^2 = R_1\left(R_1 - r_1 - \sqrt{(R_1-r_1)^2 - d_1^2}\right),$$

$$\rho_0^2 = (R_1 - r_1)^2 - d_1^2,$$

and

$$\mathcal{R}_1^2 = R_1\left(R_1 + r_1 + \sqrt{(R_1+r_1)^2 - d_1^2}\right),$$

$$\delta_1^2 = R_1\left(R_1 + r_1 - \sqrt{(R_1+r_1)^2 - d_1^2}\right),$$

$$\rho_1^2 = (R_1 + r_1)^2 - d_1^2.$$

Then $\mathcal{R}_0^2 - \delta_0^2 - \mathcal{R}_0\rho_0 = 0$ and $F_6(\mathcal{R}_1, \delta_1, \rho_1) = 0$.

The proof is easy.

Here let us remark that in this case when $n = 3$ it holds $(\hat{\mathcal{R}}_0, \hat{\delta}_0, \hat{\rho}_0) = (R_1, d_1, r_1)$.

**i**$_2$) Let $n = 5$ and let $(R_i, d_i, r_i)$, $i = 1, 2$, be such that

(2.13a) $$F_5^{(1)}(R_1, d_1, r_1) = 0, \quad F_5^{(2)}(R_2, d_2, r_2) = 0$$

and

(2.13b) $$(\hat{R}_1, \hat{d}_1, \hat{r}_1) = (R_2, d_2, r_2), \quad (\hat{R}_2, \hat{d}_2, \hat{r}_2) = (R_1, d_1, r_1).$$

Let $\mathcal{R}_1$, $\delta_1$, $\rho_1$ and $\mathcal{R}_3$, $\delta_3$, $\rho_3$ be given by

(2.14a) $$\mathcal{R}_1^2 = R_1\left(R_1 + r_1 + \sqrt{(R_1+r_1)^2 - d_1^2}\right),$$

(2.14b) $$\delta_1^2 = R_1\left(R_1 + r_1 - \sqrt{(R_1+r_1)^2 - d_1^2}\right),$$

(2.14c) $$\rho_1^2 = (R_1 + r_1)^2 - d_1^2,$$

(2.15a) $$\mathcal{R}_3^2 = R_2\left(R_2 - r_2 + \sqrt{(R_2-r_2)^2 - d_2^2}\right),$$

(2.15b) $$\delta_3^2 = R_2\left(R_2 - r_2 - \sqrt{(R_2-r_2)^2 - d_2^2}\right),$$

(2.15c) $$\rho_3^2 = (R_2 - r_2)^2 - d_2^2.$$

Then

(2.16) $$F_{10}^1(\mathcal{R}_1, \delta_1, \rho_1) = 0, \quad F_{10}^3(\mathcal{R}_3, \delta_3, \rho_3) = 0,$$

and

(2.17a) $\mathcal{R}_1^2 + \delta_1^2 - \rho_1^2 = \mathcal{R}_3^2 + \delta_3^2 - \rho_3^2 = R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2,$

(2.17b) $\mathcal{R}_1\delta_1 = \mathcal{R}_3\delta_3 = R_1 d_1 = R_2 d_2,$

(2.17c) $(\hat{\mathcal{R}}_1, \hat{\delta}_1, \hat{\rho}_1) = (R_1, d_1, r_1), \quad (\hat{\mathcal{R}}_3, \hat{\delta}_3, \hat{\rho}_3) = (R_2, d_2, r_2).$

*Proof.* Using computer algebra it is not difficult to prove that holds (2.16). The proof that holds (2.17) is straightforward. □

Here is an example. Let $R_1 = 7$, $d_1 = 2$, $r_1 = 4.789111662\ldots$. Then $F_5^{(1)}(R_1, d_1, r_1) = 0$, $(R_2, d_2, r_2) = (\hat{R}_1, \hat{d}_1, \hat{r}_1)$,

$\mathcal{R}_1 = 12.800443630\ldots, \quad \delta_1 = 1.093712093\ldots, \quad \rho_1 = 11.618225070\ldots,$

$\mathcal{R}_3 = 5.327840993\ldots, \quad \delta_3 = 2.627706048\ldots, \quad \rho_3 = 2.286114440\ldots.$

It is not difficult to check that (2.14)–(2.17) is valid.

**i**$_3$) Let $n = 7$ and let $(R_i, d_i, r_i) \in \mathbb{R}_+^3$, $i = 1, 2, 3$, be such that

$F_7^{(1)}(R_1, d_1, r_1) = 0, \quad F_7^{(2)}(R_2, d_2, r_2) = 0, \quad F_7^{(3)}(R_3, d_3, r_3) = 0$

and

$(\hat{R}_1, \hat{d}_1, \hat{r}_1) = (R_2, d_2, r_2),\ (\hat{R}_2, \hat{d}_2, \hat{r}_2) = (R_3, d_3, r_3),\ (\hat{R}_3, \hat{d}_3, \hat{r}_3) = (R_1, d_1, r_1).$

Let $(\mathcal{R}_i, \delta_i, \rho_i)$, $i = 1, 5, 3$, be given by

$$\mathcal{R}_1^2 = R_1\left(R_1 + r_1 + \sqrt{(R_1 + r_1)^2 - d_1^2}\right),$$
$$\delta_1^2 = R_1\left(R_1 + r_1 - \sqrt{(R_1 + r_1)^2 - d_1^2}\right),$$
$$\rho_1^2 = (R_1 + r_1)^2 - d_1^2,$$

$$\mathcal{R}_5^2 = R_2\left(R_2 - r_2 + \sqrt{(R_2 - r_2)^2 - d_2^2}\right),$$
$$\delta_5^2 = R_2\left(R_2 - r_2 - \sqrt{(R_2 - r_2)^2 - d_2^2}\right),$$
$$\rho_5^2 = (R_2 - r_2)^2 - d_2^2,$$

$$\mathcal{R}_3^2 = R_3\left(R_3 + r_3 + \sqrt{(R_3 + r_3)^2 - d_3^2}\right),$$
$$\delta_3^2 = R_3\left(R_3 + r_3 - \sqrt{(R_3 + r_3)^2 - d_3^2}\right),$$
$$\rho_3^2 = (R_3 + r_3)^2 - d_3^2.$$

Then

$$F_{14}^{(1)}(\mathcal{R}_1,\delta_1,\rho_1)=0,\ F_{14}^{(5)}(\mathcal{R}_5,\delta_5,\rho_5)=0,\ F_{14}^{(3)}(\mathcal{R}_3,\delta_3,\rho_3)=0 \tag{2.18}$$

and

$$\mathcal{R}_1^2+\delta_1^2-\rho_1^2=\mathcal{R}_5^2+\delta_5^2-\rho_5^2=\mathcal{R}_3^2+\delta_3^2-\rho_3^2=R_1^2+d_1^2-r_1^2 = R_2^2+d_2^2-r_2^2=R_3^2+d_3^2-r_3^2, \tag{2.19a}$$

$$\mathcal{R}_1\delta_1=\mathcal{R}_5\delta_5=\mathcal{R}_3\delta_3=R_1d_1=R_2d_2=R_3d_3, \tag{2.19b}$$

(2.19c)
$$(\hat{\mathcal{R}}_1,\hat\delta_1,\hat\rho_1)=(R_1,d_1,r_1),\ (\hat{\mathcal{R}}_5,\hat\delta_5,\hat\rho_5)=(R_2,d_2,r_2),\ (\hat{\mathcal{R}}_3,\hat\delta_3,\hat\rho_3)=(R_3,d_3,r_3).$$

*Proof.* Using computer algebra it is not difficult to prove that holds (2.18). The proof that holds (2.19) is straightforward. □

Here is an example. Let $R_1=7$, $d_1=2$, $r_1=4.979113505\ldots$. Then

$$F_7^{(1)}(R_1,d_1,r_1)=0,\quad (R_2,d_2,r_2)=(\hat R_1,\hat d_1,\hat r_1),\quad (R_3,d_3,r_3)=(\hat R_2,\hat d_2,\hat r_2),$$

$$\mathcal{R}_1=12.904674670\ldots,\quad \delta_1=1.084878182\ldots,\quad \rho_1=11.81097627\ldots,$$
$$\mathcal{R}_5=4.176948329\ldots,\quad \delta_5=3.351729276\ldots,\quad \rho_5=0.6874283825\ldots,$$
$$\mathcal{R}_3=5.250893089\ldots,\quad \delta_3=2.666213111\ldots,\quad \rho_3=2.544040464\ldots.$$

It is not difficult to check that (2.18)–(2.19) is valid.

In the same way can be found that analogously holds for $n=9$ and $n=11$. (If odd $n>11$ then the capacity of usual (standard) computer is insufficient.) In short about the case when $n=11$.

Let $F_{11}^{(i)}(R_i,d_i,r_i)=0$, $i=1,\ldots,5$, such that

$$(\hat R_i,\hat d_i,\hat r_i)=(R_{2\circ i},d_{2\circ i},r_{2\circ i}),\ i=1,\ldots,5,$$

where $2\circ 1=2$, $2\circ 2=4$, $2\circ 4=3$, $2\circ 3=6$, $2\circ 5=1$. (See later on Definition 3.)

Let

$$\mathcal{R}_i^2=R_i\left(R_i+r_i+\sqrt{(R_i+r_i)^2-d_i^2}\right),\quad i=1,3,5, \tag{2.20a}$$

$$\mathcal{R}_{11-i}^2=R_i\left(R_i-r_i+\sqrt{(R_i-r_i)^2-d_i^2}\right),\quad i=2,4. \tag{2.20b}$$

(For brevity writing we here omit writing $\delta_i$ and $\rho_i$, $i=1,3,5,2,4$.) Then

$$F_{22}^{(i)}(\mathcal{R}_i,\delta_i,\rho_i)=0,\quad i=1,3,5,$$
$$F_{22}^{(11-i)}(\mathcal{R}_{11-i},\delta_{11-i},\rho_{11-i})=0,\quad i=2,4.$$

The other relations are analogical to those given for $n = 7$. (Cf. (2.20a) and (2.20b) with (2.9) and (2.10).)

Now, before stating the following conjecture, we list some notation which will be used. Let $(R_{n,k}), d_{n,k}, r_{n,k}$ denotes a solution of Fuss' relation $F_n^{(k)}(R, d, r) = 0$. This solution, for brevity, will be often written as $(n, k)$. Let $R_{2n,k}, d_{2n,k}, r_{2n,k}$, if $k$ is odd, be given by

$$\text{(2.21a)} \qquad R^2_{2n,k} = R_{n,k}(R_{n,k} + r_{n,k} + \sqrt{(R_{n,k} + r_{n,k})^2 - d^2_{n,k}}),$$

$$\text{(2.21b)} \qquad d^2_{2n,k} = R_{n,k}(R_{n,k} + r_{n,k} - \sqrt{(R_{n,k} + r_{n,k})^2 - d^2_{n,k}}),$$

$$\text{(2.21c)} \qquad r^2_{2n,k} = (R_{n,k} + r_{n,k})^2 - d^2_{n,k}$$

and let $R_{2n,n-k}, d_{2n,n-k}, r_{2n,n-k}$, if $k$ is even, be given by

$$\text{(2.22a)} \qquad R^2_{2n,n-k} = R_{n,k}(R_{n,k} - r_{n,k} + \sqrt{(R_{n,k} - r_{n,k})^2 - d^2_{n,k}}),$$

$$\text{(2.22b)} \qquad d^2_{2n,n-k} = R_{n,k}(R_{n,k} - r_{n,k} - \sqrt{(R_{n,k} - r_{n,k})^2 - d^2_{n,k}}),$$

$$\text{(2.22c)} \qquad r^2_{2n,n-k} = (R_{n,k} - r_{n,k})^2 - d^2_{n,k}.$$

It is easy to see that from (2.21) it follows

$$\text{(2.23a)} \qquad R^2_{2n,k} + d^2_{2n,k} - r^2_{2n,k} = R^2_{n,k} + d^2_{n,k} - r^2_{n,k},$$

$$\text{(2.23b)} \qquad R_{2n,k}d_{2n,k} = R_{n,k}d_{n,k}$$

and from (2.22) it follows

$$\text{(2.23c)} \qquad R^2_{2n,n-k} + d^2_{2n,n-k} - r^2_{2n,n-k} = R^2_{n,k} + d^2_{n,k} - r^2_{n,k},$$

$$\text{(2.23d)} \qquad R_{2n,n-k}d_{2n,n-k} = R_{n,k}d_{n,k}.$$

Now from the above relations we get the following relations

$$\text{(2.23e)} \qquad R^2_{2n,n-k} + d^2_{2n,n-k} - r^2_{2n,n-k} = R^2_{2n,k} + d^2_{2n.k} - r^2_{2n.k}$$

$$\text{(2.23f)} \qquad R_{2n,n-k}d_{2n,n-k} = R_{2n,k}d_{2n,k}$$

Also can be easily seen that

$$\text{(2.23g)} \qquad r_{2n,k}r_{2n,n-k} = t_M t_m,$$

where

$$\text{(2.24)} \quad t^2_M = (R_{n,k} + d_{n,k})^2 - r^2_{n,k} = (R_{2n,k} + d_{2n,k})^2 - r^2_{2n,k}$$
$$= (R_{2n,n-k} + d_{2n,n-k})^2 - r^2_{2n,n-k},$$

$$(2.25)\quad t_m^2 = (R_{n,k} - d_{n,k})^2 - r_{n,k}^2 = (R_{2n,k} - d_{2n,k})^2 - r_{2n,k}^2$$
$$= (R_{2n,n-k} - d_{2n,n-k})^2 - r_{2n,n-k}^2.$$

Thus maximal and minimal tangent lengths $t_M$ and $t_m$ are the same for each of the classes

$$(R_{n,k}, d_{n,k}, r_{n,k}), (R_{2n,k}, d_{2n,k}, r_{2n,k}), (R_{2n,n-k}, d_{2n,n-k}, r_{2n,n-k}).$$

**Conjecture 3.** *Let $n \geq 3$ be a given integer and let $(R_{n,k}, d_{n,k}, r_{n,k})$ be a solution of Fuss' relation $F_n^{(k)}(R, d, r) = 0$. Then*

$$F_{2n}^{(k)}(R_{2n,k}, d_{2n,k}, r_{2n,k}) = 0, \text{ if } k \text{ is odd,}$$

*but*

$$F_{2n}^{(n-k)}(R_{2n,n-k}, d_{2n,n-k}, r_{2n,n-k}) = 0, \text{ if } k \text{ is even.}$$

This conjecture is easy to prove for the case when $d = 0$. The proof is as follows.

Without loss of generality we can take $R_{n,k} = 1$. Then

$$r_{n,k} = \cos\frac{360k}{2n}, \quad t_{n,k} = \sin\frac{360k}{2n}.$$

Using formula (2.21) we have

$$R_{2n,k}^2 = 2\left(1 + \cos\frac{360k}{2n}\right), \quad r_{2n,k} = 1 + \cos\frac{360k}{2n},$$

from which it follows

$$t_{2n,k}^2 = R_{2n,k}^2 - r_{2n,k}^2 = \sin^2\frac{360k}{2n} \text{ or } t_{2n,k} = t_{n,k}.$$

Thus we have

$$(2.26)\qquad \frac{t_{2n,k}}{r_{2n,k}} = \tan\frac{360k}{4n}.$$

Now using formulae (2.22) we get

$$R_{2n,n-k}^2 = 2\left(1 - \cos\frac{360k}{2n}\right), \quad r_{2n,n-k} = 1 - \cos\frac{360k}{2n},$$

from which it follows that also $t_{2n,n-k} = t_{n,k}$. Thus , in this case we have

$$(2.27)\qquad \frac{t_{2n,n-k}}{r_{2n,n-k}} = \cot\frac{360k}{4n}.$$

Here we use the following fact. If $0 < \alpha < 90$, then $\arctan\alpha + \operatorname{arccot}\alpha = 90$. Thus $\arctan\frac{360k}{4n} + \operatorname{arccot}\frac{360k}{4n} = 90$, from which it follows

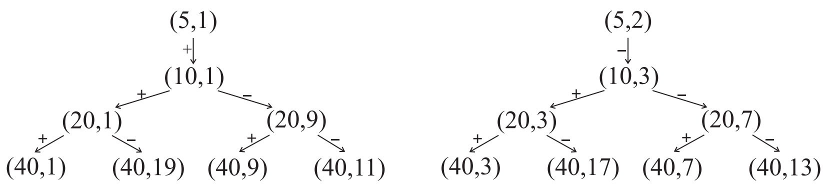
$$\operatorname{arccot}\frac{360k}{4n} = \frac{360(n-k)}{4n}.$$

Rad Hrvat. akad. znan. umjet. 515 Matematičke znanosti 17 (2013), str. 151-180


Connecting this conjecture with Conjecture 2 we get analogical situation as in the considered cases when $n = 5, 7, 9, 11$. So,

$$(\hat{R}_{2n,k}, \hat{d}_{2n,k}, \hat{r}_{2n,k}) = (\hat{R}_{2n,n-k}, \hat{d}_{2n,n-k}, \hat{r}_{2n,n-k}) = (R_{n,k}, d_{n,k}, r_{n,k}).$$

Here is an example. (Notation $(n, d)$ will be used.) Let $n = 5$. Then




and so on. and so on.

The arrow with symbol + denotes that relations (2.21) are used and the arrow with symbol − denotes that relations (2.22) are used.

Now will be something more about partition and ordering of rotation numbers.

***Definition* 1.** *Let* $\mathbb{S}$ *be the set given by* (1.1) *and let by* $\varrho$ *be denoted binary relation on* $\mathbb{S}$ *defined as follows. Let* $x$ *and* $y$ *be any given element from* $\mathbb{S}$. *Then*

$$x \,\varrho\, y \quad \text{if and only if} \quad f(x) = y.$$

For example, if $n = 11$ then $5 \,\varrho\, 1$ since $f(5) = 1$.

***Definition* 2.** *Let by* $\hat{\varrho}$ *be denoted binary relation of* $\mathbb{S}$ *defined as follows. Let* $x$ *and* $y$ *be any given elements from* $\mathbb{S}$. *Then*

$$x \,\hat{\varrho}\, y$$

*if and only if there are elements* $x_1, \dots, x_k, x_{k+1}$ *from* $\mathbb{S}$ *such that*

$$x \,\varrho\, x_1,\ x_1 \,\varrho\, x_2,\ \dots, x_k \,\varrho\, x_{k+1},\ x_{k+1} \,\varrho\, y.$$

For example, if $n = 11$ then $3 \,\hat{\varrho}\, 2$ since $3 \,\varrho\, 4$, $4 \,\varrho\, 3$, $3 \,\varrho\, 5$, $5 \,\varrho\, 1$, $1 \,\varrho\, 2$.

From Corollary A.1 it is easy to see that $x \,\hat{\varrho}\, y$ for each element $y$ from the coset $C_x$, where $C_x$ is the coset which contains the element $x$. Thus $\hat{\varrho}$ is an equivalence relation on $\mathbb{S}$. This relation determines the same partition of the set $\mathbb{S}$ as the function $f$.

It seems to be reasonably to investigate validity of the following conjectures(denoted by $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$).

$\mathbf{j}_1$) Let $n \geq 7$ be a prime number and let $\{\mathbb{C}_1, \dots, \mathbb{C}_m\}$ be partition of the set $\{1, 2, \dots, \frac{n-1}{2}\}$ determined by function $f$. Let $\mathbb{C}_i = \{a_1, a_2, \dots, a_v\}$ be a coset of this partition such that

$$f(a_1) = a_2,\ f(a_2) = a_3,\ \dots, f(a_v) = a_1.$$

Then

$$n \mid \left(a_1^2 + a_2^2 + \cdots + a_v^2\right), \tag{2.28}$$

$$n \mid \left((a_1a_2)^2 + (a_2a_3)^2 + \cdots + (a_va_1)^2\right). \tag{2.29}$$

Here let us remark that ordering $a_1 \to a_2 \to \cdots \to a_v \to a_1$ determined by function $f$ is a necessary condition. So, if $n = 11$, then

$$(1 \cdot 2)^2 + (2 \cdot 4)^2 + (4 \cdot 3)^2 + (3 \cdot 5)^2 + (5 \cdot 1)^2 = 462 = 11 \cdot 42,$$

but

$$(1 \cdot 2)^2 + (2 \cdot 3)^2 + (3 \cdot 4)^2 + (4 \cdot 5)^2 + (5 \cdot 1)^2 = 609,$$

and 609 is not divisible by 11.

**j**$_2$) Let $\{1, a_1, a_2, \ldots, a_u\}$ be a coset which contain integer 1 and

$$1 \to a_1 \to a_2 \to \cdots \to a_u \to 1.$$

Then

$$\text{either } n \mid \left((a_1a_2\cdots a_u)^2 + 1\right) \text{ or } n \mid \left((a_1a_2\cdots a_u)^2 - 1\right).$$

**j**$_3$) Let $\mathbb{C}_i = \{a_{i1}, a_{i2}, \ldots, a_{iv}\}$, $i = 1, \ldots, k$, be cosets obtained for a prime $n$, where $a_{i1} \to a_{i2} \to \cdots a_{iv} \to a_{i1}$, $i = 1, \ldots, k$. Then

$$n \mid \left((a_{11}a_{21})^2 + (a_{12}a_{22})^2 + \cdots + (a_{1v}a_{2v})^2\right),$$
$$n \mid \left((a_{11}a_{21}a_{31})^2 + (a_{12}a_{22}a_{32})^2 + \cdots + (a_{1v}a_{2v}a_{3v})^2\right),$$

and so on.

We have found that the above conjectures are true for many prime numbers.

Here are some examples.

***Example*** 1. Let $n = 17$. (See Example A1.) Then

$$17 \mid \left(1^2 + 2^2 + 4^2 + 8^2\right), \qquad 17 \mid \left(3^2 + 6^2 + 5^2 + 7^2)\right),$$
$$17 \mid \left((1 \cdot 2)^2 + (2 \cdot 4)^2 + (4 \cdot 8)^2 + (8 \cdot 1)^2\right),$$
$$17 \mid \left((3 \cdot 6)^2 + (6 \cdot 5)^2 + (5 \cdot 7)^2 + (7 \cdot 3)^2\right),$$
$$17 \mid \left((1 \cdot 3)^2 + (2 \cdot 6)^2 + (4 \cdot 5)^2 + (8 \cdot 7)^2\right).$$

***Example*** 2. Let $n = 31$. (See Example A3.) Then

$$\begin{aligned}
&31 \big| \left(1^2 + 2^2 + 4^2 + 8^2 + 15^2\right), \quad 31 \big| \left(3^2 + 6^2 + 12^2 + 7^2 + 14^2)\right), \\
&31 \big| \left(5^2 + 10^2 + 11^2 + 9^2 + 13^2\right), \\
&31 \big| \left((1 \cdot 2)^2 + (2 \cdot 4)^2 + (4 \cdot 8)^2 + (8 \cdot 15)^2 + (15 \cdot 1)^2\right), \\
&31 \big| \left((3 \cdot 6)^2 + (6 \cdot 12)^2 + (12 \cdot 7)^2 + (7 \cdot 14)^2) + (14 \cdot 3)^2\right), \\
&31 \big| \left((5 \cdot 10)^2 + (10 \cdot 11)^2 + (11 \cdot 9)^2 + (9 \cdot 13)^2) + (13 \cdot 5)^2\right), \\
&31 \big| \left((1 \cdot 3)^2 + (2 \cdot 6)^2 + (4 \cdot 12)^2 + (8 \cdot 7)^2 + (15 \cdot 14)^2\right), \\
&31 \big| \left((3 \cdot 5)^2 + (6 \cdot 10)^2 + (12 \cdot 11)^2 + (7 \cdot 9)^2) + (14 \cdot 13)^2\right), \\
&31 \big| \left((1 \cdot 5)^2 + (2 \cdot 10)^2 + (4 \cdot 11)^2 + (8 \cdot 9)^2) + (15 \cdot 13)^2\right).
\end{aligned}$$

***Example*** 3. Let $n = 19$. Here is only one coset and it is valid

$$1 \to 2 \to 4 \to 8 \to 3 \to 6 \to 7 \to 5 \to 9 \to 1,$$

$$19 \big| \left(1^2 + 2^2 + 4^2 + 8^2 + 3^2 + 6^2 + 7^2 + 5^2 + 9^2\right), \tag{2.30}$$

$$\begin{aligned}
&19 \big| \big((1 \cdot 2)^2 + (2 \cdot 4)^2 + (4 \cdot 8)^2 + (8 \cdot 3)^2 + (3 \cdot 6)^2 \\
&\quad + (6 \cdot 7)^2 + (7 \cdot 5)^2 + (5 \cdot 9)^2 + (9 \cdot 1)^2\big).
\end{aligned} \tag{2.31}$$

***Definition 3.*** *Let on the set $\mathbb{S}$ be defined binary operation $\circ$ in the following way. Let $a$ and $b$ be any given element from $\mathbb{S}$ and let $ab = nq + r$, where $q \geq 0$ and $0 < r < n$. Then*

$$a \circ b = r \text{ if } r \leq \frac{n-1}{2}, \tag{2.32a}$$

$$a \circ b = n - r \text{ if } r > \frac{n-1}{2}. \tag{2.32b}$$

For example, let $n = 17$. Then we have cosets $\{1, 2, 4, 8\}$ and $\{3, 6, 5, 7\}$ and it is valid

| ∘ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 8 |
| 2 | 2 | 4 | 8 | 1 |
| 4 | 4 | 8 | 1 | 2 |
| 8 | 8 | 1 | 2 | 4 |

| ∘ | 3 | 6 | 5 | 7 |
|---|---|---|---|---|
| 3 | 8 | 1 | 2 | 4 |
| 6 | 1 | 2 | 4 | 8 |
| 5 | 2 | 4 | 8 | 1 |
| 7 | 4 | 8 | 1 | 2 |

| ∘ | 1 | 2 | 4 | 8 | 3 | 6 | 5 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 8 | 3 | 6 | 5 | 7 |
| 2 | 2 | 4 | 8 | 1 | 6 | 5 | 7 | 3 |
| 4 | 4 | 8 | 1 | 2 | 5 | 7 | 3 | 6 |
| 8 | 8 | 1 | 2 | 4 | 7 | 3 | 6 | 5 |
| 3 | 3 | 6 | 5 | 7 | 8 | 1 | 2 | 4 |
| 6 | 6 | 5 | 7 | 3 | 1 | 2 | 4 | 8 |
| 5 | 5 | 7 | 3 | 6 | 2 | 4 | 8 | 1 |
| 7 | 7 | 3 | 6 | 5 | 4 | 8 | 1 | 2 |

**THEOREM 3.** *The binary operation $\circ$ is commutative and associative.*

*Proof.* Let $a, b, c$ be any given element from $\mathbb{S}$. The proof that $a \circ b = b \circ a$ follows from the equality $ab = ba$ and Definition 3. The proof that $(a \circ b) \circ c = a \circ (b \circ c)$ follows from relations (for usual multiplication and addition)

$$(ab)c = a(bc) = nq + r, \quad ab = nq_1 + r_1, \quad bc = nq_2 + r_2,$$
$$(nq_1 + r_1)c = nq + r, \quad a(nq_2 + r_2) = nq + r,$$

where $q, q_1, q_2$ are nonnegative integers and $r, r_1, r_2$ are positive integers less than $n$. Namely, from these relations it follows

$$r_1 c \equiv a r_2 (\mathrm{mod}\ n),$$

that is, $r_1 c = nq_3 + r_3$, $ar_2 = nq_4 + r_3$ where $q_3$ and $q_4$ are nonnegative integers and $r_3$ is a positive integer less than $n$. Thus

$$(a \circ b) \circ c = a \circ (b \circ c) = r_3 \qquad \text{if} \quad r_3 \leq \frac{n-1}{2},$$
$$(a \circ b) \circ c = a \circ (b \circ c) = n - r_3 \quad \text{if} \quad r_3 > \frac{n-1}{2}.$$

□

**THEOREM 4.** *Let $a$ and $b$ be any given elements from $\mathbb{S}$. Then*

$$a \to b \text{ if and only if } b = 2 \circ a.$$

*Proof.* If $2 \circ a = 2a$ then by Definition A we have $a \to 2a$. But if $2 \circ a > \frac{n-1}{2}$ then $a \to n - 2a$. □

**Corollary 4.1.** *Let $a$ and $b$ be any given elements from $\mathbb{S}$ such that*

$$a \to b.$$

*Then for each element $k$ from $\mathbb{S}$ it is valid*

$$k \circ a \to k \circ b.$$

*Proof.* By Theorem 4 and Theorem 3 it is valid

$$k \circ a \to 2 \circ (k \circ a) = k \circ (2 \circ a) = k \circ b.$$

□

***Definition* 4.** *Let $n \geq 3$ be an odd integer. Let $\mathbb{C} = \{a_1, a_2, \ldots, a_v\}$ be a coset obtained starting from $n$ and using function $f$. Let $k$ be any given integer from the set $\mathbb{S}$. Then the product $k \circ \mathbb{C}$ is given by*

$$k \circ \mathbb{C} = \{k \circ a_1, \ldots, k \circ a_v\}.$$

**THEOREM 5.** *Let $\mathbb{C}_1 = \{a_1, a_2, \ldots, a_v\}$ and $\mathbb{C}_2 = \{b_1, b_2, \ldots, b_v\}$ be any given two cosets obtained starting from a prime number $n \geq 5$. If $\mathbb{C}_1$ is the coset which contain integer 1, say, $a_1 = 1$, then for each $i = 1, \ldots, v$ it is valid*

$$a_i \circ \{a_1, \ldots, a_v\} = \{a_1, \ldots, a_v\},$$
$$b_i \circ \{a_1, \ldots, a_v\} = \{b_1, \ldots, b_v\}.$$

*Proof.* By Definition 4 we have that

$$a_i \circ \{1, a_2, \ldots, a_v\} = \{a_i, a_i \circ a_2, \ldots, a_i \circ a_v\}$$

and by Corollary 4.1 it is valid

$$a_i \to a_i \circ a_2 \to \cdots \to a_i \circ a_v.$$

From this, by properties of function $f$, it is clear that

$$a_i \circ \{1, a_2, \ldots, a_v\} = \{a_1, a_2, \ldots, a_v\}.$$

In the same way it can be concluded that the second assertion of Theorem 5 also holds good. □

For example, let $n = 31$. Then we have cosets
(2.33)

$$\mathbb{C}_1 = \{1, 2, 4, 8, 15\},\ \mathbb{C}_2 = \{3, 6, 12, 7, 14\},\ \mathbb{C}_3 = \{5, 10, 11, 9, 13\}$$

and it is valid

$$15 \circ \{1, 2, 4, 8, 15\} = \{1, 2, 4, 8, 15\},$$
$$3 \circ \{1, 2, 4, 8, 15\} = \{3, 6, 12, 7, 14\},$$
$$5 \circ \{1, 2, 4, 8, 15\} = \{5, 10, 11, 9, 13\}.$$

**Corollary 5.1.** *Let $\mathbb{C}_1$ be coset which contains integer 1. Then $(\mathbb{C}_1, \circ)$ is an Abelian group.*

***Definition* 5.** *Let $n$ and $k$ be as in Definition 4. Let $\mathbb{C}_1 = \{a_1, a_2, \ldots, a_v\}$ and $\mathbb{C}_2 = \{b_1, b_2, \ldots, b_v\}$ be any given cosets obtained starting from prime $n$ and using function $f$. Then product $\mathbb{C}_1 \circ \mathbb{C}_2$ is given by*

$$\mathbb{C}_1 \circ \mathbb{C}_2 = \{a_1 \circ b_1, \ldots, a_1 \circ b_v, \ldots, a_v \circ b_1, \ldots, a_v \circ b_v\}.$$

For example, let $n = 17$. Then we have cosets $\mathbb{C}_1 = \{1, 2, 4, 8\}$ and $\mathbb{C}_2 = \{3, 6, 5, 7\}$ and it is valid

| $\circ$ | $\mathbb{C}_1$ | $\mathbb{C}_2$ |
|---|---|---|
| $\mathbb{C}_1$ | $\mathbb{C}_1$ | $\mathbb{C}_2$ |
| $\mathbb{C}_2$ | $\mathbb{C}_2$ | $\mathbb{C}_1$ |

Here is one more example. Let $n = 31$. Then we have cosets given by (2.33) and it is valid

| $\circ$ | $\mathbb{C}_1$ | $\mathbb{C}_2$ | $\mathbb{C}_3$ |
|---|---|---|---|
| $\mathbb{C}_1$ | $\mathbb{C}_1$ | $\mathbb{C}_2$ | $\mathbb{C}_3$ |
| $\mathbb{C}_2$ | $\mathbb{C}_2$ | $\mathbb{C}_3$ | $\mathbb{C}_1$ |
| $\mathbb{C}_3$ | $\mathbb{C}_3$ | $\mathbb{C}_1$ | $\mathbb{C}_2$ |

**THEOREM 6.** *Let $n \geq 5$ be an odd integer with property that each coset obtained starting from $n$ has the same number of elements. Let*

$$\mathbb{C}_1 = \{a_1, \ldots, a_v\}, \quad \mathbb{C}_2 = \{b_1, \ldots, b_v\}$$

*be any given cosets obtained starting from $n$ and let*

$$a_1 \to a_2 \to \cdots \to a_v \to a_1, \quad b_1 \to b_2 \to \cdots \to b_v \to b_1.$$

*Then for each $i = 1, \ldots, v$ we get the same coset*

$$a_i \circ \{b_1, b_2, \ldots, b_v\}.$$

*Proof.* By Corollary 4.1 we can write

$$\begin{aligned}
&a_1 \circ b_1 \to a_1 \circ b_2 \to \cdots \to a_1 \circ b_v, \\
&a_2 \circ b_1 \to a_2 \circ b_2 \to \cdots \to a_2 \circ b_v, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_v \circ b_1 \to a_v \circ b_2 \to \cdots \to a_v \circ b_v.
\end{aligned}$$

Now from the element $a_1 \circ b_1$ in the first row and the element $a_2 \circ b_1$ in the second row, since $a_1 \to a_2$, can be concluded that

$$a_1 \circ b_1 \to a_2 \circ b_1.$$

From this, by properties of function $f$, it is clear that the coset which refers to the first row is the same as the coset which refers to the second row. In the same way can be concluded that the coset which refers to the second row is the same as the coset which refers to the third row. And so on.

This proves Theorem 6. □

Now can be easily seen that the following corollary of Theorem 6 is also true.

**Corollary 6.1.** *Let $n$ be as in Theorem 6 and let $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m$ be all cosets obtained starting from this $n$. Then*

$$(\{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m\}, \circ)$$

*is an Abelian group.*

***Notice*** 3. We have calculated all cosets for each odd $n$, prime and not prime, between 2 and 1000. For each prime integer we found that each coset has the same number of elements. Also we found that the same holds for each odd $n$ between 2 and 1000 if from the set $\{1, 2, \ldots, \frac{n-1}{2}\}$ are eliminated elements, which are not relatively prime to $n$. We believe that every odd $n$ has this property and that this can be proved. In this connection, here, in short, concerning cosets whose elements are not relatively prime to $n$.

Concerning elements which are not relatively prime to $n$ we prove the following theorem.

**THEOREM 7.** *Let $n \geq 3$ be any given odd integer and let $Q$ be the set given by*

$$\mathbb{Q} = \left\{ x : x \in \left\{ 1, 2, \ldots, \frac{n-1}{2} \right\} \text{ and } \mathrm{GCD}(x, n) > 1 \right\}. \tag{2.34}$$

*Let $f_1 : \mathbb{Q} \to \mathbb{Q}$ be mapping given by*

$$f_1(x) = 2x \text{ if } 2x \in \mathbb{Q}, \text{ but } f_1(x) = n - 2x \text{ if } 2x \notin \mathbb{Q}.$$

*Then function $f_1$ determines a partition of the set $\mathbb{Q}$.*

*Proof.* It is easy to see that

$$\mathrm{GCD}(x, n) > 1 \implies \mathrm{GCD}(2x, n) > 1,$$
$$\mathrm{GCD}(x, n) > 1 \implies \mathrm{GCD}(n - 2x, n) > 1.$$

□

Let, for example, $n = 63$. Then we have cosets

$\mathbb{C}_1 = \{1, 2, 4, 8, 16, 31\}$ where $1 \to 2 \to 4 \to 8 \to 16 \to 31 \to 1$,

$\mathbb{C}_2 = \{5, 10, 20, 23, 17, 29\}$ where $5 \to 10 \to 20 \to 23 \to 17 \to 29 \to 5$,

$\mathbb{C}_3 = \{11, 22, 19, 25, 13, 26\}$ where $11 \to 22 \to 19 \to 25 \to 13 \to 26 \to 11$, whose elements are relatively prime to 63 and the following cosets

$\mathbb{C}_4 = \{7, 14, 28\}$ where $7 \to 14 \to 28 \to 7$,

$\mathbb{C}_5 = \{3, 6, 12, 24, 15, 30\}$ where $3 \to 6 \to 12 \to 24 \to 15 \to 30 \to 3$,

$\mathbb{C}_6 = \{9, 18, 27\}$ where $9 \to 18 \to 27 \to 9$,

$\mathbb{C}_7 = \{21\}$ where $21 \to 21$,

whose elements are not relatively prime to 63.

**THEOREM 8.** *Let $n \geq 3$ be any given odd integer and let $\mathbb{T}$ be the set given by*

$$\mathbb{T} = \left\{1, 2, \dots, \frac{n-1}{2}\right\}.$$

*Let $f_2 : \mathbb{T} \to \mathbb{T}$ be mapping such that*

$$f_2(x) = \begin{cases} 2x & \text{if } 2x \in \mathbb{T}, \\ n - 2x & \text{if } 2x \notin \mathbb{T}. \end{cases}$$

*Then $f_2$ determines a partition of the set $\mathbb{T}$.*

*Proof.* It follows from Theorem A.1 and Theorem 7 since

$$f_2(x) = f(x) \text{ if } x \in \mathbb{S}, f_2(x) = f_1(x) \text{ if } x \in \mathbb{Q}.$$

□

**Corollary 8.1.** *If partition of the set $\mathbb{T}$ determined by function $f_2$ has only one coset then $n$ is a prime number.*

Of course, conversely is not always valid and the following conjecture is strongly suggested.

**j**$_4$) If the partition of the set $\mathbb{T}$ determined by function $f_2$ has the property that each coset has the same number of elements then $n$ is a prime number. (Before we have state conjecture $j_1$, $j_2$, $j_3$.)

We have found that this conjecture is a true one for each odd $n$ between 2 and 200.

**THEOREM 9.** *Let $n \geq 5$ be any given prime number and let $h$ be given by*

$$h = \frac{n-1}{2}.$$

*Let the partition of the set $\mathbb{S}$ determined by function $f$ has only one coset and let it be denoted by $\mathbb{C}_1$. Then this coset can be written as $\mathbb{C}_1 = \{1, 2, 2^2, \dots, 2^{h-1}\}$ and ordering of its elements is given by*

$$1 \to 2 \to 2^2 \to \cdots \to 2^{h-1} \to 2^h, \text{ where } 2^h = 1.$$

*Proof.* The proof is easy and can be as follows. By Definition 3 we have

$$2^k \to 2 \circ 2^k = 2^{k+1} \text{ if } 2^{k+1} \leq \frac{n-1}{2},$$

$$2^k \to n - 2^{k+1} \text{ if } 2^{k+1} > \frac{n-1}{2}.$$

Thus in each case $2^k \to 2^{k+1}$ since in the second case by Definition 3 we have $2^{k+1} = n - 2^{k+1}$. □

So, if $n = 11$ then $1 \to 2 \to 2^2 \to 2^3 \to 2^4 \to 2^5$ since $2^3 = 3$, $2^4 = 5$, $2^5 = 1$.

**Corollary 9.1.** *If there is no integer $j > 1$ such that $2^{h/j} = 1$ then we get only one coset. But if there is an integer $j > 1$ such that $2^{h/j} = 1$ then we get more then one coset.*

For example, if $n = 13$ then there is no integer $j > 1$ such that $2^{6/j} = 1$. But if $n = 17$ then there is $j = 2$ such that $2^{8/2} = 1$.

***Notice*** 4. It is easy to see that analogously holds for odd $n$ which is not a prime. So in this case, if by $q$ is denoted the number of elements from the set $\{1, 2, \dots, \frac{n-1}{2}\}$ which are not relatively prime to $n$, then instead of relations $2^h = 1$ and $2^{h/j} = 1$ we have relations

$$2^{h-q} = 1, \quad 2^{(h-q)/j} = 1.$$

For example, if $n = 9$ then $q = 1$ and $2^{4-1} = 1$ (since $2 \circ 2 \circ 2 = 4 \circ 2 = 9 - 8 = 1$). If $n = 15$ then $q = 3$ and $2^{7-3} = 1$. If $n = 63$, then $q = 13$ and $j = 3$. In this case we have $2^{\frac{31-13}{3}} = 2^6 = 1$.

The end of the article we shall finish by establishing some groups whose elements are certain classes of bicentric $n$-gons.

Using operation $\circ$ given by (2.32) the following group concerning bicentric $n$-gons can be defined. For brevity writing and without loss of generality we can take $n = 17$. Then coset which contain integer 1 is given by $\mathbb{C}_1 = \{1, 2, 4, 8\}$ where $1 \to 2 \to 4 \to 8 \to 1$. Let $(R_1, d_1, r_1)$ be a solution of Fuss' relation $F_{17}^{(1)} = 0$ and let $(R_i, d_i, r_i)$, $i = 2, 4, 8$ be given by

$$(R_2, d_2, r_2) = (\hat{R}_1, \hat{d}_1, \hat{r}_1),$$

$$(R_4, d_4, r_4) = (\hat{R}_2, \hat{d}_2, \hat{r}_2),$$

$$(R_8, d_8, r_8) = (\hat{R}_4, \hat{d}_4, \hat{r}_4),$$

where we used notation given by (1.9).

Thus, each of the triples $(R_i, d_i, r_i)$, $i = 2, 4, 8$ is determined by triple $(R_1, d_1, r_1)$. Let, for brevity, by $T_i$ be denoted triple $(R_i, d_i, r_i)$, $i =$

$1, 2, 4, 8$ and let in the set $\{T_1, T_2, T_4, T_8\}$ be defined binary operation $\Delta$ given by

$$T_i \,\Delta\, T_j = T_{i\circ j},\ i, j = 1, 2, 4, 8.$$

Then $(\{T_1, T_2, T_4, T_8\}, \Delta)$ becomes a group isomorphic with the group $(\{1, 2, 4, 8\}, \circ)$. The isomorphism is given by $k \to T_k,\ k = 1, 2, 4, 8$. This property can be interesting since $(R_i, d_i, r_i),\ i = 1, 2, 4, 8$, are classes of bicentric 17-gons relevant to coset $\mathbb{C}_1$. The class determined by triple $(R_i, d_i, r_i)$ has the property that

$R_i$**:** radius of circumcircle of the class,
$r_i$**:** radius of incircle of the class,
$d_i$**:** distance between centers of circumcircle and incircle.

Also let us remark that this property is in some way connected with Conjecture 2.

Concerning groups $(\{T_1, T_2, T_4, T_8\}, \Delta)$ and $(\{1, 2, 4, 8\}, \circ)$ the following group may also be interesting. Let $(R_1, d_1, r_1)$ be a solution of Fuss' relation $F_{17}(R, d, r) = 0$ such that $C^1_{17}(R_1, d_1, r_1)$ is a class of bicentric 17-gons whose rotation number is 1. Let $t_M$ and $t_m$ be given by

$$(2.35)\qquad t_M = \sqrt{(R_1 + d_1)^2 - r_1^2}, \qquad t_m = \sqrt{(R_1 - d_1)^2 - r_1^2}.$$

As can be easily seen, $t_M$ and $t_m$ are maximal and minimal tangent lengths of the class $C^1_{17}(R_1, d_1, r_1)$. Thus, for every length $t_1$ such that $t_M \geq t_1 \geq t_m$ there is a bicentric 17-gon whose first tangent has the length $t_1$. Now let $(R_2, d_2, r_2)$, $(R_4, d_4, r_4)$ and $(R_8, d_8, r_8)$ be such that holds relation (1.9), that is

$$(R_2, d_2, r_2) = (\hat{R}_1, \hat{d}_1, \hat{r}_1), (R_4, d_4, r_4) = (\hat{R}_2, \hat{d}_2, \hat{r}_2), (R_8, d_8, r_8) = (\hat{R}_4, \hat{d}_4, \hat{r}_4)$$

Then $C^k_{17}(R_k, d_k, r_k),\ k = 2, 4, 8$ are classes of bicentric 17-gons whose rotation numbers are $2, 4, 8$. From relation (2.1) in Theorem 1 it follows that also as in the case (2.35) we have

$$(2.36)\qquad \sqrt{(R_i + d_i)^2 - r_i^2} = t_M \text{ and } \sqrt{(R_i - d_i)^2 - r_i^2} = t_m \text{ for each } i = 2, 4, 8.$$

From relations (2.35) and (2.36) it is clear that there are bicentric 17-gons $P^1_{17}, P^2_{17}, P^4_{17}, P^8_{17}$ such that first tangent in each of them has the length $t_1$. Using computer algebra we have found that for numerous examples the following is valid. If $t_1, t_2, \ldots, t_{17}$ are tangent lengths of the 17-gon $P^1_{17}$ whose rotation number is 1, then tangent lengths of the 17-gon $P^k_{17}$ whose rotation number is $k \in \{2, 4, 8\}$, is given by

$$(2.37)\qquad t_1, t_{1+k}, t_{1+2k}, \ldots, t_{1+16k}.$$

It seems that this property can be proved using relation (1.9) and computer with larger capacity. Something more in this connection will be below concerning bicentric pentagons.

Here we restrict ourselves to the obvious fact that permutations

$$p_k = (t_1, t_{1+k}, t_{1+2k}, \dots, t_{1+16k}), ; k = 1, 2, 4, 8$$

form a group with respect to composition which is isomorphic with the group $(\{1, 2, 4, 8\}, \circ)$. (Cf. with Corollary 2.1.2 in [3]).

As can be seen the relation (1.9) has one of the key role in the present article. Using this relation we shall now show that for bicentric pentagons the following is valid.

Let $(R_1, d_1, r_1)$ be a solution of Fuss' relation $F_5^{(1)}(R, d, r) = 0$ and let $(R_2, d_2, r_2)$ be a solution of Fuss' relation $F_5^{(2)}(R, d, r) = 0$ such that $(R_2, d_2, r_2) = (\hat{R}_1, \hat{d}_1, \hat{r}_1)$. Let $t_1$ be any given length such that $t_M \geq t_1 \geq t_m$, where

$$t_M^2 = (R_1 + d_1)^2 - r_1^2, \quad t_m^2 = (R_1 - d_1)^2 - r_1^2.$$

(By Theorem 1 it is also valid $t_M^2 = (R_2 + d_2)^2 - r_2^2, \quad t_m^2 = (R_2 - d_2)^2 - r_2^2$.)

Let $A_1 \dots A_5$ be a pentagon from the class $C_5^{(1)}(R_1, d_1, r_1)$ whose first tangent has the length $t_1$, and let $B_1, \dots, B_5$ be a pentagon from the class $C_5^{(2)}(R_2, d_2, r_2)$ whose first tangent has also length $t_1$. For brevity writing we shall take $t_1 = t_M$. Let by $t_1, t_2, \dots, t_5$ be denoted tangent lengths of the pentagon $A_1 \dots A_5$ and by $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_5$ be denoted tangent lengths of the pentagon $B_1 \dots B_5$, where $t_1 = \hat{t}_1 = t_M$. Using formula for calculation tangent lengths given by (1.15) in [5] we get the following expressions

$$\hat{t}_2 = \frac{R_2 - d_2}{R_2 + d_2} t_M,$$

$$\hat{t}_3 = \frac{(R_2^2 - d_2^2)^2 - 4R_2 d_2 r_2^2}{(R_2^2 - d_2^2)^2 + 4R_2 d_2 r_2^2} t_M.$$

$$\hat{t}_4 = \frac{q\left(p^4q^4 + 2p^4q^2 - 3p^4 - 2p^2q^4 + 2p^2q^2 + q^4\right)}{p\left(p^4q^4 - 2p^4q^2 + p^4 + 2p^2q^4 + 2p^2q^2 - 3q^4\right)} t_M,$$

$$\hat{t}_5 = \frac{\nu}{\delta} t_M,$$

where

$$\begin{aligned}\nu &= p^8q^8 - 4p^8q^6 + 6p^8q^4 - 4p^8q^2 + p^8 + 4p^6q^8 + 4p^6q^6 - 4p^6q^4\\&\quad - 4p^6q^2 - 10p^4q^8 + 4p^4q^6 + 6p^4q^4 + 4p^2q^8 - 4p^2q^6 + q^8,\\ \delta &= p^8q^8 + 4p^8q^6 - 10p^8q^4 + 4p^8q^2 + p^8 - 4p^6q^8 + 4p^6q^6 + 4p^6q^4\\&\quad - 4p^6q^2 + 6p^4q^8 - 4p^4q^6 + 6p^4q^4 - 4p^2q^8 - 4p^2q^6 + q^8.\end{aligned}$$

Replacing $R_2$, $d_2$ $r_2$ in the expressions for $\hat{t}_2$, $\hat{t}_3$, $\hat{t}_4$, $\hat{t}_5$ by $\hat{R}_1$, $\hat{d}_1$, $\hat{r}_1$, respectively, we get $\hat{t}_2 = t_3$, $\hat{t}_3 = t_5$, $\hat{t}_4 = t_2$, $\hat{t}_5 = t_4$, that is,

$$\hat{t}_i = t_{1+(i-1)2}.$$

Here let us remark that the expressions for $t_2$, $t_3$, $t_4$, $t_5$ are obtained in the same way as the expressions for $\hat{t}_2$, $\hat{t}_3$, $\hat{t}_4$, $\hat{t}_5$ only writing $R_1$, $d_1$ $r_1$ instead of $R_2$, $d_2$, $r_2$. Also let us remark that, for example, in the proof that $\hat{t}_4 = t_2$ we get $\hat{t}_4 - t_2 = cF_5^{(1)}(R_1, d_1, r_1)$, where $c \neq 0$ and $F_5^{(1)}(R_1, d_1, r_1) = 0$.

Here is an example. Let $R_1 = 7$, $d_1 = 2$, $r_1 = 4.789111662\ldots$ and $R_2 = 4.698157318\ldots$, $d_2 = 2.979891701\ldots$, $r_2 = 0.942351978\ldots$. Then $t_1 = \hat{t}_1 = t_M = 7.720000623\ldots$ and

$$\hat{t}_2 = t_3 = 1.705275004\ldots, \qquad t_3 = t_5 = 4.233333683\ldots,$$
$$\hat{t}_4 = t_2 = 4.233333683\ldots, \qquad t_5 = t_4 = 1.705275004\ldots.$$

In the same way can be found that analogously holds for $n = 7$ and $n = 9$. (For odd $n > 9$ needs computer with large capacity.)

## Appendix

In order that the article be convenient for reading and useful in further investigation, here is an appendix where for some odd $n$ we state partition and ordering.

**1:** Let $n = 3$. Then $\mathbb{S} = \{1\}$ and $1 \to 1$ since $f\left(\frac{3-1}{2}\right) \to 1$.
**2:** Let $n = 5$. Then $\mathbb{S} = \{1, 2\}$ and $1 \to 2 \to 1$.
**3:** Let $n = 7$. Then $\mathbb{S} = \{1, 2, 3\}$ and $1 \to 2 \to 3 \to 1$.
**4:** Let $n = 9$. Then $\mathbb{S} = \{1, 2, 4\}$ and $1 \to 2 \to 4 \to 1$. Since $\mathbb{Q} = \{3\}$ we have $3 \to 3$.
**5:** Let $n = 11$. Then $\mathbb{S} = \{1, \ldots, 5\}$ and $1 \to 2 \to 4 \to 3 \to 5 \to 1$.
**6:** Let $n = 13$. Then $\mathbb{S} = \{1, \ldots, 6\}$ and $1 \to 2 \to 4 \to 5 \to 3 \to 6 \to 1$.
**7:** Let $n = 15$. Then $\mathbb{S} = \{1, 2, 4, 7\}$ and $1 \to 2 \to 4 \to 7 \to 1$. Since $\mathbb{Q} = \{3, 5, 6\}$ we have $3 \to 6 \to 3$ and $5 \to 5$.
**8:** Let $n = 17$. Then $1 \to 2 \to 4 \to 8 \to 1$ and $3 \to 6 \to 5 \to 7 \to 3$.
**9:** Let $n = 19$. Then $1 \to 2 \to 4 \to 8 \to 3 \to 6 \to 7 \to 5 \to 9 \to 1$.

**10:** Let $n = 21$. Then $\mathbb{S} = \{1, 2, 4, 5, 8, 10\}$ and $\mathbb{Q} = \{3, 6, 7, 9\}$ and we have $1 \to 2 \to 4 \to 8 \to 5 \to 10 \to 1$, $\quad 3 \to 6 \to 9 \to 3$.

**11:** Let $n = 23$. Then $1 \to 2 \to 4 \to 8 \to 7 \to 9 \to 5 \to 10 \to 3 \to 6 \to 11 \to 1$.

**12:** Let $n = 25$. Then $\mathbb{S} = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12\}$ and $\mathbb{Q} = \{5, 10\}$ and we have $1 \to 2 \to 4 \to 8 \to 9 \to 7 \to 11 \to 3 \to 6 \to 12 \to 1$, and $5 \to 10 \to 5$.

**13:** Let $n = 27$. Then $\mathbb{S} = \{1, 2, 4, 5, 7, 8, 10, 11, 13\}$ and $\mathbb{Q} = \{3, 6, 9, 12\}$ and we have $1 \to 2 \to 4 \to 8 \to 11 \to 5 \to 10 \to 7 \to 13 \to 1$,
$3 \to 6 \to 12 \to 3$, $\quad 9 \to 9$ since $\frac{27-9}{2} = 9$.

**14:** Let $n = 29$. Then $1 \to 2 \to 4 \to 8 \to 13 \to 3 \to 6 \to 12 \to 5 \to 10 \to 9 \to 11 \to 7 \to 14 \to 1$.

**15:** Let $n = 31$. See Example A.3.

**16:** The case when $n = 2^{2^k} + 1$ where $k = 1, 2, 3, \ldots$, can be interesting in itself.

If $k = 1$ then $n = 5$ and we have cosset $\mathbb{C}_1 = \{1, 2\}$.

If $k = 2$ then $n = 17$ and we have cosets $\mathbb{C}_1 = \{1, 2, 4, 8\}$ and $\mathbb{C}_2 = \{3, 6, 5, 7\}$.

If $k = 3$ then $n = 257$ and we have cosets

$$\mathbb{C}_1 = \{1, 2, 4, 8, 16, 32, 64, 128\},$$
$$\mathbb{C}_2 = \{3, 6, 12, 24, 48, 96, 65, 127\},$$
$$\mathbb{C}_3 = \{5, 10, 20, 40, 80, 97, 63, 126\},$$
$$\mathbb{C}_4 = \{7, 14, 28, 56, 112, 33, 66, 125\},$$
$$\mathbb{C}_5 = \{9, 18, 36, 72, 113, 31, 62, 124, \},$$
$$\mathbb{C}_6 = \{11, 22, 44, 88, 81, 95, 67, 123\},$$
$$\mathbb{C}_7 = \{13, 26, 52, 104, 49, 98, 61, 122\},$$
$$\mathbb{C}_8 = \{15, 30, 60, 120, 17, 34, 68, 121\},$$
$$\mathbb{C}_9 = \{19, 38, 76, 105, 47, 94, 69, 119\},$$
$$\mathbb{C}_{10} = \{21, 42, 84, 89, 79, 99, 59, 118\},$$
$$\mathbb{C}_{11} = \{23, 46, 92, 73, 111, 35, 70, 117\},$$
$$\mathbb{C}_{12} = \{25, 50, 100, 57, 114, 29, 58, 116\},$$
$$\mathbb{C}_{13} = \{27, 54, 108, 41, 82, 93, 71, 115\},$$
$$\mathbb{C}_{14} = \{29, 58, 116, 75, 50, 100, 57, 114\},$$
$$\mathbb{C}_{15} = \{37, 74, 109, 39, 78, 101, 55, 110\},$$
$$\mathbb{C}_{16} = \{43, 86, 85, 87, 83, 91, 75, 107\},$$

where we write $\mathbb{C}_i = \{a_{i1}, a_{i2}, \ldots, a_{i8}\}$, $i = 1, \ldots, 16$, if

$$a_{i1} \to a_{i2} \to \cdots \to a_{i8} \to a_{i1}.$$

It seems that for $n = 2^{2^k} + 1$, where $k = 1, 2, 3, \ldots$, there are $2^{2^k - k - 1}$ classes and that each class has $2^k$ elements. Thus, by Corollary 9.1, we have the equality $2^{h/j} = 1$, where $j = 2^{2k-k-1}$.

Also it seems that in the case when $n = 2^{2^k} + 1$ can be proved before the stated conjectures (denoted by $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$).

The possibility of construction polygons whose rotation numbers are from the class $\mathbb{C}_1$ (which contain integer 1) deserve to be investigated in connection with the ordering in this class.

Generally, can be said that it remains much more for investigation about partition and ordering in connection with bicentric polygons.

## References

[1] N. Fuss, *De quadrilateris quibus circulum tam inscribere quam cicumscribere licet*, NAAPS 1792 (Nova acta), t. X(1797), 103–125 (14.VII.1794).

[2] N. Fuss, *De poligonis simmetrice irregularibus calculo simul inscriptis et circumscriptis*, NAAPS 1792 (Nova acta), t. XIII(1802), 168–189 (19.04.1798)

[3] B. Mirman, *Short cycles of Poncelet's conics*, Linear algebra and its applications, Volume 432, Issue 10, 2543-2564.

[4] J. V. Poncelet, *Traité des propriétés projectives des figures*, Paris 1865 (first ed. in 1822).

[5] M. Radić, *Certain relation concerning bicentric polygons and 2-parametric presentation of Fuss' relation*, Mathematica Pannonica, 20/2(2009), 1–30.

[6] M. Radić, *An improved method for establishing Fuss' relations for bicentric polygons*, Comptes Rendus mathèmatique, Volume 348, Issues 7-8, April 2010, Pages 415-417.

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