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INEQUALITIES FOR CONVEX AND 3-LOG-CONVEX FUNCTIONS

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Abstract. In this article some new inequalities for convex functions are proved from which some other known inequalities for log-convex, 3-convex and 3-log-convex functions are derived.

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Introduction

The function f is called n-convex on the interval (a,b) if its n-th derivative $f^{(n)}(t)$ is positive for all $t \in (a,b)$. Especially, using this terminology, convex function is called 2-convex function. Moreover, the function is called n-log-convex function if f is positive and $(\ln f(t))^{(n)}$ is positive for all $t \in (a,b)$.

Also, let's introduce the usual notation $g(a+)$ for $\lim_{x\to a+} g(x)$ and $g(b-)$ for $\lim_{x\to b-} g(x)$.

The aim of this article is to establish some basic result for convex functions which can be easily used for obtaining many other results.

In the introduction, let's remind on some results for 3-log-convex functions given in [1]:

Theorem A. Suppose that $f(x) > 0$ for $x \in (a,b)$ and let $h = \frac{f'}{f}$ is twice differentiable and $h''(x) > 0$. Set $R(x) = \frac{f(a+b-x)}{f(x)}$. Then, for all $x \in (a,b)$, the following inequalities hold

$$R(b-)e^{2h(\frac{a+b}{2})(b-x)} \leq R(x) \leq R(a+)e^{2h(\frac{a+b}{2})(a-x)}.$$

and

$$R(a+)e^{(h(a+)+h(b-))(a-x)} \leq R(x) \leq R(b-)e^{(h(a+)+h(b-))(b-x)}.$$

This result will be also obtained, in a different way, as a consequence of our main result theorem (Theorem 1) for convex function.

Main results

Let's state and prove the main result.

Theorem 1. Let f be convex on (a,b). Then the following inequalities

$$\max\{f(a+)-f'(a+)(a-x), f(b-)-f'(b-)(b-x)\} \le f(x)$$
$$\le \min\{f(a+)-f'(b-)(a-x), f(b-)-f'(a+)(b-x)\}$$

hold for all $x \in (a,b)$.

Proof. If f is convex on (a,b), then f' is increasing on (a,b).

Let's define $F_1(x) = f(x) - f'(a+)x$ and $F_2(x) = f(x) - f'(b-)x$. Then

$$F_1'(x) = f'(x) - f'(a+) \ge 0,$$
$$F_2'(x) = f'(x) - f'(b-) \le 0,$$

for all $x \in (a,b)$. Hence, F_1 is increasing on (a,b) and F_2 is decreasing on (a,b). Hence, for all $x \in (a,b)$, we have

$$F_1(a+) \le F_1(x) \le F_1(b-)$$

and

$$F_2(b-) \le F_2(x) \le F_2(a+).$$

So, we have

$$f(a+) - f'(a+)a \le f(x) - f'(a+)x \le f(b-) - f'(a+)b$$

or equivalently,

$$f(a+) - f'(a+)(a-x) \le f(x) \le f(b-) - f'(a+)(b-x). \tag{1}$$

Also, we have

$$f(b-) - f'(b-)b \le f(x) - f'(b-)x \le f(a+) - f'(b-)a$$

$$f(b-) - f'(b-)(b-x) \le f(x) \le f(a+) - f'(b-)(a-x). \tag{2}$$

The result of Theorem 1 follows from (1) and (2).

By putting $\ln f$ instead of f in Theorem 1, we immediately obtain the following result:

Corollary 2. Let f be log-convex on (a,b). Then the following inequalities hold

$$\max\{f(a+)\cdot e^{-\frac{f'(a+)}{f(a+)}(a-x)}, f(b-)\cdot e^{-\frac{f'(b-)}{f(b-)}(b-x)}\} \leq f(x)$$
$$\leq \min\{f(a+)\cdot e^{-\frac{f'(b-)}{f(b-)}(a-x)}, f(b-)\cdot e^{-\frac{f'(a+)}{f(a+)}(b-x)}\}.$$

Now, let's suppose that f is 3-convex on (a,b). Then, by well-known result (see [2], page 72), the function $F(x) = f(a+b-x) - f(x)$ is convex on $(a, \frac{a+b}{2})$. By applying Theorem 1 for function F on an interval $(a, \frac{a+b}{2})$, we obtain the following result:

Theorem 3. Let f be 3-convex function on (a,b). Then, the following inequalities hold

$$\begin{aligned}
&\max\{f(b-) - f(a+) + (f'(a+) + f'(b-))(a-x),\\
&2f'\left(\frac{a+b}{2}\right)(\frac{a+b}{2} - x)\}\\
&\leq f(a+b-x) - f(x)\\
&\leq \min\{(f'(a+) + f'(b-))(\frac{a+b}{2} - x),\\
&f(b-) - f(a+) + 2f'\left(\frac{a+b}{2}\right)(a-x)\}.
\end{aligned}$$

Proof. As we already mentioned, function $F(x) = f(a+b-x) - f(x)$ is convex on $(a, \frac{a+b}{2})$. Now we have $F'(x) = -f'(a+b-x) - f'(x)$ and it is easy to compute the following values:

$$\begin{aligned}
F(a+) &= f(b-) - f(a+), & F\left(\frac{a+b}{2}-\right) &= 0,\\
F'(a+) &= -(f'(a+) + f'(b-)), & F'\left(\frac{a+b}{2}-\right) &= -2f'\left(\frac{a+b}{2}\right).
\end{aligned}$$

Now, by applying Theorem 1 for function F on an interval $(a, \frac{a+b}{2})$, we have the following pair of inequalities

$$f(b-) - f(a+) + (f'(a+) + f'(b-))(a-x)$$
$$\leq f(a+b-x) - f(x) \leq (f'(a+) + f'(b-))(\frac{a+b}{2} - x). \tag{3}$$

and

$$\begin{aligned}2f'\big(\frac{a+b}{2}\big)\big(\frac{a+b}{2}-x\big) &\le f(a+b-x)-f(x)\\ &\le f(b-)-f(a+)+2f'\left(\frac{a+b}{2}\right)(a-x), \qquad (4)\end{aligned}$$

which is equivalent to the result of Theorem 3.

From Theorem 3 we obtain the following result:

Corollary 4. Let f be 3-convex function on (a,b). Then, the following inequalities hold

$$\begin{aligned}&\max\{f(b-)-f(a+)+(f'(a+)+f'(b-))(a-x),\\ &f(a+)-f(b-)+2f'\left(\frac{a+b}{2}\right)(b-x)\}\\ &\le f(a+b-x)-f(x)\\ &\le \min\{f(a+)-f(b-)+(f'(a+)+f'(b-))(b-x),\\ &f(b-)-f(a+)+2f'\left(\frac{a+b}{2}\right)(a-x)\}.\end{aligned}$$

Proof. Taking into account what we proved in Theorem 3, we have to prove two more inequalities:

$$f(a+b-x)-f(x)\le f(a+)-f(b-)+(f'(a+)+f'(b-))(b-x) \qquad (5)$$

and

$$f(a+)-f(b-)+2f'\left(\frac{a+b}{2}\right)(b-x)\le f(a+b-x)-f(x). \qquad (6)$$

From (3) we have

$$f(a+b-x)-f(x)\le (f'(a+)+f'(b-))\big(\frac{a}{2}-\frac{x}{2}\big)+(f'(a+)+f'(b-))\big(\frac{b}{2}-\frac{x}{2}\big)$$

and

$$(f'(a+)+f'(b-))(a-x)\le f(a+)-f(b-)+f(a+b-x)-f(x).$$

Using these inequalities we finally obtain

$$\begin{aligned}&f(a+b-x)-f(x)\\ &\le \frac{1}{2}(f(a+)-f(b-))\\ &+\frac{1}{2}(f(a+b-x)-f(x))+(f'(a+)+f'(b-))(\frac{b}{2}-\frac{x}{2})\end{aligned}$$

from which inequality (5) follows immediately.

The proof of an inequality (6) can be done in the same way using inequality (4).

At the end, let's come back to the result of Theorem A.

Remark. By putting $\ln f$ instead of f in Corollary 4, we immediately obtain the result of Theorem A. Here, this result is obtained as a consequence of Theorem 1 as a basic result.

References:

[1] Kenneth S.Berenhaut, Dongui Chen: Inequalities for 3-log-convex functions, JIPAM, Vol 9 (2008), 4, Article 97, 9 pp.

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