Dimensional Analysis of non-Newtonian Fluids

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The article provides a new definition of rheological similarity and thus a new basis for the modeling of non-Newtonian fluids: Fluids, whose rheological functions $\eta=f(\dot\gamma)$ can be twist-free shifted in a log/log field to a common master curve are rheologically similar and can be mutually used as model fluids in model experiments. This thesis is theoretically explained and confirmed by evaluation of experimental data published in the literature. The rheological similarity, so far explained by the congruency of rheological functions, is now replaced by a less crucial condition of a flexible congruency.

Key words:
Model theory, non-Newtonian fluids, rheology, transport processes, viscoelasticity

Introduction

A dimensional-analytical description of flow processes with non-Newtonian, particularly pseudo-plastic, fluids has already been presented in part.[1–4] This study deals with the topic as a whole, uniformly and comprehensively, in mathematical terms. It is based on two fundamental facts:

– First, it is a logical extension of classical dimensional analysis that, according to the pi theorem, all physico-technical facts that can be described with dimensionally homogeneous relations can also be represented with correlations exclusively between dimensionless quantities, the so-called pi quantities,[4,5] including the geometry of equipment and rheological behaviour. These procedures which will be dealt with later on, correspond fully with the mathematical structure of the problem under discussion and therefore exclude in principle any alternative procedure to the contrary as inadequate.

– And second, it is isomorphy[6] between the actual course of a process and its mathematical simulation, according to which any physical process can be at least in principle simulated with adequate mathematical means. This paper is concerned primarily with momentum, energy and mass transfer.

The rheological impact of non-Newtonian, particularly pseudo-plastic, fluids on flow processes is therefore in the dimensional-analytical sense not expressed with primary known dimensional flow curves $\eta=f(\dot\gamma)$, but with their dimensionless representations, the master curves $\eta/\mathrm{H} = \mathrm{F}(\dot\gamma/\Gamma)$; see symbols at the end of the paper.

$$\eta = f(\dot\gamma) \rightarrow \eta/\mathrm{H} = \mathrm{F}(\dot\gamma/\Gamma) \qquad (1)$$

These contain twodimensional parameters (Γ, H). These parameters have in fact the same dimensions as $\dot\gamma$ and/or η, which is emphasized by their notations, although they represent an entirely different category of quantities, independent of $\dot\gamma$ and η.

Relation 1 represents shift transformation in the coordinate field (log x, log y), in which $y = f(x)$ is shifted *twist-free* by $\Delta x = (-\log\Gamma)$ and $\Delta y = (-\log \mathrm{H})$ to $y = \mathrm{F}(x)$. The (Γ, H) parameters function here as shift parameters and are as such subject to no restrictions. If any reference values such as ($\dot\gamma_b$, η_b) are selected as shift parameters, $y = \mathrm{F}(x)$ proceeds through zero-point of the log/log field. The availability of the (Γ, H) parameters is of great significance for the extended concept of rheological similarity and for its applicability in model experiments.

However, it also brings about an ambiguity as to process characteristics. (See Section Transformation behavior in transport characteristics). Between functions $\eta = f(\dot\gamma)$ and $\eta/\mathrm{H} = \mathrm{F}(\dot\gamma/\Gamma)$ there are reciprocally inverse functional relations with $x \equiv \dot\gamma/\Gamma$ and $y \equiv \dot\gamma$ as follows

$$\mathrm{F}(x) = \frac{f(\Gamma x)}{\mathrm{H}}; \quad f(y) = \mathrm{H}\cdot\mathrm{F}(y/\Gamma). \qquad (2)$$

Compared with the Newtonian analogy, the parameters (Γ, H) result in modification and extension of the pi space,[4,7] in which the problem is described.

– Newtonian viscosity η no longer constitutes a problem and is replaced by parameter H, with which the dimensionless numbers $\mathrm{Re_H} \equiv \rho n d^2/\mathrm{H}$ and $\mathrm{Pr_H} \equiv c_p\mathrm{H}/\lambda$ are formed.

– Further, the pi space is enlarged by an additional pi number. According to the process in question, the pi number can be formed as $\mathrm{B} \equiv \rho d^2\Gamma/\mathrm{H}$ or $\mathrm{A} \equiv \Gamma(\mathrm{H}/\rho g^2)^{1/3}$ or $\mathrm{Q} \equiv n/\Gamma$.

Consequently, in pseudo-plastic fluids, characteristics of the following processes are altered: mixing power (3a), heat transfer in vessels (3b), homogenization of miscible liquids (3c), intensively formulated mass transfer in two-phase systems (3d) and extruders under creeping flow conditions (3e), as compared with Newtonian fluids, as follows:

Newtonian fluid	Pseudoplastic fluid		
$Ne(Re_\eta)$	$Ne(Re_H, B)$	(3a)	
$Nu(Re_\eta, Pr_\eta)$	$Nu(Re_H, Pr_\eta, B)$	(3b)	
$Ho(Re_\eta)$	$Ho(Re_H, B)$	(3c)	(3)
$(k_La)^*((P/V)^*, v^*)$	$(k_La)^*((P/V)^*, v^*, A)$	(3d)	
$\Delta pd/\eta nL\ (q/nd^3)$	$\Delta pd/HnL\ (q/nd^3, Q)$	(3e)	

(The pi space in viscoelastic fluids is further enlarged by additional material quantities[3,5,11])

This fundamental enlargement of the pi space is in disagreement with findings concerning some evaluation techniques published in literature, which assign an equivalent pi space to analogous processes in Newtonian as well as non-Newtonian fluids by introducing "representative viscosity" η_{rep}.[8,10] In such a case, η_{rep} is usually defined according to the Metzner-Otto concept.[8] (See its criticism in.[9])

Rheological similarity

In process engineering research, experiments are often carried out with a model fluid and the results obtained are then transferred to the original fluid.

In devices of similar geometry, dimensionless formulated process equations agree in both cases as long as the fluids display a similar flow behavior, where a special emphasis is related to the problem of rheological similarity of non-Newtonian and particularly pseudo-plastic fluids.

Fluids are usually considered rheologically similar to each other if their flow curves $\eta = f(\dot\gamma)$ are congruent in the log/log field.[3] The core of the present investigation is the thesis that this definition is to close and that the concept of rheological similarity should be extended.

At given geometry of the device, the pi simulation of the transport characteristics depends in addition only on the master curve (1). (The result of the simulation is therefore a functional of the master curve.) Therefore, fluids are rheologically similar, if their flow curves can be depicted with suitable (Γ, H)-parameters on a common master curve. This results in coressponding transport characteristics.

Master curves can be gradually designed from available flow curves as well as put down *a priori* and *ad hoc* and afterwards proved by experimentally determined flow curves $\eta = f(\dot\gamma)$. The prospect to cover the master curve by real flow curves $\eta = f(\dot\gamma)$ increases considerably with an appropriate shape of the master curve.

Thus the request for congruency of flow curves is replaced by a minor condition of a flexible congruency (see Fig. 1). Flexible congruency is given, if flow curves can be depicted in a master curve by shifting.

Master curves can be congruently shifted twist-free in the log/log field. Displaced by $\log(\Delta\Gamma)$, $\log(\Delta H)$, the original (Γ_1, H_1) shift parameters are changed into (Γ_2, H_2) according to the following relationship

$$\begin{aligned} \log \Gamma_2 &= \log \Gamma_1 + \log (\Delta\Gamma) \\ \log H_2 &= \log H_1 + \log (\Delta H). \end{aligned} \tag{4}$$

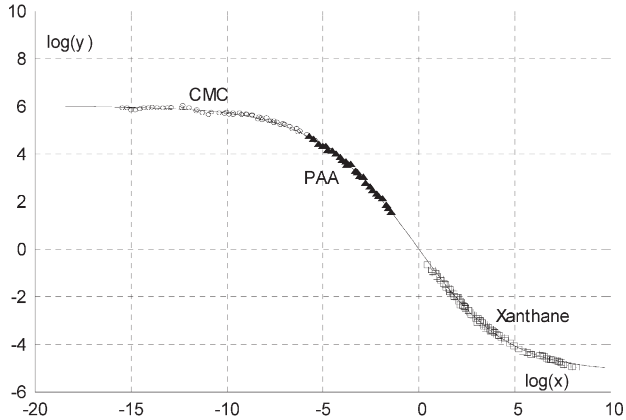
The eligibility of (Γ, H) is, however, insignificant because in this case the master curve and the process equations are shifted as a whole congruently in the log/log field (see Section Transformation behavior of the transport characteristics).

Process characteristics

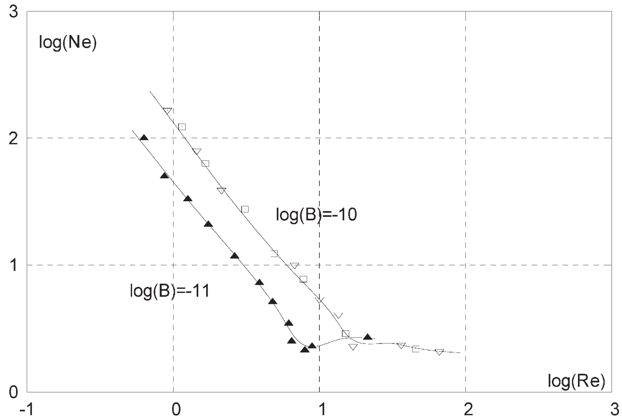
In the following, stirrer power characteristics $Ne(Re_H, B)$[3] and heat transfer characteristics $Nu/Pr_H^{1/3}(Re_H, B)$[10] are evaluated according to the theory presented. In principle, for its verification the hypothesis depends on experimental proof, that these characteristics coincide at different (Γ, H) values, i.e. at different rheologically similar fluids but at the same B values. This B = idem proof can be carried out – with the exception of Γ/H = idem – practically only in measuring devices of different size with a suitable d. Such a verification has been carried out for $Ne(Re_H, B)$ using three geometrically similar measuring appliances with aqueous polyacrylamide solutions (PAA) in.[3,Fig.8] Unfortunately, the relationship $Ne/Pr_H^{1/3} = Y(Re_H, B)$ with aq. Carboxymethylcellulose (CMC) and aq. Xanthane solutions could not be verified in this way because appliances of the same size were used.

Fig. 1 presents the master curve of three material groups. They show that these groups are rheologically similar in an extended sense (though the flow curves of CMC and Xanthane are not congruent even within a group and thus, in a narrow sense, they are not similar). The corresponding (Γ, H) as well the B parameters, which depend on them, are given in Table 1.

As a master curve y(x), the function with three parameters (p_3, p_4, p_5) has been chosen:



F i g . 1 – *Master curve with pictures of aqueous CMC, PAA and Xanthane solutions. (Γ, Η) parameters are listed in Table 1*



F i g . 2 – *Stirrer power characteristics in PAA solutions. With the right curve (log B = –10) covered twice the connection Ne(Re_H, B) is basically confirmed.*
log(B) = –10:
d = 72 mm, η_0 = 19.5 Pa s
d =144 mm, η_0 = 38.6 Pa s
d = 47 mm, η_0 = 41.6 Pa s
log(B) = –11:
d = 47 mm, η_0 = 41.6 Pa s

T a b l e 1 – *Shift parameters (Γ, Η) and B(Γ, Η) parameters*

	log(Γ)	log(Η)	log(B)
	PAA		
η_0 = 19.5 Pas	–6.29	5.02	–10.00
η_0 = 38.6 Pas	–6.59	5.32	–10.00
η_0 = 41.6 Pas	–6.62	5.35	–11.02
	CMC		
c = 2 g/l	14.38	–8.10	25.29
c = 5 g/l	10.48	–7.17	20.46
c = 8 g/l	9.58	–6.61	19.00
c = 14 g/l	7.85	–5.48	16.14
	Xanthane		
c = 0.25 g/l	–6.23	2.60	–6.02
c = 0.6 g/l	–4.24	2.60	–4.03
c = 1.0 g/l	–2.69	2.19	–2.07
c = 3.0 g/l	–1.22	1.88	–0.29
c = 6.0 g/l	–0.93	2.07	–0.19

$$\log(y/h) = -\tanh(\log(x/b))$$

$$\begin{aligned}&\text{at } \log x<0\text{: } b=p_3 \ h=p_5;\\ &\text{at } \log x>0\text{: } b=p_4,\ h=p_5p_4/p_3\end{aligned} \qquad (5)$$

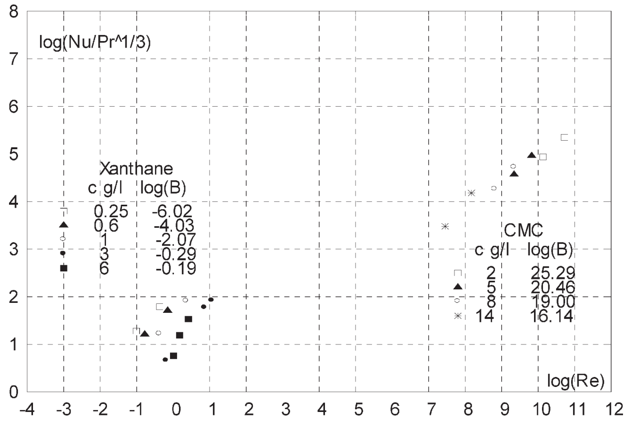
the form of which corresponds with wings both asymptotically ending to the common course of flow curves of pseudoplastic fluids. The optimized p parameters are: p_3=5.402, p_4=4.625, p_5=5.996.

Fig. 2 shows a theoretically consistent evaluation of stirrer power characteristics Ne(Re_H, B) in PAA solutions, from.[3] Measurements were conducted in three geometrically similar measuring devices. The results obtained were verified by B=idem proof. In the course of a transition to a creeping flow, these characteristics – unlike the Newtonian case – do not pass into a log/log strait line with a slope k = – 1: In the case of Newtonian fluids, due to irrelevance of material density ρ, the dependence of $Ne(Re)_\eta$ results in $NeRe_\eta$ = const, in which ρ no longer appears. On the other hand, in non-Newtonian fluids the analogous transformation of Ne(Re_H, B) leads to Ne · Re_H = f(B) where ρ (which is present at B) is not eliminated. The irrelevance of ρ in creeping flow is expressed first in equivalent description of Ne(Re_H, Q) with Q = Re_H/B ≡ n/Γ. To verify this, it would be necessary to adjust the measurements according to Q =const.

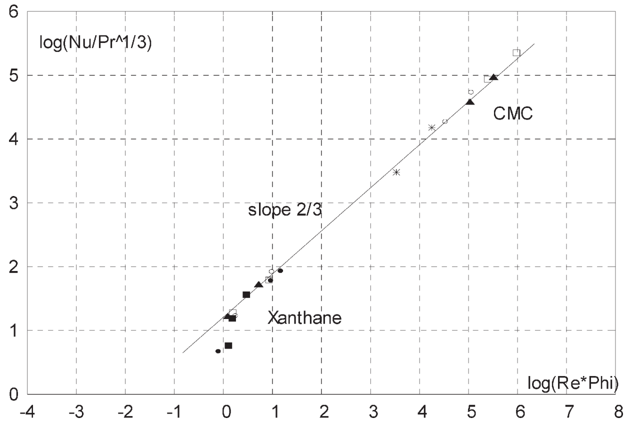
In Fig. 3, the heat transfer characteristics are presented in a compressed form analogous to the Newtonian case as follows:

$$Y \equiv Nu/Pr_H^{1/3} = f(Re_H, B) \qquad (6)$$

Because both groups of the aq. CMC and Xanthane solutions can be plotted on a common master curve (Fig. 1), the whole complex of curves is to be regarded as a uniform characteristic. The question, whether this is also true in case of a considerable gap between the measuring points would have to be examined using measurements by B=idem proof. The results of the present measurements, which were obtained using only one measuring device, cannot be used to verify this finding. The fact that the correlation of characteristics shown in Fig. 4



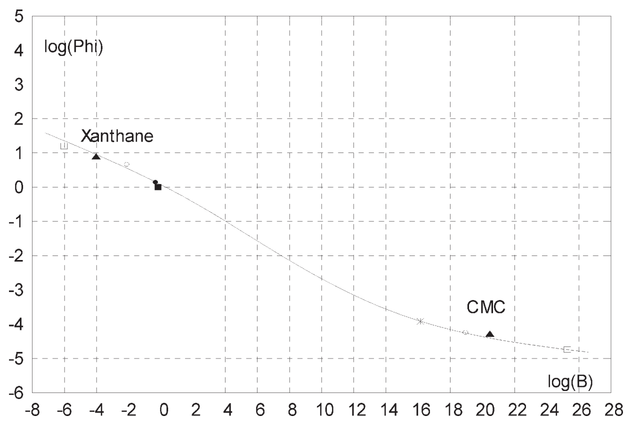
F i g . 3 – *Common heat transfer characteristics $Nu/Pr_H^{1/3}(Re_H, B)$ for CMC and Xanthane*



F i g . 4 – *Correlated characteristics $Nu/Pr_H^{1/3}\{Re_H \cdot \Phi(B)\}$ for both groups. Symbols as in Fig. 3*

$$Y \equiv Nu \cdot /Pr_H^{1/3} = f\{\Phi(B) \cdot Re_H\}. \tag{7}$$

can be presented with a smooth form of correlation function $\Phi(B)$ in Fig. 5 may confirm this expectation (see next section). It should, however, be pointed out that the relationship (7) is only approximative because the reduction is not a condition required by theory of similarity.



F i g . 5 – *$\Phi(B)$ function used in correlation in Fig. 4.*

Transformation behavior of transport characteristics

The freedom of positioning master curves in the log/log field is linked with corresponding ambiguity concerning shift parameters (Γ, H). To deal with this problem, transformation behavior of transport characteristics by varying of (Γ, H) is of significance, which is discussed using heat transfer characteristics $Y \equiv Nu/Pr_H^{1/3} = f(Re_H, B)$ in Fig. 3 as an example:

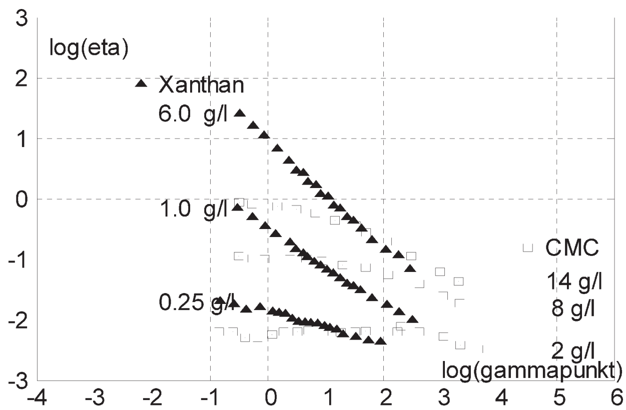
We start from a given fluid curve $\eta = f(\dot{\gamma})$ and arbitrary (Γ, H) parameters. By change from (Γ_1, H_1) to (Γ_2, H_2) the values Re_H, Y and B satisfy the following transformation equations:

$$Re_{H,2} = Re_{H,1} \cdot e^{-1},\ Y_2 = Y_1 \cdot e^{-1/3},\ B_2 = B_1 \cdot g e^{-1} \tag{8}$$

with $g \equiv \Gamma_2/\Gamma_1$ and $e \equiv H_2/H_1$. The heat transfer characteristics $Y(Re_H, B)$ are therefore in a log/log field congruently displaced along a fluid specific strait line with a slope of 1/3. It remains only to examine, whether the $\Phi(B)$ function in Fig. 5 is physically acceptable or not.

Conclusions

In Fig.6 some flow curves $\eta = f(\dot{\gamma})$ are given which are depicted on the master curve (Fig.1). In their totality, they are not congruent within homologous material groups and are according to the customary concept are looked upon as rheologically dissimilar. With the extended concept of flexible congruency (s. Section Rheological similarity) these fluids can be in their entirety regarded as rheologically similar to each other and the crowd of the curves in Fig. 3 can be interpreted as a common characteristics $Y \equiv Nu/Pr_H^{1/3} = f(Re_H, B)$ – with a $\Phi(B)$-function (Fig. 5). With that an entirely new basis is given for the rheological similarity of



F i g . 6 – *On the master curve (Fig.1) depicted flow curves $\eta = f(\dot{\gamma})$ of CMC and Xanthane are depicted here in the log/log field [10]* non-Newtonian pseudo-plastic fluids and the choice of rheologically similar fluids for model measurements is considerably broadened (provided that a verification with the B = idem test was affirmative).

Symbols

Dimensional quantities:

c_p – specific heat [W kg^{-1} K^{-1}]
d – diameter [m]
k_La – volume-related liquid-side mass transfer coefficient [s^{-1}]
L – length [m]
n – stirrer speed [s^{-1}]
P – stirrer power [W]
P/V – volume-related stirrer power [W m^{-3}]
Δp – pressure buildup [Pa m^{-1}]
q – fluid throughput [m^3 s^{-1}]
v – superficial velocity of gas [m s^{-1}]
α – heat transfer coefficient [W m^{-2} K^{-1}]
$\dot{\gamma}$ – shear rate [s^{-1}]
Γ – shift parameter [s^{-1}]
η – viscosity [Pa s]
η_{rep} – representative, effective viscosity [Pa s]
H – shift parameter, [Pa s]
θ – mixing time [s]
λ – heat conductivity [W m^{-1} K^{-1}]
ρ – material density [kg m^{-3}]

Dimensionless quantities:

$A \equiv \Gamma(H/\rho g^2)^{1/3}$ – dimensionless number
$B \equiv \rho d^2\Gamma/H$ – dimensionless number
$e \equiv H_2/H_1$
$g \equiv \Gamma_2/\Gamma_1$
$Ho \equiv n\theta$ resp. $\Gamma\theta$ – mixing (homogenization) time number
$(k_La)^* \equiv k_La\,(\eta/\rho g^2)^{1/3}$ resp. $k_La\,(H/\rho g^2)^{1/3}$
$Ne \equiv P/\rho n^3 d^3$ – Newton number
$Nu \equiv \alpha d/\lambda$ – Nusselt number
$(P/V)^* \equiv (P/V)/(\eta\rho^2 g^4)^{1/3}$ resp. $(P/V)/(H\rho^2 g^4)^{1/3}$
$Pr_\eta \equiv c_p\eta/\lambda$ resp. $Pr_H \equiv c_pH/\lambda$ – Prandtl number
$Q = n/\Gamma$
$Re_\eta \equiv \rho nd^2/\eta$ resp. $Re_H \equiv \rho nd^2/H$ – Reynolds number
$v^* \equiv v\,(\rho/\eta g)^{1/3}$ resp. $v\,(\rho/Hg)^{1/3}$
$\Phi(B)$ – function used for correlation

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