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TORSION OF THIN-WALLED BEAMS OF SYMMETRICAL OPEN CROSS-SECTIONS WITH INFLUENCE OF SHEAR

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Summary

The theory of torsion of thin walled beams of open cross-sections with influence of shear on the basis of the classical Vlasov's theory of thin-walled beams of open cross-sections for beams with single and double symmetrical cross-sections is developed. It is proved that the beam subjected to torsion with influence of shear exerted by couples in the beam cross-sections is also subjected to bending due to shear in the plane orthogonal to the plane of symmetry. The beam is subjected to torsion with influence of shear only in the case of double symmetrical cross-sections. The principal cross-section axes as well as the principal pole are defined according to the classical theory of thin-walled beams of open sections. Illustrative examples are given, as well as a comparison with the finite element method.

Key words: *torsion of thin-walled beams, influence of shear, open sections, single and double symmetrical section, analytic method; FEM*

1. Introduction

In classical theories of torsion of thin-walled beams with open cross-sections the warping of the cross-section due to shear is neglected [1-4].

By analogy to the advanced theories of bending, in an engineering approach [5-11], the concept of shear factors is considered in torsion [12-20].

In the case of single symmetrical cross-sections, the effect of bending due to shear in the plane orthogonal to the beam plane of symmetry, as the result of torsion with influence of shear, has not been sufficiently investigated [12,13,16,20]. Still, some results are available where numerical methods are applied [21-23].

In this paper, an analytical solution for shear factors in torsion will be investigated; the stress distributions along the beam cross-section contour will be given in the analytic form, as well as the stresses and displacements along the beam length. Beams with single and double symmetrical cross-sections under various torsion load conditions will be considered. The results will be compared with the finite element method.

2. Strains and displacements

The displacement of an arbitrary point $S(x,s)$ of the middle surface of a thin-walled beam of open cross-section with one axis of symmetry subjected to torsion can be expressed as

$$u_S = -\frac{d\alpha}{dx}\omega - \frac{dv}{dx}y + \int_0^s \gamma_{x\xi}\, ds\,, \tag{1}$$

where $\alpha = \alpha(x)$ is the angle of torsion, i.e. the rotation of the cross-section middle line as a rigid line with respect to a cross-section pole P, in the axis of symmetry, $v = v(x)$ is the displacement of the pole P in the y-direction, $y = y(s)$ is the orthogonal coordinate, $\gamma_{x\xi} = \gamma_{x\xi}(x,s)$ is the shear strain in the middle surface, s is the curvilinear coordinate of the middle line, ξ is the tangential axis on the curvilinear coordinate s; Oxyz is the orthogonal coordinate system, where the z-axis is the axis of symmetry (Fig. 1);

$$\omega = \int_0^s h_P\, ds\,, \quad d\omega = h_P ds\,, \tag{2}$$

where $\omega = \omega(s)$ is the sectorial coordinate for the pole P and $h_P = h_P(s)$ is the distance of the tangent through the arbitrary point S at the middle line from the pole P.

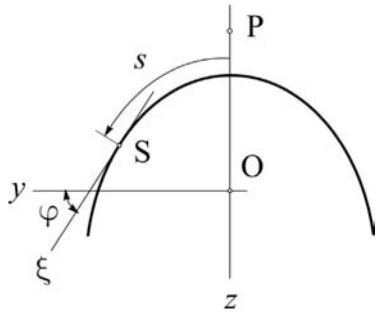

Fig. 1 Cross-section middle-line

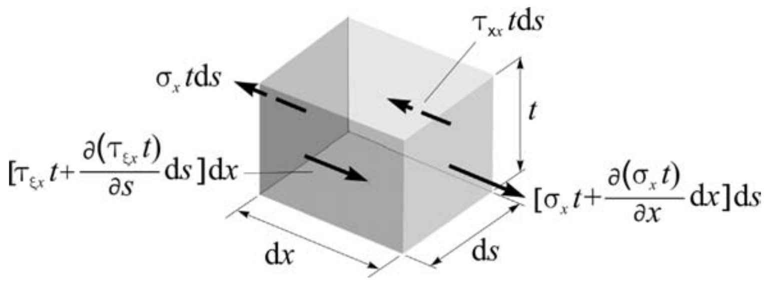

Fig. 2 The equilibrium of the element of the wall

Here $\omega(s=0)=0$, so Eq. (1) may be expressed as

$$u_S = \vartheta\omega - \gamma y + \int_0^s \gamma_{x\xi}\, ds\,, \quad \vartheta = -d\alpha/dx\,, \quad \gamma = dv/dx\,; \tag{3}$$

where $\vartheta = \vartheta(x)$ is the relative angular displacement of the middle line as rigid line with respect to the pole P and $\gamma = \gamma(x)$ is the angular displacement of the middle line as rigid line with respect to the z-axis.

Thus, it is assumed that the cross-section middle line is displaced in the longitudinal direction due to warping, as in the case of the ordinary theory of torsion, expressed by the first member of Eq. (3), and in addition, due to the influence of shear, expressed by the second and third members of Eq. (3).

The displacements may be separated as follows

$$\alpha = \alpha_t + \alpha_a\,, \quad v = v_a\,, \tag{4}$$

where $\alpha_t = \alpha_t(x)$ is the angular displacement of the cross-section as plane section with respect to the pole P, as in the case of the classical theories of thin-walled beams of open cross-sections, $\alpha_a = \alpha_a(x)$ and $v_a = v_a(x)$ are the additional displacements due to shear.

Then

$$\vartheta = \vartheta_t + \vartheta_a\,,\quad \vartheta_t = -\mathrm{d}\alpha_t/\mathrm{d}x\,,\quad \vartheta_a = -\mathrm{d}\alpha_a/\mathrm{d}x\,,\quad \gamma = \gamma_a = \mathrm{d}v_a/\mathrm{d}x\,. \tag{5}$$

The strain in the beam longitudinal direction may be then expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\omega - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}y + \int_0^s \frac{\partial \gamma_{x\xi}}{\partial x}\mathrm{d}s\,. \tag{6}$$

3. Stresses and displacement

By ignoring the normal stresses in the transverse directions, Hooke's law may be written as

$$\sigma_x = E\varepsilon_x \qquad \tau_{x\xi} = G\gamma_{x\xi}\,, \tag{7}$$

where E is the modulus of elasticity and G is the shear modulus.

Thus

$$\sigma_x = -E\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\omega - E\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}y + \frac{E}{G}\int_0^s \frac{\partial \tau_{x\xi}}{\partial x}\mathrm{d}s\,. \tag{8}$$

From the equilibrium of a differential portion of the beam wall, it may be written

$$\tau_{x\xi} = \frac{1}{t}\left[-\int_0^s \frac{\partial\left(\sigma_x t\right)}{\partial x}\mathrm{d}s + f(x)\right],\quad f = f(M)\tau_{\xi x}(x,M) = T_M(x)\,;$$

$t = t(s)$ is the wall thickness (Fig. 2).

If $\partial\tau_{x\xi}/\partial x = const.$, the shear stress, referring to (8), can be expressed as

$$\tau_{x\xi} = \frac{1}{t}\left[T_M + E\left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}S_z(s) + \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3}S_\omega(s)\right)\right],\; S_z(s) = \int_0^s y\,\mathrm{d}A\,,\; S_\omega(s) = \int_0^s \omega\,\mathrm{d}A\,,\; \mathrm{d}A = t\mathrm{d}s\,. \tag{9}$$

Eq. (9) may be rewritten as

$$\tau_{x\xi} = -\frac{E}{t}\left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}S_z^* + \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3}S_\omega^*\right),\; S_z^* = \int_{s^*} y\,\mathrm{d}A^*\,,\; S_\omega^* = \int_{s^*} \omega\,\mathrm{d}A^*\,,\; \mathrm{d}A^* = t\mathrm{d}s^*,\; \mathrm{d}s^* = -\mathrm{d}s\,, \tag{10}$$

where $S_z^* = S_z^*(s)$ is the moment of the cut-off portion of area with respect to the z-axis, $S_\omega^* = S_\omega^*(s)$ is the sectorial moment of the cut-off portion of area with respect to the sectorial coordinate ω, s^* is the curvilinear coordinate of the cut-off portion of the beam wall of area $A^* = A^*(s)$, from the free edge, i.e. where $\tau_{x\xi} = 0$.

It is assumed that the normal stress given by Eq. (8) and the shear stresses given by Eqs. (9) and (10) are constant across the wall thickness. According to the assumption that cross-sections maintain their shape during deformations, the St. Venant pure torsion may be included by the linearly distributed component $\tau_{x\xi,V}^{\mathrm{V}} = \tau_{x\xi}^{\mathrm{V}}(x,s)$,

$$\tau_{x\xi}^{\mathrm{V}} = M_t\eta/I_t\,, \tag{11}$$

where $M_t = M_t(x)$ is the moment of pure torsion, given as

$$M_t = GI_t \frac{\mathrm{d}\alpha_t}{\mathrm{d}x} = -GI_t \vartheta_t, \qquad I_t = \frac{1}{3}\int_L t^3 \mathrm{d}s. \tag{12}$$

Thus, the total shear stress $\tau_{x\xi}^{\mathrm{tot}} = \tau_{x\xi}^{\mathrm{tot}}(x,s)$ is

$$\tau_{x\xi}^{\mathrm{tot}} = \tau_{x\xi} + \tau_{x\xi}^{\mathrm{V}}. \tag{13}$$

4. Equilibrium equations

It is assumed that the beam is loaded by couples in the cross-section planes, i.e. by moments per unit length $m_P = m_P(x)$:

$$m_P = \int_L p_z y \, \mathrm{d}s, \tag{14}$$

where $p_z = p_z(x,s)$ is the surface loads with respect to the *z*-axes and *L* is the cross-section middle line length.

For a portion of the beam wall, the following equilibrium equations can be written

$$\sum F_y = \int_L \frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} \cos\varphi \, \mathrm{d}x \, \mathrm{d}s = 0, \quad \sum M_P = \int_L \frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} \mathrm{d}x \, h \mathrm{d}s + \frac{\mathrm{d}M_t}{\mathrm{d}x}\mathrm{d}x + m_P \mathrm{d}x = 0 \tag{15}$$

where

$$\int_L \left(\frac{\partial \tilde{M}_t}{\partial x}\right) \mathrm{d}s = \frac{\partial}{\partial} \int_L \tilde{M}_t \mathrm{d}s \, \mathrm{d}x = \frac{\mathrm{d}M_t}{\mathrm{d}x}\mathrm{d}x.$$

Taking into account Eqs. (2), Eqs. (15) can be rewritten as

$$\int_L \frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} \mathrm{d}y = 0, \quad \int_L \frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} \mathrm{d}\omega + \frac{\mathrm{d}M_t}{\mathrm{d}x} + m_P = 0. \tag{16}$$

By integrating by parts one has

$$\frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} y \Bigg|_{e_1}^{e_2} - \int_L y \frac{\partial}{\partial s}\left[\frac{\partial\left(\tau_{x\xi} t\right)}{\partial x}\right] \mathrm{d}s = 0, \quad \frac{\partial\left(\tau_{x\xi} t\right)}{\partial x} \omega \Bigg|_{e_1}^{e_2} - \int_L \omega \frac{\partial}{\partial s}\left[\frac{\partial\left(\tau_{x\xi} t\right)}{\partial x}\right] \mathrm{d}s + \frac{\mathrm{d}M_t}{\mathrm{d}x} + m_P = 0 \tag{17}$$

where e_1 and e_2 are the boundaries, where $\tau_{x\xi} = 0$.

Thus,

$$\int_L y \frac{\partial}{\partial x}\left[\frac{\partial\left(\tau_{x\xi} t\right)}{\partial s}\right] \mathrm{d}s = 0, \quad \int_L \omega \frac{\partial}{\partial s}\left[\frac{\partial\left(\tau_{x\xi} t\right)}{\partial x}\right] \mathrm{d}s - \frac{\mathrm{d}M_t}{\mathrm{d}x} - m_P = 0 \cdot \tag{18}$$

By substituting Eqs. (9) and (12) one has

$$EI_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + EI_{z\omega} \frac{\mathrm{d}^4 \alpha}{\mathrm{d}x^4} = 0, \quad EI_{\omega z} \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + EI_{\omega} \frac{\mathrm{d}^4 \alpha}{\mathrm{d}x^4} = m_{\omega}, \tag{19}$$

where: $I_z = \int_A y^2 \mathrm{d}A$, $I_{z\omega} = I_{\omega z} = \int_A y\omega \, \mathrm{d}A$, $I_\omega = \int_A \omega^2 \mathrm{d}A$, $m_\omega = m_P + \mathrm{d}M_t / \mathrm{d}x$;

i.e. recalling Eq. (12)

$$m_\omega = m_P + GI_t \frac{\mathrm{d}^2 \alpha_t}{\mathrm{d}x^2} = m_P - GI_t \frac{\mathrm{d}\vartheta_t}{\mathrm{d}x}. \tag{20}$$

If y and ω are the principal coordinate, i.e. if the pole P is the principal pole, when $I_{z\omega} = I_{\omega z} = 0$, Eqs. (19) take the following simple forms

$$\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = 0, \qquad EI_\omega \frac{\mathrm{d}^4 \alpha}{\mathrm{d}x^4} = m_\omega, \tag{21}$$

5. Internal forces and shear stresses

Integration of the shear stress components $\tau_{x\xi}$ over the cross-sections gives

$$\int_A \tau_{x\xi} \cos\varphi \, \mathrm{d}A = 0, \qquad M_\omega = \int_A \tau_{x\xi} h_P \, \mathrm{d}A, \tag{22}$$

where $M_\omega = M_\omega(x)$ is the sectorial moment of torsion with respect to the pole P.

Substitution of Eq. (10) into Eqs. (22) gives

$$\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = 0, \qquad M_\omega = -EI_\omega \frac{\mathrm{d}^3 \alpha}{\mathrm{d}x^3}, \tag{23}$$

where: $\cos\varphi \, \mathrm{d}A = \cos\varphi \, t\mathrm{d}s = t\mathrm{d}y$, $h_P \, \mathrm{d}A = h_P \, t\mathrm{d}s = t\mathrm{d}\omega$, $\int_L S_z^* \, \mathrm{d}\omega = I_{z\omega} = 0$, $\int_L S_\omega^* \, \mathrm{d}\omega = I_\omega$.

Referring to Eqs. (21) and (23) gives

$$\mathrm{d}M_\omega / \mathrm{d}x = -m_\omega . \tag{24}$$

Thus, by substituting Eq. (19) into (10), the shear stress component $\tau_{x\xi}$ can finally be written as

$$\tau_{x\xi} = \frac{M_\omega S_\omega^*}{I_\omega t}. \tag{25}$$

According to the assumption $\partial\tau_{x\xi} / \partial x = const.$, it follows, referring to (24), that $m_\omega = const$. Referring to (20), if $m_P = const$, then $\mathrm{d}\vartheta/\mathrm{d}x = const.$, $\mathrm{d}M_t/\mathrm{d}x = const$.

That could be accepted only as an approximation; or the case when the St. Venant shear component (11) may be ignored with respect to the warping component (25).

6. Internal forces and normal stresses

Integration of the normal stresses over the cross-sections gives

$$B = \int_A \sigma_x \omega \, \mathrm{d}A, \qquad \int_A \sigma_x y \, \mathrm{d}A = 0, \tag{26}$$

where $B = B(x)$ is the bimoment.

By substituting Eq.(8) into Eqs. (26), the following can be written

$$B = -EI_\omega \frac{\mathrm{d}^2 \alpha}{\mathrm{d}x^2} - B^\omega, \qquad 0 = EI_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - M_z^\omega, \tag{27}$$

where

$$B^\omega = -\frac{E}{G} \int_A \omega \, \mathrm{d}A \int_0^s \frac{\partial \tau_{x\xi}}{\partial x} \mathrm{d}s, \qquad M_z^\omega = \frac{E}{G} \int_L y \, \mathrm{d}A \int_0^s \frac{\partial \tau_{x\xi}}{\partial x} \mathrm{d}s, \tag{28}$$

i.e. referring to Eq. (25)

$$B^\omega = m_\omega \frac{E}{GI_\omega} \int_A \left(\frac{S_\omega^*}{t} \right)^2 \mathrm{d}A, \qquad M_z^\omega = -m_\omega \frac{E}{GI_\omega} \int_L \frac{S_z^* S_\omega^*}{t} \mathrm{d}s. \tag{29}$$

Referring to Eqs. (23) and (27), it follows

$$-EI_\omega \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3} = \frac{\mathrm{d}B}{\mathrm{d}x} + \frac{\mathrm{d}B^\omega}{\mathrm{d}x} = M_\omega , \qquad EI_z \frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = \frac{\mathrm{d}M_z^\omega}{\mathrm{d}x} = 0 , \tag{30}$$

and according to Eq.(21)

$$-EI_\omega \frac{\mathrm{d}^4\alpha}{\mathrm{d}x^4} = \frac{\mathrm{d}^2 B}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 B^\omega}{\mathrm{d}x^2} = \frac{\mathrm{d}M_\omega}{\mathrm{d}x} = -m_\omega , \qquad EI_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = 0 , \tag{31}$$

where according to Eqs. (13) and (20)

$$M_P = M_\omega + M_t , \qquad m_P = -\mathrm{d}M_P / \mathrm{d}x . \tag{32}$$

It is assumed that $m_\omega = const.$; for $m_\omega \neq const.$, Eqs. (30) and (31) give an approximate solution to the problem.

The normal stress given by Eq. (8), referring to Eqs. (24), (25) and (27), can be expressed as

$$\sigma_x = \frac{B}{I_\omega}\omega + \frac{B^\omega}{I_\omega}\omega - \frac{E}{G}\cdot\frac{m_\omega}{I_\omega}\int_0^s \frac{S_\omega^*}{t}\mathrm{d}s - \frac{M_z^\omega}{I_z} y . \tag{33}$$

The internal forces given by Eq. (29) can also be written as

$$B^\omega = \frac{EI_\omega}{GI_P}\kappa_{\omega\omega} m_\omega , \qquad M_z^\omega = -\frac{EI_z}{GW_P}\kappa_{y\omega} m_\omega , \tag{34}$$

where

$$\kappa_{\omega\omega} = \frac{I_P}{I_\omega^2}\int_A \left(\frac{S_\omega^*}{t}\right)^2 \mathrm{d}A , \qquad \kappa_{y\omega} = \frac{W_P}{I_z I_\omega}\int_A \frac{S_z^* S_\omega^*}{t^2}\mathrm{d}A \tag{35}$$

are the shear factors with respect to the α -displacements and to the v-displacements during the α -displacements, respectively;

$$I_P = \int_A h_P^2 \,\mathrm{d}A , \qquad W_P = \frac{I_P}{h_0} \tag{36}$$

are the polar second moment of area with respect to the principal pole P and the polar section modulus with respect to the pole, respectively; h_0 is the distance of the tangent through the arbitrary starting point M_0 (where the principal coordinate ω is equal to zero) from the principal pole P.

Hence, the normal stresses given by (33) can also be written as

$$\sigma_x = \frac{B}{I_\omega}\omega + \frac{E\kappa_{\omega\omega}}{GI_P} m_\omega - \frac{E}{GI_\omega} m_\omega \int_0^s \frac{S_\omega^*}{t}\mathrm{d}s + \frac{E\kappa_{y\omega}}{GW_P} m_\omega y . \tag{37}$$

7. Differential equations with separated displacements

Eqs. (27), according to Eqs. (34) and (35) can be expressed as

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} = -\frac{B}{EI_\omega} - \frac{m_\omega \kappa_{\omega\omega}}{GI_P} , \qquad \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{\kappa_{y\omega}}{GW_P} m_\omega . \tag{38}$$

Eqs. (38), referring to Eqs. (4) can be separated as

$$\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2} = -\frac{B}{EI_\omega} , \quad \frac{\mathrm{d}^2\alpha_a}{\mathrm{d}x^2} = -\frac{\kappa_{\omega\omega}}{GI_P} m_\omega , \quad \frac{\mathrm{d}^2 v_a}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{\kappa_{y\omega}}{GW_P} m_\omega . \tag{39}$$

Integrating the 2nd and 3rd of Eqs. (39), taking into account Eq. (24) gives

$$\frac{\mathrm{d}\alpha_a}{\mathrm{d}x} = -\vartheta_a = -\frac{M_\omega \kappa_{\omega\omega}}{GI_P}, \qquad \frac{\mathrm{d}v_a}{\mathrm{d}x} = \gamma_a = \frac{M_\omega \kappa_{y\omega}}{GW_P} \tag{40}$$

where the integration constants are ignored; it is assumed that the angular displacements ϑ_a and γ_a do not depend on the boundary conditions.

The first of Eqs. (39) is the well known equation of the classical theory of torsion of thin-walled beams, where

$$EI_\omega \frac{\mathrm{d}^3\alpha_t}{\mathrm{d}x^3} = -\frac{\mathrm{d}B}{\mathrm{d}x} = -M_\omega, \quad EI_\omega \frac{\mathrm{d}^4\alpha_t}{\mathrm{d}x^4} = -\frac{\mathrm{d}^2B}{\mathrm{d}x^2} = -\frac{\mathrm{d}M_\omega}{\mathrm{d}x} = m_\omega, \quad \frac{\mathrm{d}\alpha_t}{\mathrm{d}x} = -\vartheta_t. \tag{41}$$

Eqs. (40) take into account the displacement due to shear. Integrating Eqs. (40) gives

$$\alpha_a = \frac{\kappa_{\omega\omega}}{GI_P} B + C_\alpha, \quad v_a = \frac{\kappa_{y\omega}}{GW_P} B + C_v, \tag{42}$$

where C_α and C_v are the integration constants, with respect to the α and v-displacements, respectively.

Eqs. (42) can also be written as

$$\alpha_a = \frac{B}{GI_{Pr}} + C_\alpha, \quad v_a = \frac{B}{GW_{Py}} + C_v, \quad I_{Pr} = \frac{I_P}{\kappa_{\omega\omega}}, \quad W_{Py} = \frac{W_P}{\kappa_{y\omega}} \tag{43}$$

where I_{Pr} and W_{Py} are the reduced polar second moment of area and the reduced polar modulus of area due to shear, respectively.

The normal stresses may then be written as

$$\sigma_x = \frac{B}{I_\omega}\omega + \frac{E}{GI_{Pr}} m_\omega - \frac{E}{GI_\omega} m_\omega \int_0^s \frac{S_\omega^*}{t}\mathrm{d}s + \frac{E}{GW_{Py}} m_\omega y. \tag{44}$$

8. Shear strain energy

According to Hooke's law, taking into account Eq. (5), the second of Eqs. (36) and the first of Eqs. (40), the average shear stresses with respect to the displacements α_s, i.e. $\alpha_s h_0$, can be expressed as

$$\tau_{x\xi,av} = G\gamma_{x\xi,av} = G\frac{d}{\mathrm{d}x}\left(\alpha_a h_0\right) = \frac{\kappa_{\omega\omega}}{W_P} M_\omega, \tag{45}$$

where $\gamma_{x\xi,av}$ is the average shear strain with respect to the displacements α_s, i.e. $\alpha_a h_0$.

The average shear stresses can also be expressed as

$$\tau_{x\xi,av} = M_\omega / W_{P\omega}, \quad W_{P\omega} = W_P / \kappa_{\omega\omega}. \tag{46}$$

The shear energy of the beam element may be expressed as

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G}\int_A \tau_{x\xi}^2 \mathrm{d}A, \tag{47}$$

i.e. according to Eq. (25)

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G}\frac{M_\omega^2}{I_\omega^2}\int_A \left(\frac{S_\omega^*}{t}\right)^2 \mathrm{d}A. \tag{48}$$

The shear energy can be expressed also by average shear deformations as

$$\mathrm{d}U = \frac{\mathrm{d}x}{2}\left(-\vartheta_a\right)M_\omega = \frac{\mathrm{d}x}{2}\gamma_{x\xi,av}\frac{M_\omega}{h_0} = \frac{\mathrm{d}x}{2G}\tau_{x\xi,av}\frac{M_\omega}{h_0}. \tag{49}$$

i.e. taking into account the second of Eqs. (36) and Eq. (44)

$$\mathrm{d}U = \frac{\mathrm{d}x}{2G}\frac{\kappa_{\omega\omega}}{I_P}M_\omega^2. \tag{50}$$

The shear factor $\kappa_{\omega\omega}$ can be obtained by equating (50) and (48). The result will be equal to the obtained shear factor, given by (35).

9. Boundary conditions

Boundary conditions can be defined as follows, at the staring section D,

$$\alpha_a = 0, \quad v_a = 0. \tag{51}$$

Hence, referring to Eq. (43),

$$C_\alpha = -\frac{B_D}{GI_{Pr}}, \quad C_v = -\frac{B_D}{GW_{Py}}, \quad B_D = B(x = x_D) \tag{52}$$

The total displacements then are

$$\alpha = \alpha_t + \frac{B - B_D}{GI_{Pr}}, \quad v = \frac{B - B_D}{GW_{Py}}, \tag{53}$$

For the hinged sections it may be written

$$\alpha\big|_{x=x_D} = \alpha_t\big|_{x=x_D} = 0,\ \left.\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2}\right|_{x=x_D} = 0\ \left(B_D = 0\right);\ \alpha\big|_{x=x_E} = \alpha_t\big|_{x=x_E} = 0,\ \left.\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2}\right|_{x=x_E} = 0.\ \left(B_E = 0\right) \tag{54}$$

For the clamped sections:

$$\alpha\big|_{x=x_D} = \alpha_t\big|_{x=x_D} = 0, \quad \left.\frac{\mathrm{d}\alpha_t}{\mathrm{d}x}\right|_{x=x_D} = 0 \quad \left(\vartheta_D^t = 0\right);$$

$$\alpha\big|_{x=x_E} = \alpha_t\big|_{x=x_E} + \frac{1}{GI_{Ps}}\left(-EI_\omega\left.\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2}\right|_{x=x_E} + EI_\omega\left.\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2}\right|_{x=x_D}\right) = 0, \quad \left.\frac{\mathrm{d}\alpha_t}{\mathrm{d}x}\right|_{x=x_E} = 0 \quad \left(\vartheta_E = 0\right). \tag{55}$$

For the free section:

$$\left.\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2}\right|_{x=x_D} = 0 \quad \left(B_D = 0\right); \quad \left.\frac{\mathrm{d}^3\alpha_t}{\mathrm{d}x^3}\right|_{x=x_D} = 0 \quad \left(M_{D\omega} = 0\right). \tag{56}$$

10. Double symmetrical cross-section

For double symmetrical cross-sections the normal stresses given by Eq. (44) become

$$\sigma_x = \frac{B}{I_\omega}\omega + \frac{E}{GI_{Pr}}m_\omega - \frac{E}{GI_\omega}m_\omega\int_0^s \frac{S_\omega^*}{t}\mathrm{d}s, \tag{57}$$

where due to symmetry

$$\kappa_{y\omega} = 0 \quad \left(W_{Py} = \propto\right), \tag{58}$$

i.e. referring to the second equation of Eqs. (35)

$$\int_A \frac{S_z^* S_\omega^*}{t^2}\mathrm{d}A = 0. \tag{59}$$

The total displacements given by Eq. (53) become

$$\alpha = \alpha_t + \frac{B - B_D}{I_{Pr}}, \quad v = 0, \tag{60}$$

11. Illustrative examples

The I-section with two axes of symmetry (Fig. 3.a) and a symmetrical U-section (Fig. 3.b) are considered.

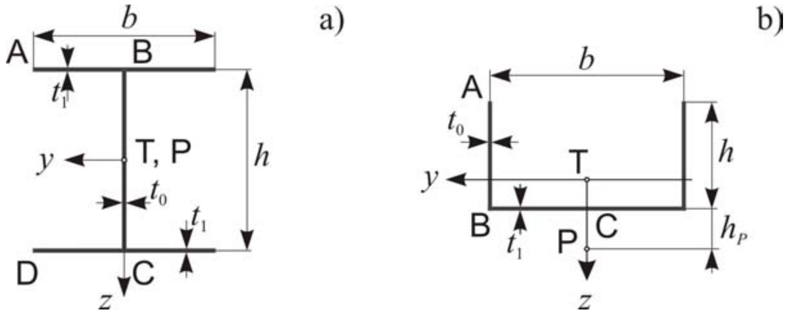

Fig. 3 Analysed cross-sections: a) double symmetrical I-section; b) symmetrical U-section

The shear factors for the double symmetrical I-section, according to Eqs. (58) and (35) are:

$$\kappa_{\omega y} = 0, \quad \kappa_{\omega\omega} = 6/5 . \tag{61}$$

The shear factors for the symmetrical U-section, according to Eqs. (35) are:

$$\kappa_{\omega y} = -\frac{\left[18\psi + \rho^2 (1+6\psi)^2\right] \cdot \left[10(5+6\psi) - 2\rho^2\right]}{20\rho^2 (2+3\psi)(1+6\psi)^2},$$

$$\kappa_{\omega\omega} = \frac{3\left[18\psi + \rho^2 (1+6\psi)^2\right] \cdot \left[2\left(8+21\psi+18\psi^2\right) + 3\psi\rho^2\right]}{10\rho^2 (1+6\psi)^2 (2+3\psi)^2}, \tag{62}$$

where: $A_1 = bt_1$, $A_0 = ht_0$, $\psi = A_0/A_1$, $\rho = b/h$, $L_s = h$, $h_P = 3h\psi/(1+6\psi)$.

Shear factors given by (62), for the U-section beam ($b = 5$ m, $h = 3.5$ m, $t = 0.2$ m) are indirectly compared with those presented in [21] as shown in Table 1.

Table 1 Comparison of shear factors for the U-section

Presented theory	Kim [21]
$W_{Py} = \dfrac{W_P}{\kappa_{\omega y}} = -5.77375 \quad \text{m}^3$	$A_{3r} = -W_{Py} = 5.77375 \quad \text{m}^3$
$I_{Pr} = \dfrac{I_P}{\kappa_{\omega\omega}} = 7.28238 \quad \text{m}^4$	$A_r = I_{Pr} = 7.28238 \quad \text{m}^4$

The comparison of the angle of torsion and the lateral displacement, given by (53), at the free end of the cantilevered U-section beam (l = 18 m, b = 5 m, h = 3.5 m, t = 0.2 m) subjected to the end moment of torsion M_t = 1000 kNm, with material properties $E = 30000$ kN/m^2 and $G = 13000$ kN/m^2 is given in Table 2.

Table 2 Comparison of the angle of torsion and the lateral displacement of the U-section beam

	Presented theory	Kim [21]	El Fatmi [23]
$\alpha \cdot 10^{-3}$ rad	4.253	4.236	4.203
$v \cdot 10^{-4}$ m	2.172	2.163	2.369

The range of examples has been carried out by the FEM using Autodesk Algor Simulation Pro in order to compare the results with those obtained analytically, by the presented theory (TIS – **T**orsion with **I**nfluence of **S**hear). Shell elements with 5 DOF are used. Mesh was generated with square elements with sides of $h/40$.

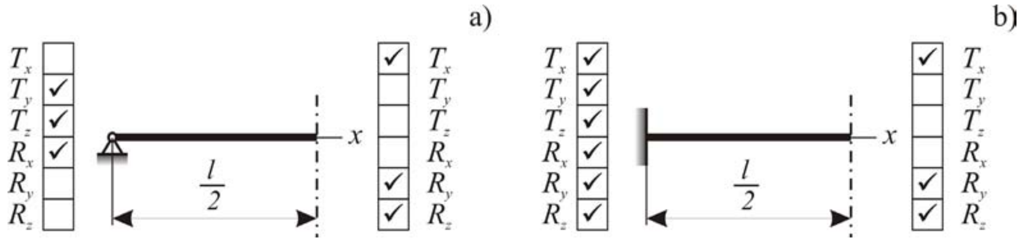

Fig. 4 The boundary conditions: a) a simply supported beam; b) a clamped beam

Due to symmetry, only one half of the beam is modelled. Fig. 4 shows the boundary conditions that are used: at the simply supported end and at $x = l/2$ (Fig. 4.a), at the clamped end and at $x = l/2$ (Fig. 4.b). The sign ☑ means that a certain displacement, translation *T* or rotation *R*, is constrained.

The beams under uniformly distributed moments of torsion per unit length m_P were analysed, where:

$$h = 400 \text{ mm},\ b = h \text{ (I-sec.)},\ b = 2h \text{ (U-sec.)},\ t_1 = t_2 = t_0 = h/40,\ E = 210 \text{ GPa},\ \nu = 0.3 .$$

Some results in comparison with the FEM analysis are presented in Tables 3 and 4 and in Figs. 5 and 6.

The normal stresses in the *x*-direction at the selected point of the beam cross-section are normalised as follows: $\sigma_x / \sigma_{x,\max}^{Vlasov}$, $\sigma_x^{FEM} / \sigma_{x,\max}^{Vlasov}$, where σ_x is the normal stress in the *x*-direction at the selected point obtained analytically by Eq. (75), σ_x^{FEM} is the maximal normal stress in the *x*-direction at that point obtained by the FEM, and $\sigma_{x,\max}^{Vlasov}$ is the maximal normal stress in the *x*-direction at the point A (Fig. 3) according to the classical Vlasov's theory.

Table 3 Normalised maximal normal stresses

	Double sym. I-section (point A, Fig. 8.a)				Symmetrical U-section (point A, Fig. 8.b)			
	Simply supported		Clamped		Simply supported		Clamped	
L/h	TIS	FEM	TIS	FEM	TIS	FEM	TIS	FEM
3	1.039	1.041	1.116	1.091	1.025	1.026	1.073	1.057
5	1.014	1.016	1.042	1.032	1.009	1.010	1.026	1.012

The maximal angles of torsion are normalised as: α / α^{Vlasov} and $\alpha^{FEM} / \alpha^{Vlasov}$, where α is the maximal angle of torsion obtained analytically by Eq. (78), $\alpha^{FEM} = v_B^{FEM} / l_P$ is the maximal angle of torsion obtained indirectly by the FEM, where v_B^{FEM} is the horizontal displacement of the point B obtained by the FEM with $l_P = h/2$ for a double symmetrical I-section and $l_P = h_P$ for a symmetrical U-section; $\alpha^{Vlasov} = \alpha_t$ is the maximal angle of torsion according to the classical Vlasov's theory.

Table 4 Normalised maximal angles of torsion for various ratios L/h

	Double symmetrical I-section				Symmetrical U-section			
	Simply supported		Clamped		Simply supported		Clamped	
L/h	TIS	FEM	TIS	FEM	TIS	FEM	TIS	FEM
3	1.277	1.295	2.387	2.359	1.129	1.123	1.644	1.596
5	1.100	1.105	1.500	1.483	1.046	1.044	1.232	1.206

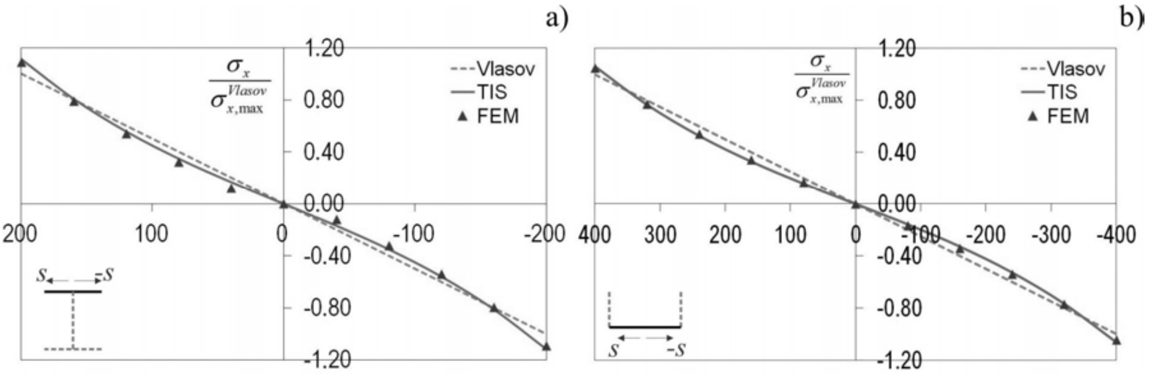

Fig. 5 Normalised normal stresses at the clamped beam midspan (L=3h):
a) top flange of the double symmetrical I-section, b) horizontal wall of the symmetrical U-section

The normalised horizontal displacements of the point B, in the case of double symmetrical I-section, are expressed as $v_B / v_{B,\max}^{Vlasov}$ and $v_B^{FEM} / v_{B,\max}^{Vlasov}$, where v_B is the total horizontal displacements of the point B obtained analytically and $v_B^{Vlasov} = \alpha_t h_P^B$ is the horizontal displacement of the point B according to the classical Vlasov's theory, whereas h_P^B is the distance between points B and P.

The normalised horizontal displacements of the point C, in the case of symmetrical U-sections, are expressed as $v_C / v_{C,\max}^{Vlasov}$ and $v_C^{FEM} / v_{C,\max}^{Vlasov}$, where v_C is the total horizontal displacements of the point C obtained analytically, v_C^{FEM} is the horizontal displacement of the point C by the FEM, and $v_C^{Vlasov} = \alpha_t h_P$ is the horizontal displacement of the point C according to the classical Vlasov's theory. Then the total horizontal displacement of the point C of the cross-section can be expressed as $v_C = v_C^{Vlasov} + v_C^a + v_C^{M_z}$ where v_C^a is the horizontal displacement of the point C due to shear, and $v_C^{M_z}$ is the horizontal displacement of the point C due to bending caused by shear.

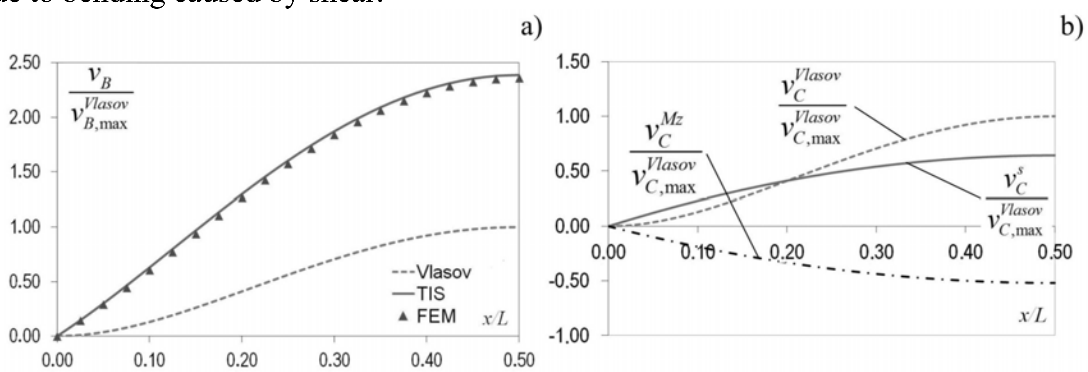

Fig. 6 Normalised horizontal displacements at the clamped beam (L=3h): a) total displacement of the point B of double symmetrical I-section, b) component displacements of the point C of symmetrical U-section

12. Conclusion

A theory of torsion of thin-walled beams with influence of shear for open cross-sections with one axis and two axes of symmetry is developed. The theory is based on the classical Vlasov's theory. The shear factors with respect to the torsion are given in the analytic form.

It is proved that the beam with single symmetrical sections, loaded by couples in the plane of the cross-sections, is also subjected to bending due to shear in the plane orthogonal to the plane of symmetry.

Thus, a new shear factor is introduced, given by (35) and (62), with respect to bending due to shear, as a result of torsion, which vanishes for double symmetrical cross-sections.

For various types of cross-sections with one and two axes of symmetry, the shear factors are given in the parametric forms.

The normal stress can be obtained in the analytic form both along the cross-section middle line and the beam length. Both simply supported and clamped beams under uniformly distributed moments of torsion per unit length are considered.

Several examples are analyzed in comparison with the finite element method. Excellent agreements of the results for displacements are obtained, as well as for the normal stresses. Some discrepancies for normal stresses are noticed by the presented theory and the finite element method at the beam ends in the case of clamped ends, as a result of different boundary conditions.

Appendix A: Cross-section functions

Cross-section functions for the double symmetrical I-section (Fig. A1) are:

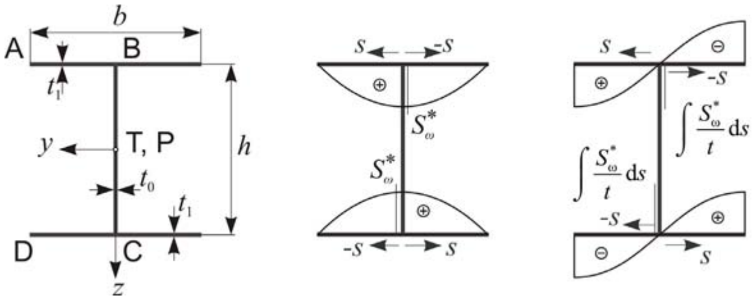

Fig. A1 Double symmetrical I-section: Cross-section functions

$$S^*_\omega = \frac{h}{4} t_1 \left(\frac{b^2}{4} - s^2 \right), \quad \int_0^s \frac{S^*_\omega}{t} \mathrm{d}s = \frac{h}{4} s \left(\frac{b^2}{4} - \frac{s^2}{3} \right), \qquad (-b/2 \le s \le b/2)$$

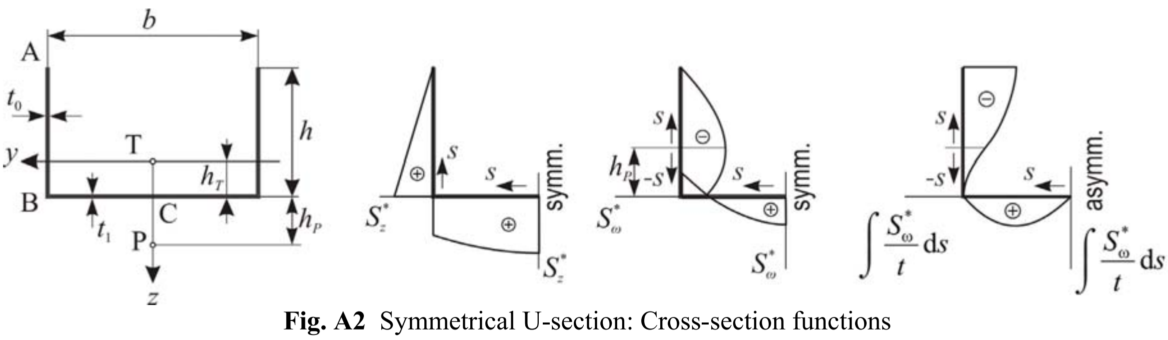

Fig. A2 Symmetrical U-section: Cross-section functions

Cross-section functions for the symmetrical U-section (Fig. A2) are:

$$S_z^* = \frac{bt_0}{2}\left(h-s\right) \quad \left(0 \le s \le h\right), \quad S_\omega^* = \frac{t_0 b}{4}\left[s^2 - \left(h-h_P\right)^2\right] \quad \left(-h_P \le s \le h-h_P\right);$$

$$\int_0^s \frac{S_\omega^*}{t} ds = \frac{b}{12}s\left[s^2 - 3\left(h-h_P\right)^2\right] - \frac{bh_P}{12}\left[2h_P^2 - 6hh_P + 3h^2\right] \quad \left(-h_P \le s \le h-h_P\right);$$

$$S_z^* = \frac{A_0 b}{2} + \frac{t_1}{2}\left(\frac{b^2}{4} - s^2\right), \quad S_\omega^* = -\frac{bA_0}{4}\left(h-2h_P\right) + \frac{t_1 h_P}{2}\left(\frac{b^2}{4} - s^2\right)$$

$$\int_0^s \frac{S_\omega^*}{t_1} ds = -\frac{bA_0}{4t_1}\left(h-2h_P\right)s + \frac{h_P b^2}{8}s - \frac{h_P}{6}s^3 \quad \left(0 \le s \le b/2\right);$$

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