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GENERALIZATION OF PERTURBED TRAPEZOID FORMULA AND RELATED INEQUALITIES

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Abstract. We derive some new inequalities for perturbed trapezoid formula and give some sharp and best possible constants.

1. Introduction

A.McD. Mercer has proved the following identity ([1])

$$\begin{aligned}\int_{-1}^{1} f(x)\mathrm{d}x \quad &+ \quad \frac{2^n n!}{(2n)!}\sum_{q=0}^{n-1}(-1)^{q+1}\left\{\left[f^{(q)}(1)+(-1)^q f^{(q)}(-1)\right]P_n^{(n-1-q)}(1)\right\}\\ (1.1) \qquad &= \quad \frac{(-1)^k}{(2n)!}\int_{-1}^{1} f^{(2n-k)}(x)D^k[(x^2-1)^n]\mathrm{d}x,\end{aligned}$$

with $k = 0, 1, \ldots, n$, where $f : [-1, 1] \to \mathbf{R}$ possesses continuous derivatives of all orders which appear, D denotes differentiation with respect to x, and $P_n(x)$ is the Legendre polynomial of degree n.

Pečarić and Varošanec ([3]) have considered the following. Let

$$\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$$

be a subdivision of the interval $[a, b]$ for some $m \in \mathbf{N}$. Set

$$(1.2) \qquad S_n(t,\sigma) = \begin{cases} P_{1n}(t), & t \in [a, x_1] \\ P_{2n}(t), & t \in (x_1, x_2] \\ \vdots & \\ P_{mn}(t), & t \in (x_{m-1}, b], \end{cases}$$

where $\{P_{jn}\}_n$ are the sequences of harmonic polynomials, i.e. $P'_{jk}(t) = P_{j,k-1}(t)$, for $k = 1, \ldots, n$ and $P_{j0}(t) = 1$. By successive integration by

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parts they have proved that

$$(-1)^n \int_a^b S_n(t,\sigma)\mathrm{d}f^{(n-1)}(t) = \int_a^b f(t)\mathrm{d}t + \sum_{k=1}^{n}(-1)^k \Big[P_{mk}(b)f^{(k-1)}(b)$$

$$+\sum_{j=1}^{m-1}(P_{jk}(x_j)-P_{j+1,k}(x_j))f^{(k-1)}(x_j) - P_{1k}(a)f^{(k-1)}(a)\Big] \tag{1.3}$$

whenever the integrals exist. Formula (1.3) is generalized in the following way in [2]. Let us consider subdivision

$$\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$$

of the interval $[a,b]$. Further, set

$$T_n(t,\sigma) = \begin{cases} M_{1n}(t), & t \in [a,x_1] \\ M_{2n}(t), & t \in (x_1,x_2] \\ \vdots & \\ M_{mn}(t), & t \in (x_{m-1},b], \end{cases} \tag{1.4}$$

where M_{jn} are monic polynomials of degree n, for $j = 1,\ldots,m$. The next theorem has been proved.

Theorem 1. *Let $f : [a,b] \to \mathbf{R}$ be $(n-1)-$times differentiable function, for some $n \in \mathbf{N}$. Then the next identity holds*

$$\int_a^b f(t)\mathrm{d}t \;+\; \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\cdot\Big[M_{mn}^{(n-k-1)}(b)f^{(k)}(b) + \sum_{j=1}^{m-1}\big(M_{jn}^{(n-k-1)}(x_j)$$

$$-\; M_{j+1,n}^{(n-k-1)}(x_j)\big)f^{(k)}(x_j) - M_{1n}^{(n-k-1)}(a)f^{(k)}(a)\Big] \tag{1.5}$$

$$=\; \frac{(-1)^n}{n!}\int_a^b T_n(t,\sigma)\mathrm{d}f^{(n-1)}(t),$$

whenever the integrals exist.

If we put in (1.5) $M_{jn} = n!\cdot P_{jn}$, where $\{P_{jn}\}$ are harmonic polynomials with leading coefficient $\frac{1}{n!}$, then we will recover relation (1.3), since

$$P_{jn}^{(n-k-1)}(t) = P_{j,k+1}(t),$$

for $0 \le k \le n-1$.
In this paper we will use the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t,$$

where $x \in \mathbf{R}_+$ and the incomplete Beta function

$$B(x,a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t,$$

where $x, a, b > 0$. In this paper we will show that identity (1.1) is a special case of Theorem 1. Further, we will obtain some sharp and best possible L_p inequalities for quadrature formula in (1.1).

2. Perturbed trapezoid identity

Let us define polynomial

$$M_{1n}(t) = \frac{(n!)^2}{(2n)!} 2^n P_n(t), \quad t \in [-1,1]. \tag{2.1}$$

Since the leading coefficient of $P_n(t)$ equals to $\frac{(2n)!}{2^n (n!)^2}$, the polynomial M_{1n} is monic, so we can apply Theorem 1 with $m = 1$ for some function $f : [-1,1] \to \mathbf{R}$ with continuous $n-$th derivative . Using the property of the Legendre polynomials

$$P_n^{(k)}(-t) = (-1)^{n+k} P_n^{(k)}(t),$$

and Rodrigues formula

$$D^n[(t^2-1)^n] = 2^n n! P_n(t),$$

we get from the relation (1.5)

$$\int_{-1}^1 f(x)\mathrm{d}x \quad + \quad \frac{2^n n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^{q+1} \left\{ \left[f^{(q)}(1) + (-1)^q f^{(q)}(-1) \right] P_n^{(n-1-q)}(1) \right\}$$

$$= \quad \frac{(-1)^n}{(2n)!} \int_{-1}^1 f^{(n)}(x) D^n[(x^2-1)^n]\mathrm{d}x. \tag{2.2}$$

In ([1]) is obtained that

$$(-1)^k \int_{-1}^1 f^{(2n-k)}(x) D^k[(x^2-1)^n]\mathrm{d}x = \int_{-1}^1 f^{(2n)}(x)(x^2-1)^n \mathrm{d}x,$$

for $k = 0, 1, \ldots, n$, so (2.2) becomes (1.1).

3. Some inequalities

Theorem 2. *Let us suppose $f : [-1,1] \to \mathbf{R}$ is $(2n-k)-$times differentiable function for some $n \in \mathbf{N}$ and some $k = 0, 1, 2, \ldots, n$. Further,* *let us assume that* $f^{(2n-k)} \in L_p[-1,1]$, *for some* $1 \le p \le \infty$. *Then the following inequality holds*

$$\begin{aligned} \Big| \int_{-1}^{1} f(x)\mathrm{d}x &+ \frac{2^n n!}{(2n)!} \sum_{j=0}^{n-1} (-1)^{j+1} \Big\{ \Big[f^{(j)}(1) \\ &+ (-1)^j f^{(j)}(-1) \Big] P_n^{(n-1-j)}(1) \Big\} \Big| \\ &\le C(n,k,q) \| f^{(2n-k)} \|_p, \end{aligned} \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$C(n,k,q) = \begin{cases} \frac{1}{(2n)!} \left[\int_{-1}^{1} \left| D^k[(x^2-1)^n] \right|^q \mathrm{d}x \right]^{\frac{1}{q}}, & 1 \le q < \infty \\ \frac{1}{(2n)!} \sup_{x \in [-1,1]} |D^k[(x^2-1)^n]|, & q = \infty. \end{cases}$$

The inequality is the best possible for $p = 1$ *and sharp for* $1 < p \le \infty$. *In the last case equality is attained for the functions of the form*

$$f(x) = M f_*(x) + r_{2n-k-1}(x),$$

where $M \in \mathbf{R}$, r_{2n-k-1} *is an arbitrary polynomial of degree at most* $2n-k-1$ *and function* $f_* : [-1,1] \to \mathbf{R}$ *is defined by*

$$f_*(x) := \int_{-1}^{x} \frac{(x-\xi)^{2n-k-1}}{(2n-k-1)!} \mathrm{sgn} D^k[(\xi^2-1)^n] \mathrm{d}\xi, \ \text{for } p = \infty \tag{3.2}$$

and for $1 < p < \infty$

$$f_*(x) := \int_{-1}^{x} \frac{(x-\xi)^{2n-k-1}}{(2n-k-1)!} \mathrm{sgn} D^k[(\xi^2-1)^n] |D^k[(\xi^2-1)^n]|^{\frac{1}{p-1}} \mathrm{d}\xi \tag{3.3}$$

Proof. We apply Hölder inequality to the relation (1.1) to get

$$\begin{aligned} \Big| \int_{-1}^{1} f(x)\mathrm{d}x &+ \frac{2^n n!}{(2n)!} \sum_{j=0}^{n-1} (-1)^{j+1} \Big\{ \Big[f^{(j)}(1) \\ &+ (-1)^j f^{(j)}(-1) \Big] P_n^{(n-1-j)}(1) \Big\} \Big| \\ &\le \frac{1}{(2n)!} \| D^k[(x^2-1)^n] \|_q \| f^{(2n-k)} \|_p. \end{aligned}$$

Obviously, $C(n,k,q) = \frac{1}{(2n)!} \| D^k[(x^2-1)^n] \|_q$, so we obtain relation (3.1). For the proof of sharpness we need to find function f such that

$$\frac{1}{(2n)!} \left| \int_{-1}^{1} D^k[(x^2-1)^n] f^{(2n-k)}(x) \mathrm{d}x \right| = C(n,k,q) \cdot \| f^{(2n-k)} \|_p,$$

where $1 < p \leq \infty$. The function f_* defined by (3.2) and (3.3) is $(2n-k)-$times differentiable and $f_*^{(2n-k)} \in L_p[-1,1]$. Further, f_* is a solution of the differential equation

$$D^k[(x^2-1)^n]f^{(2n-k)}(x) = |D^k[(x^2-1)^n]|^q,$$

so the above identity holds.
For $p = 1$ we shall prove that
(3.4)

$$\left|\int_{-1}^{1} D^k[(x^2-1)^n]f^{(2n-k)}(x)\mathrm{d}x\right| \leq \sup_{x\in[-1,1]} |D^k[(x^2-1)^n]| \cdot \int_{-1}^{1} |f^{(2n-k)}(x)|\mathrm{d}x$$

is the best possible inequality. Suppose that $|D^k[(x^2-1)^n]|$ attains its maximum at point $x_0 \in [-1,1]$. First, let us assume that $D^k[(x_0^2-1)^n] > 0$. For ϵ small enough define $f_\epsilon^{(2n-k-1)}(x)$ by

$$f_\epsilon^{(2n-k-1)}(t) = \begin{cases} 0, & x \leq x_0 \\ \frac{x-x_0}{\epsilon}, & x \in [x_0, x_0+\epsilon] \\ 1, & x \geq x_0+\epsilon. \end{cases}$$

Then, for ϵ small enough,

$$\begin{aligned} & \left|\int_{-1}^{1} D^k[(x^2-1)^n]f_\epsilon^{(2n-k)}\mathrm{d}x\right| \\ = & \left|\int_{x_0}^{x_0+\epsilon} D^k[(x^2-1)^n]\frac{1}{\epsilon}\mathrm{d}x\right| = \frac{1}{\epsilon}\int_{x_0}^{x_0+\epsilon} D^k[(x^2-1)^n]\mathrm{d}x. \end{aligned}$$

Now, relation (3.4) implies

$$\frac{1}{\epsilon}\int_{x_0}^{x_0+\epsilon} D^k[(x^2-1)^n]\mathrm{d}x \leq \frac{1}{\epsilon}D^k[(x_0^2-1)^n]\int_{x_0}^{x_0+\epsilon} \mathrm{d}t = D^k[(x_0^2-1)^n].$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_0}^{x_0+\epsilon} D^k[(x^2-1)^n]\mathrm{d}x = D^k[(x_0^2-1)^n],$$

the statement follows. The case $D^k[(x_0^2-1)^n] < 0$ follows similarly.

□

Remark 1. *For $n \in \mathbf{N}$ we have by direct calculation*

$$C(n,0,q) = \frac{1}{(2n)!}\left[\frac{\sqrt{\pi}\Gamma(nq+1)}{\Gamma(\frac{3}{2}+nq)}\right]^{\frac{1}{q}}, \quad 1 \leq q < \infty, \quad C(n,0,\infty) = \frac{1}{(2n)!}$$

and

$$C(n,n,2)=\frac{2^{n+1}n!}{(2n+1)!},\quad C(n,n,\infty)=\frac{2^n n!}{(2n)!}.$$

Further,

$$C(1,1,q)=\left(\frac{2}{q+1}\right)^{\frac{1}{q}},\quad 1\le q<\infty,\quad C(1,1,\infty)=1,$$

$$C(2,1,q)=\frac{1}{3\cdot 2^{1/q}}\left(\frac{\Gamma(\frac{1+q}{2})\Gamma(1+q)}{\Gamma(\frac{3(1+q)}{2})}\right)^{\frac{1}{q}},\quad 1\le q<\infty,\quad C(2,1,\infty)=\frac{\sqrt{3}}{27}$$

and

$$C(2,2,q)=\frac{1}{6}\left(\frac{(-1)^q\left((-1+(-1)^q)\sqrt{\pi}\Gamma(1+q)+B(3,\frac{1}{2},1+q)\Gamma(\frac{3}{2}+q)\right)}{\sqrt{3}\Gamma(\frac{3}{2}+q)}\right)^{\frac{1}{q}},$$

for $1\le q<\infty$*, and*

$$C(2,2,\infty)=\frac{1}{3}.$$

Specially,

$$C(2,2,1)=\frac{4\sqrt{3}}{27},$$

which coincides with constants obtained in [4]*. For* $n=3$ *we have the following constants*

$$C(3,1,q)=\frac{1}{120}\left(\frac{\Gamma(\frac{1+q}{2})\Gamma(1+q)}{\Gamma(\frac{3+3q}{2})}\right)^{\frac{1}{q}},\quad 1\le q<\infty$$

and $C(3,1,\infty)=\frac{2\sqrt{5}}{1875}$.

The case $k=0$ in (1.1) is of special interest since function $(x^2-1)^n$ doesn't change sign on $[-1,1]$ for every $n\in\mathbf{N}$. More precisely, $(x^2-1)^n\ge 0$ for even n and $(x^2-1)^n\le 0$ for odd n. So we have the following

Theorem 3. *Let us suppose* $f:[-1,1]\to\mathbf{R}$ *is such that* $f^{(2n)}$ *is continuous function on* $[-1,1]$ *for some* $n\in\mathbf{N}$*. Then there exists* $\eta\in(-1,1)$ *such that*

$$\int_{-1}^{1}f(x)\mathrm{d}x+\frac{2^n n!}{(2n)!}\sum_{q=0}^{n-1}(-1)^{q+1}\left\{\left[f^{(q)}(1)+(-1)^q f^{(q)}(-1)\right]P_n^{(n-1-q)}(1)\right\}$$

$$=\frac{(-1)^n\sqrt{\pi}n!}{(2n)!\Gamma(\frac{3}{2}+n)}\cdot f^{(2n)}(\eta). \tag{3.5}$$

Proof. The proof follows from the integral mean value theorem applied to the right-hand side of (1.1) with $k=0$, since $(x^2-1)^n$ does not change sign on $[-1,1]$. So there exists some $\eta \in (-1,1)$ such that

$$\frac{1}{(2n)!}\int_{-1}^{1} f^{(2n)}(x)(x^2-1)^n \mathrm{d}x = \frac{f^{(2n)}(\eta)}{(2n)!}\cdot\int_{-1}^{1}(x^2-1)^n \mathrm{d}x$$

$$= \frac{(-1)^n\sqrt{\pi}n!}{(2n)!\Gamma(\frac{3}{2}+n)}\cdot f^{(2n)}(\eta).$$

□

Remark 2. *Applying previous theorem for $n=1,2,3$ respectively, we get the following identities:*

$$\int_{-1}^{1} f(x)\mathrm{d}x - [f(1)+f(-1)] = -\frac{2}{3}f''(\eta), \tag{3.6}$$

which is identity related to the famous trapezoid formula,

$$\int_{-1}^{1} f(x)\mathrm{d}x - [f(1)+f(-1)] + \frac{1}{3}[f'(1)-f'(-1)] = \frac{2}{45}f^{(4)}(\eta), \tag{3.7}$$

and

$$\int_{-1}^{1} f(x)\mathrm{d}x - [f(1)+f(-1)] + \frac{2}{5}[f'(1)-f'(-1)]$$

$$- \frac{1}{15}[f''(1)+f''(-1)] = -\frac{2}{1575}f^{(6)}(\eta).$$

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