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Butterfly Lines' Curve in pseudo-Euclidean Plane

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Abstract. Up till now the validity of the Butterfly theorem has been verified in the Euclidean, isotropic, hyperbolic and pseudo-Euclidean plane. Furthermore, it has been shown that an infinite number of butterfly points, located on a conic, is associated with any quadrangle inscribed into a circle.

In the present paper we study the curve formed by butterfly lines. In the Euclidean plane this curve is always a curve of order four and class three having one real cusp while in the pseudo-Euclidean plane it can also be a curve of order four and class three having three real cusps or a special parabola.

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1 Introduction

The basic object of the Butterfly theorem is a complete quadrangle inscribed into a circle. The numerous proofs, many variants and generalizations of the theorem in the Euclidean plane have been published, see for instance [2], [9], [14]. The Butterfly theorem is also valid in the isotropic, hyperbolic and pseudo-Euclidean plane, see [1], [3], [11], [12].

A *pseudo-Euclidean plane* can be defined as a real projective plane $\mathrm{PG}(2,\mathbb{R})$ where the metric is induced by an absolute figure $\{f, F_1, F_2\}$ in the sense of Cayley-Klein, consisting of a real line f and two real points F_1 and F_2 incidental with it, [4], [6]. We assume that $\mathrm{PG}(2,\mathbb{R})$ is embedded into its complexification $\mathrm{PG}(2,\mathbb{R}\subset\mathbb{C})$.

An involution of points on the absolute line f having the absolute points F_1, F_2 for the fixed points is called the *absolute involution.* Two lines are *perpendicular* if they meet f in a pair of points corresponding in the absolute involution. The *midpoint* of the segment AB is the point P such that the pair A, B is harmonically separated by the pair P, O, where $O = f \cap AB$.

The proper conics in the pseudo-Euclidean plane are in [4] classified into: ellipses, hyperbolas, parabolas, special hyperbolas, special parabolas and circles. The pole of the absolute line with respect to the conic c is called the *center* of the conic c and all lines through the center are its *diameters*, [6].

Butterfly Points

The Butterfly Theorem in the pseudo-Euclidean plane says the following: *Let the complete quadrangle $ABCD$ be inscribed into the circle c and let l be a line perpendicular to the diameter o of the circle c. If $L = o \cap l$ is the midpoint of one of the segments formed on l by the pairs of the opposite sides of the quadrangle, then it is also the midpoint of the other two.*

This theorem has been proved synthetically in [12]. The proof is based on the fact that L and the isoptropic point O of the line l are fixed points of the involution determined on l by the conics of the pencil $[ABCD]$, [5].

The point L with described property is called a *butterfly point* and the line l is called a *butterfly line* of the quadrangle $ABCD$.

It has been shown in the papers [3], [9], [11] and [12] that in all four mentioned planes there is a butterfly point on every diameter of c. All of them lie on a conic, the so-called *butterfly points' curve*, passing through the center of the circle, three diagonal points of the quadrangle and six midpoints of its sides, Figure 1. In the Euclidean plane this conic is always a rectangular hyperbola while in the pseudo-Euclidean plane three cases are possible:

- k is a rectangular hyperbola if the pencil $[ABCD]$ contains the circle c, ellipses, hyperbolas and two parabolas.
- k is an ellipse if the pencil $[ABCD]$ contains, apart from the circle c, only hyperbolas.
- k is a special parabola if the pencil $[ABCD]$ contains, apart from the circle c and one special parabola, only special hyperbolas.

2 Butterfly Lines

In the present paper we move our interest from the butterfly points to the butterfly lines. It has been shown in [10] that in the Euclidean plane the envelope of the butterfly lines, so-called *butterfly lines' curve*, is a curve of order four and class three having one real cusp and touching absolute line at two real points. In our study we will use the approach introduced in that paper. It is natural to expect that in the pseudo-Euclidean plane the situation is similar, but not exactly the same.

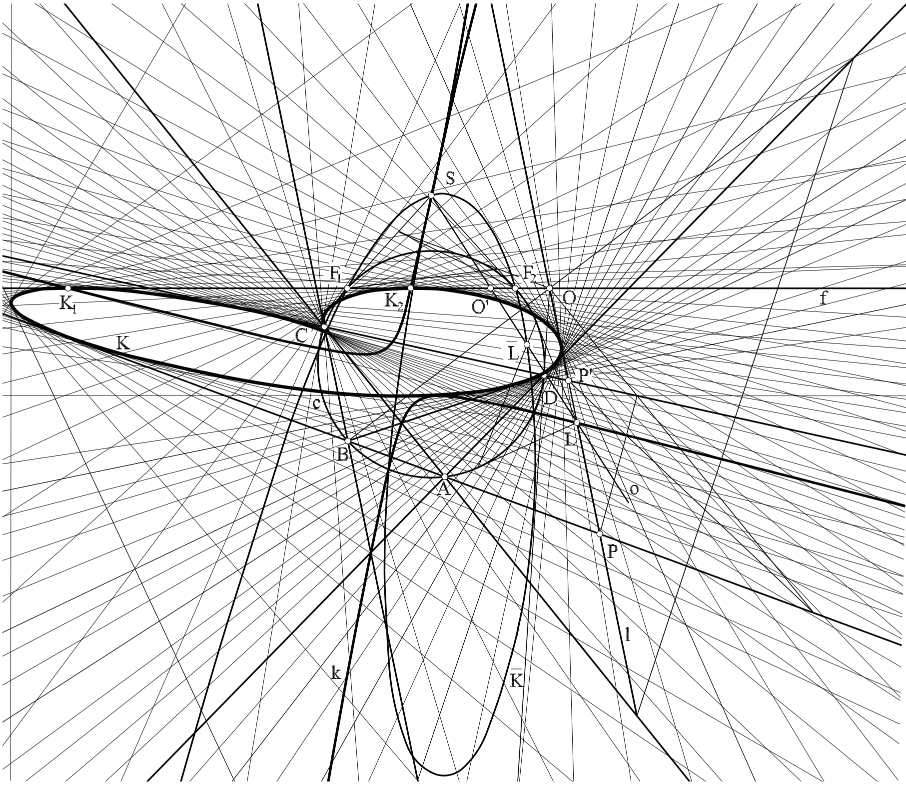

Figure 1.

Some of the butterfly lines of the quadrangle $ABCD$ are: six sides of the quadrangle, perpendiculars at the diagonal points to the joints of the diagonal points and the center of the circle c, and the absolute line.

Pedal and *negative pedal transformations* in the pseudo-Euclidean plane can be defined analogously as in the Euclidean plane.

Definition 1. *The pedal K_P of a curve K with respect to a point P is the locus of the foot of the perpendicular from P to the tangent to the curve K.*
The curve K is called the negative pedal of K_P.

If the complete quadrangle $ABCD$ is inscribed into the circle c with the center S, then the butterfly points are feet from S to the butterfly lines. So, it follows:

Theorem 1. *Let the complete quadrangle $ABCD$ be inscribed into the circle c with the center S. The butterfly lines' curve K of the quadrangle $ABCD$ is the negative pedal of the butterfly points' curve k with respect to the pole S.*

This theorem allows us one construction of the butterfly lines, Figure 1. The another construction can be made by applying Pascal's theorem: choose a point O on the absolute line and construct the tangent l of the conic determined by A, B, C, D, O at O.

The main result of our work is stated in the following theorem:

Theorem 2. *Let the complete quadrangle $ABCD$ be inscribed into the circle c. Let k be the butterfly points' curve and K be the butterfly lines' curve of the quadrangle $ABCD$.*

- *K is a curve of order four and class three with one real cusp if k is a hyperbola.*
- *K is a curve of order four and class three with three real cusps if k is an ellipse.*
- *K is a special parabola if k is a special parabola.*

Proof. Let us first suppose that k is a hyperbola or an ellipse, Figure 1. For every isotropic point O there is a butterfly line l passing through it. On the other hand, every point L of the butterfly points' curve k has the butterfly property on only one butterfly line l. Therefore, the butterfly lines' curve K is a curve of class three since it is the result of the projectively linked pencil of points of the first order on the line f and pencil of points of the second order on the conic k. If k is a hyperbola or an ellipse, then it intersects f in two points: real or imaginary, respectively. They both play the role of butterfly point on f. Hence, the absolute line is a double tangent of K, ordinary or isolated, respectively. According to Plücker's equations[1] (see [7], pp. 64-65. or [8], p. 24.) K has to be a curve of order four with three cusps and no stationary tangent. Furthermore, by using Klein's equation[2] ([8], p. 24.) we get that

[1] $k = n(n-1) - 2d - 3r, \quad n = k(k-1) - 2t - 3w,$
$w = 3n(n-2) - 6d - 8r, \quad r = 3k(k-2) - 6t - 8w,$
where: n - the order of the curve, k - the class of the curve, d - the number of its double points (nodes and isolated), r - the number of its cusps, t - the number of its double tangents, w - the number of its points of inflexion.

[2] $w_1 = k + r_1 + 2d_1 - 2t_1 - n,$
where: r_1 - the number of the real cusps of the curve, d_1 - the number of its real double points, t_1 - the number of its isolated double tangents, w_1 - the number of its real points of inflexion.

if K has an isolated double tangent, all three cusps are real, otherwise, only one of them is real.

For further investigation of K we will study its polar curve $\overline{K}$ with respect to the circle c (the curve consisting of the poles of the butterfly lines with respect to c). It is obvious from the construction that $\overline{K}$ is also an inverse image of k with respect to the same circle c, [4]. An inversion maps a conic to a curve of order four. In our case the conic k passes through the fundamental point S and its inverse splits onto a cubic $\overline{K}$ and the corresponding fundamental line f. $\overline{K}$ passes through the absolute points F_1, F_2 and has a node or an isolated double point at S depending on whether k is a hyperbola or an ellipse. The tangents at the cusps to K correspond to the stationary points of $\overline{K}$.

If k is a special parabola touching the absolute line at e.g. F_1, then its inverse $\overline{K}$ splits onto a conic passing through S and F_2, and two fundamental lines of inversion: the absolute line f and the isotropic line SF_1, Figure 2. It follows immediately that the polar curve K of the conic $\overline{K}$ is a conic touching f and SF_2. This finishes the proof. □

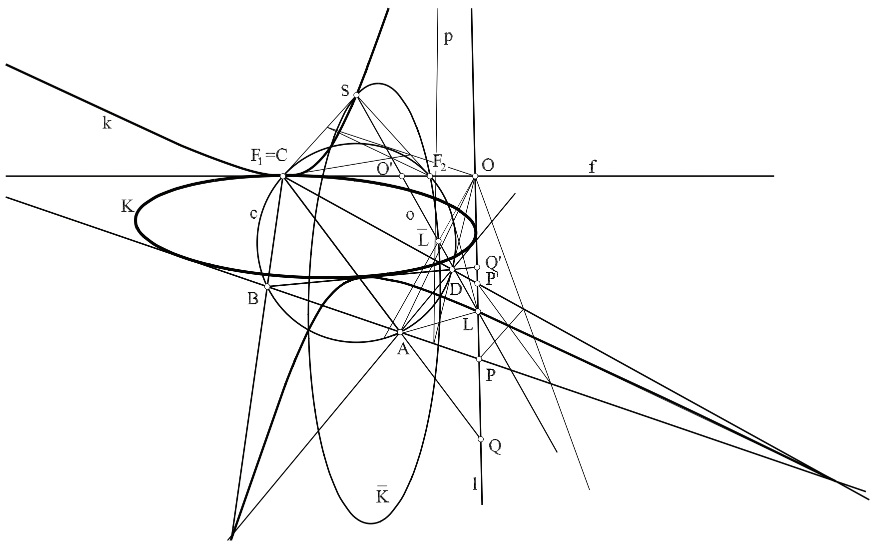

Figure 2.

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