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ISSN 1333-1124

DETERMINATION OF THE PARETO FRONTIER FOR MULTI-OBJECTIVE OPTIMIZATION PROBLEM

UDC 519.853:519.863:621.01

Summary

Engineering design problems often occur as multi-objective optimization problems related to the Pareto frontier determination problem. One of the objectives of this article is to clearly demonstrate the determination, meaning and characteristics of the Pareto frontier by means of several insightful examples of a two dimensional problem, both with and without constraints. In addition to its educational value, the main contributions of the article are summarized as follows. Firstly, we demonstrate the importance of presenting results both in the design variable and the objective space in order to become acquainted with all the data required for correct decision making and interpretation of optimization results. Secondly, we present an approach to explicit (analytical) determination of the Pareto frontier equation in the design space. In this case, the objective functions have to be differentiable. Thirdly, we propose a suitable modification of the Gsoal attainment method as an efficient way to determine the Pareto frontier for some classes of multi-optimization problems.

Key words: *Multi-objective optimization, Pareto frontier, Pareto frontier equation*

1. Introduction

1.1 Genuine Pareto frontier

Multi-objective optimization can be formulated as a decision making problem of simultaneous optimization of two or more design objectives that are conflicting in nature, [1, 3, 4, 5, 7, 8, 11]. This optimization problem can be written as:

Minimize (or Maximize) $F(X)=\left[f_1(X),f_2(X),...,f_m(X)\right]$,

limited by inequality constraints: $G(X)=\left[g_1(X),g_2(X),...,g_p(X)\right]\geq 0$ (or ≤ 0),

by equality constraints: $H(X)=\left[h_1(X),h_2(X),...,h_r(X)\right]=0$,

and by lower and upper requirements for each decision variable x_i: $x_{i,L}\leq x_i\leq x_{i,U}$, $i=1,2,...,n$;

where: n is the number of variables; m is the number of objectives; p is the number of inequality constraints; r is the number of equality constraints.

Single objective function optimization problems have a single design space. On the other hand, a multi-criteria optimization problem incorporates two design spaces, the design variable space and the design objective space. In order to decide correctly, both design spaces and the link between them need to be defined.

Multi-objective optimization concept is linked to the following definitions, [5]:

1. *"A Pareto optimal solution is one for which any improvement in one objective will result in the worsening of at least one other objective"*,
2. *"A dominated point is a point in the design objective space, for which there exists a point in the feasible set that is better (lower, in the case of minimization) in all objectives"*,
3. *"The Pareto frontier is the set of all Pareto optimal solutions represented in the design objective (f) space"*.

Definitions 1 and 2 indicate that the Pareto frontier can be defined as a set of points in the design variable space from which an elementary shift along that set of points causes one of the objective functions to decrease while increasing the other one. If the goal is to obtain the Pareto frontier for two differentiable objective functions with two variables, ($f_1(x_1, x_2)$ and $f_2(x_1, x_2)$), then the given definition leads to the following: the Pareto frontier is a geometrical position of points for which the gradients of the objective functions are collinear and with opposing directions, i.e., the gradient angle difference is 180°, and the contour lines of objective functions are in contact along the Pareto frontier, Fig. 1.

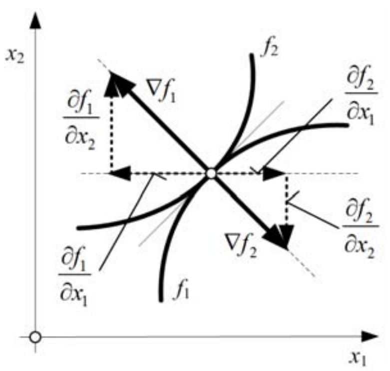

Fig. 1 Genuine Pareto frontier point characteristic

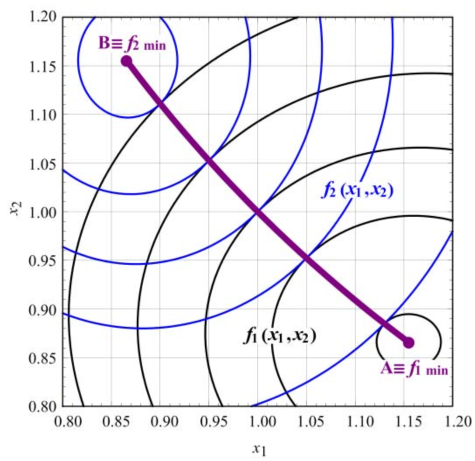

Fig. 2 Pareto frontier A-B

If analytically possible, the Pareto frontier equation for two objective functions with two variables is derived from the following requirement:

$$\frac{\dfrac{\partial f_1(x_1,x_2)}{\partial x_2}}{\dfrac{\partial f_1(x_1,x_2)}{\partial x_1}} - \frac{\dfrac{\partial f_2(x_1,x_2)}{\partial x_2}}{\dfrac{\partial f_2(x_1,x_2)}{\partial x_1}} = 0 \text{ or } \frac{\partial f_1(x_1,x_2)}{\partial x_1}\frac{\partial f_2(x_1,x_2)}{\partial x_2} = \frac{\partial f_1(x_1,x_2)}{\partial x_2}\frac{\partial f_2(x_1,x_2)}{\partial x_1}, \qquad (1)$$

which corresponds to the optimality requirements [1, Eq. (2.9)].

The solution to Eq. (1) presents an analytical description of the Pareto frontier in the design variable space which is valid for the interior of the feasible design variable set. The Pareto frontier derived from the requirements in (1) will be further addressed as the **Genuine Pareto frontier**.

Depending on the objective functions, such a Pareto frontier, as a continuous function, may directly link individual objective function minima, Fig. 2, or, in the case of combined lines, along the contact points of contour lines (line segments A-E and E-F), Fig. 3. In the case when a particular minimum is not placed on the Genuine Pareto frontier (point B, Fig. 3), the Pareto frontier will follow the boundary determined by the active constraints (arc B-F, constraint g_1, Fig. 3).

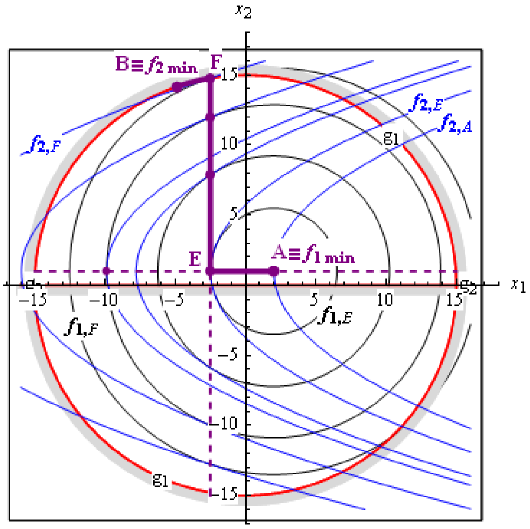

Fig. 3 Pareto frontier A-E-F-B

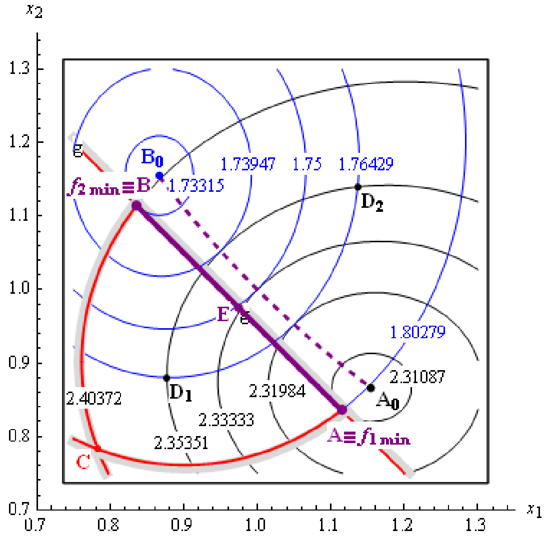

Fig. 4 Pareto frontier A-B in the feasible design variable space A-B-C-A

In the case when the Genuine Pareto frontier lies completely outside the feasible design variable set (dashed line A_0-B_0, Fig. 4), the Pareto frontier will be located on the active constraints closest to the Genuine Pareto frontier (line A-B Fig. 4).

1.2 Result verification

To verify the results of the Pareto frontier determination procedure described in the previous section, the Modified Goal attainment method, Figs 5a and 5b, which is based on the known Goal attainment method, [6], is presented in this section. In order to apply the procedure, individual minima of each objective function ($f_{1\min}$ in point A, $f_{2\min}$ in point B) must be initially determined, as well as the associated values of the other objective function at the same individual minimum position (f_{2A} in point A, f_{2B} in point B). The method consists of searching the Pareto frontier from the chosen reference point R by means of exploring the area between points A and B through the variation of the angle $\varphi_A \le \varphi \le \varphi_B$. The reference point R can be chosen in such a manner that either $f_{1R} \le f_{1\min}, f_{2R} \le f_{2\min}$ (i.e., so that point R is to the left of A and under B), Fig. 5b, or so that $f_{1R} \ge f_{1B}, f_{2R} \ge f_{2A}$, Fig. 5a.

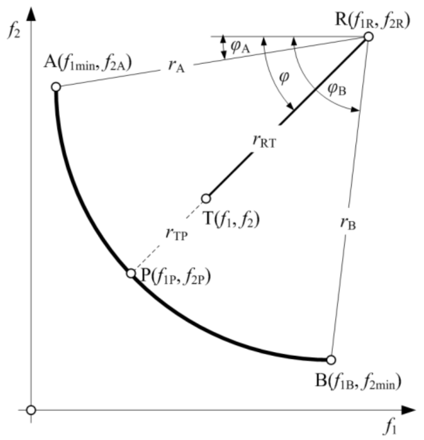

Fig. 5a Modified Goal attainment method principle, searching max r_{RT}

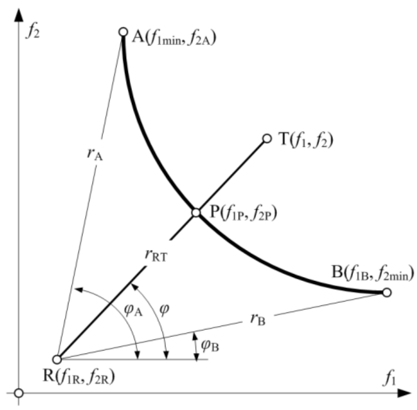

Fig. 5b Modified Goal attainment method principle, searching min r_{RT}

The position of the reference point R should be chosen in such a manner that by dividing the section φ_A - φ_B into a chosen number of sections, an approximately equal distribution of points P on the future Pareto frontier is achieved. Angles φ_A and φ_B, as well as lengths r_A and r_B, are defined by the choice of the reference point R.

The weighted sum method [1, page 48], converts the initial two function problem min $f(x) = (f_1(x), f_2(x))$; $x = (x_1, x_2)$ into a single objective function which is a linear combination of given objective functions, i.e.,

$$\min f(x) = w_1 \cdot f_1(x) + w_2 \cdot f_2(x), \text{ with constraints of } w_1 + w_2 = 1 \text{ and } 0 \le w_1 \le 1. \quad (2)$$

On the other hand, the proposed Modified Goal attainment method is based on the following observations.

Proposition 1. Let $f_{1R} \ge f_{1B}, f_{2R} \ge f_{2A}$, and a series of angles $\varphi_A \le \varphi \le \varphi_B$ be chosen as presented in Fig. 5a. Then, the calculation of the points on the Pareto frontier for the initial problem min $f(x) = (f_1(x), f_2(x))$ corresponds to the following nonlinear programming optimization:

$$\max \ r_{RT} = \sqrt{\left(f_{1R} - f_1\right)^2 + \left(f_{2R} - f_2\right)^2}\ ; \quad (3a)$$

with $\quad f_1 \le f_{1R} - r_{RT}\cos\varphi;\ \ f_2 \le f_{2R} - r_{RT}\sin\varphi$.

In the case when the chosen reference point position is $f_{1R} \le f_{1\min}, f_{2R} \le f_{2\min}$, Fig. 5b, the problem converts into the following nonlinear programming optimization problem for a series of angles $\varphi_A \le \varphi \le \varphi_B$.

$$\min \ r_{RT} = \sqrt{\left(f_1 - f_{1R}\right)^2 + \left(f_2 - f_{2R}\right)^2}\ ; \quad (3b)$$

with $\quad f_1 \le f_{1R} + r_{RT}\cos\varphi;\ \ f_2 \le f_{2R} + r_{RT}\sin\varphi$.

Proof. It is assumed that the Pareto frontier A-P-B equation is known and is given by the following expression:

$$f_2 = f\left(f_1\right) \quad (4)$$

Note that the point P in Figs 5a and 5b is on the Pareto frontier and therefore $f_{2p} = f(f_{1p})$. Furthermore, note that the values (f_{1R}, f_{2R}) define the reference point position R (see Fig 5a, b).

For the chosen angle φ, the following is valid: $\dfrac{f_{2R} - f\left(f_{1P}\right)}{f_{1R} - f_{1P}} = \tan\varphi$.

From the given equation, f_{1P} is calculated, followed by the calculation of f_{2P} from (4).

Vector r_{TP} = TP, which equals $r_{TP} = \sqrt{\left(f_1 - f_{1P}\right)^2 + \left(f_2 - f_{2P}\right)^2}$, represents the objective function for each $\varphi_A \le \varphi \le \varphi_B$ position. Since the optimization procedure should bring the T point to the Pareto frontier, i.e., it should be placed on the point P, it is necessary to calculate x_1 and x_2 variable values for which $r_{TP} = 0$. If the constraint is set somewhere along the way, the process will stop at the constraint border. Therefore, the formulation of problem is the following:

For the chosen number of positions in the area $\varphi_A \le \varphi \le \varphi_B$, calculate

$$\min \ r_{TP} = \sqrt{\left(f_1 - f_{1P}\right)^2 + \left(f_2 - f_{2P}\right)^2} \quad (5)$$

limited by $r_{TP} \ge 0$, i.e., $r_{TP}\cos\varphi = f_1 - f_{1P} \ge 0$; $\quad r_{TP}\sin\varphi = f_2 - f_{2P} \ge 0$.

The stated constraints ensure that point T stops at the Pareto frontier. Note that the problem formulation (5) assumes that the Pareto frontier equation (4) is known.

Instead of minimizing the vector r_{TP}, it is possible to conduct the maximization of the r_{RT} vector since for the same angle φ it must be valid that $r_{RP} = r_{RT} + r_{TP}$, where r_{RP} is the distance between points R and P, which is constant for the same angle φ. Thus, for the minimum r_{TP} vector size, the r_{RT} vector is at its maximum. Therefore, it is possible to replace the r_{TP} vector minimization by maximizing the vector $r_{RT} = \sqrt{(f_{1R} - f_1)^2 + (f_{2R} - f_2)^2}$.

At the same time, the constraints from (5) become $r_{RT} + r_{TP} \le r_{RP}$, from which the projection onto the axis yields:

$$(f_1 - f_{1P}) + r_{RT}\cos\varphi \le f_{1R} - f_{1P}; \quad (f_2 - f_{2P}) + r_{RT}\sin\varphi \le f_{2R} - f_{2P}.$$

$$\text{i.e., } f_1 + r_{RT}\cos\varphi \le f_{1R}; \quad f_2 + r_{RT}\sin\varphi \le f_{2R}. \tag{6}$$

This means that the Pareto frontier equation does not have to be known.

The procedure and formulation of the problem when maximizing the r_{RT} vector are as follows:

1. Calculate individual minima $f_{1\min}$ and $f_{2\min}$, their positions in the design variable space A(x_{1A}, x_{2A}) and B(x_{1B}, x_{2B}), the value of the function f_2 at point A and the value of function f_1 at point B;
2. Choose the reference point R(f_{1R}, f_{2R}) design objective space position;
3. Calculate angles φ_A and φ_B;
4. Divide the angle space $\varphi_B - \varphi_A$ into n_φ equal parts $\Delta\varphi = (\varphi_B - \varphi_A)/n_\varphi$;
5. Define the r_{RP} vector position list, $\varphi_i = \varphi_A + i\cdot\Delta\varphi$, $i = 0, \ldots, n_\varphi$;
6. Calculate factors $w_{1i} = \cos\varphi_i$, $w_{2i} = \sin\varphi_i = \sqrt{1 - w_{1i}^2}$;
7. Define the objective function $r_{RT} = \sqrt{(f_{1R} - f_1(x_1,x_2))^2 + (f_{2R} - f_2(x_1,x_2))^2}$;
8. For n_φ+1 positions, search for solutions to the problem defined in the following manner:

$$\max\ r_{RT} = \sqrt{(f_{1R} - f_1)^2 + (f_{2R} - f_2)^2} \tag{7}$$

with constraints $f_{1R} - (f_1 + r_{RT}\cdot\cos\varphi_i) \ge 0$, $f_{2R} - (f_2 + r_{RT}\cdot\sin\varphi_i) \ge 0$; $i = 1, \ldots, n_\varphi + 1$.

Additionally, constraints regarding the value limitations of particular variables, as well as other constraints in the design variable space, if they exist, should be taken into account.

Remark 1. The benefit of using the above described procedure, when compared to the approach in (2), is that by selecting the number and distribution of angles φ_i in the interval [φ_A, φ_B] we have control over the distribution of points on the Pareto frontier that we intend to calculate. On the other hand, it is unclear how an *a priori* selected distribution of numerical values of the weights w_1 and w_2 from (2) is mapped into the distribution of calculated points on the Pareto frontier. For example, the uniform distribution of weights of the form $(w_1, w_2) \in \{(0.1,0.9), (0.2,0.8),\ldots\}$ generally results in a (highly) non-uniform distribution of points on the Pareto frontier.

Remark 2. Depending on the particular problem at hand, the selection of the reference point R according to Fig. 5b has certain advantages when compared to the selection of R according to Fig. 5a, and *vice versa*. For illustration, suppose that R is selected as it is done in Fig. 5b. Then, in some cases (see Example E below in the text), the shape of the Pareto frontier is such that obtaining the uniform distribution of calculated points on the Pareto frontier requires a highly non-uniform distribution of the angles φ_i in the interval [φ_A, φ_B]. However, for the same example, when the reference point is selected according to Fig. 5a, the uniform distribution of points on the Pareto frontier is obtained by the uniform distribution of angles φ_i in the interval [φ_A, φ_B]. Along the same lines, the selection of the reference point according to Fig. 5b is preferable in some other cases, like in Example C below. Finally, selection of the reference point R has an impact on the numerical sensitivity in the calculation of points on the Pareto frontier.

2. Example A

2.1 Description of the example and individual minima of objective functions

For a simple two-bar truss from Fig. 6 determine dimensions h_1 and h_2 with the goal of minimizing the mass (i.e., volume) of the bars and vertical displacement of point C, [9]. The allowable stresses are given: σ_1 for bar 1 (tension), σ_2 for bar 2 (compression).

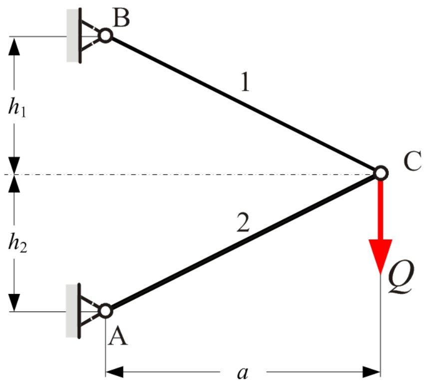

Fig. 6 Two-bar truss

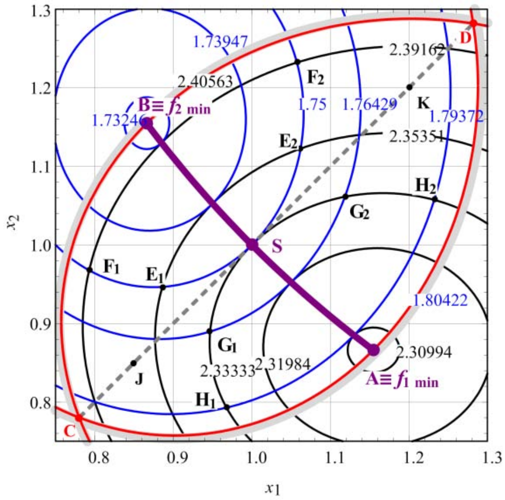

Fig. 7 Design variable space with the Pareto frontier and contour lines in contact

Dimensionless variables are defined as: $x_1 = h_1 / a$, $x_2 = h_2 / a$.

Bar cross section areas equal $A_1 = \dfrac{F_1}{\sigma_1} = \dfrac{Q}{\sigma_1}\dfrac{\sqrt{1^2+x_1^2}}{x_1+x_2}$, $A_2 = \dfrac{F_2}{\sigma_2} = \dfrac{Q}{\sigma_2}\dfrac{\sqrt{1^2+x_1^2}}{x_1+x_2}$.

Bar volume equals $V = A_1\sqrt{a^2+h_1^2} + A_2\sqrt{a^2+h_2^2}$ or $V = \dfrac{Qa}{\sigma_1}\dfrac{1+\sigma_1/\sigma_2 + x_2^2\,\sigma_1/\sigma_2 + x_1^2}{x_1+x_2}$,

which enables the objective function to be written in the dimensionless form:

$$f_1\left(x_1,x_2\right) = \frac{V\sigma_1}{Qa} = \frac{1+x_1^2+\sigma_1/\sigma_2\left(1+x_2^2\right)}{x_1+x_2}. \tag{8}$$

From $\partial f_1/\partial x_1 = 0$, $\partial f_1/\partial x_2 = 0$, the optimum solution is derived solely for the mass objective function:

$$\min f_1 = \frac{V_{\text{opt}}}{Q}\frac{\sigma_1}{a} = 2\sqrt{\frac{\sigma_1}{\sigma_2}};\quad x_{1,f1\min} = x_{1,\text{A}} = \sqrt{\sigma_1/\sigma_2};\quad x_{2,f2\min} = x_{2,\text{A}} = \sqrt{\sigma_2/\sigma_1},$$

see point A, Fig. 7.

Point C vertical displacement (Castigliano's theorem) equals, Fig. 6:

$$\delta_\text{C} = \frac{\sigma_1}{E}\frac{1+x_1^2+\left(\sigma_1/\sigma_2\right)\left(1+x_2^2\right)}{x_1+x_2},$$ and in the dimensionless form:

$$f_2\left(x_1,x_2\right)=\frac{E}{\sigma_1}\delta_C=\frac{1+x_1^2+(\sigma_2/\sigma_1)\left(1+x_2^2\right)}{x_1+x_2}. \quad (9)$$

From $\partial f_2/\partial x_1=0$, $\partial f_2/\partial x_2=0$, the optimum solution is derived solely for the displacement objective function:

$$\min f_2=\frac{E}{\sigma_1}\delta_C=2\sqrt{\frac{\sigma_2}{\sigma_1}}\,;\; x_{1,f2\min}=x_{1,B}=\sqrt{\sigma_2/\sigma_1}\,;\; x_{2,f2\min}=x_{2,B}=\sqrt{\sigma_1/\sigma_2}\,,$$

see point B, Fig. 7. In this example $\sigma_1/\sigma_2=4/3$ is chosen.

2.2 Design variable space

The Pareto frontier equation for functions (8) and (9), which follows from the requirement (1) solution, equals:

$$x_2=1/x_1. \quad (10)$$

Fig. 7 shows the design variable space including the contour lines of objective functions with contacts along the Genuine Pareto frontier (A-B) which connects the minima of individual objective functions, including the following constraints:

$$f_1\left(x_1,x_2\right)\le f_1\left(x_{1,B},x_{2,B}\right) \quad \text{and} \quad f_2\left(x_1,x_2\right)\le f_2\left(x_{1,A},x_{2,A}\right). \quad (11)$$

Verification points marked in Fig. 7, and their positions are transferred to the design objective space, Fig. 8. In this example, each pair of contour lines has two crossing points, e.g., contour lines f_1 = 2.35351 and f_2 = 1.75 cross at points E_1(0.886029, 0.946556) and at E_2(1.06209, 1.12262). In the design objective space, Fig. 8, these two points have the same position E(f_1, f_2) = E(2.35351, 1.75).

2.3 Pareto frontier in the design objective space, $f_2=f(f_1)$

Incorporating the Pareto frontier Eq. (10) into the objective function equations gives the objective function equations along the Pareto frontier as functions of x_1 coordinates, as follows:

- for the objective function $f_1\left(x_1,x_2\right)\to f_{1\mathrm{Par}}\left(x_1\right)=4/(3x_1)+x_1$, (12)
- for the objective function $f_2\left(x_1,x_2\right)\to f_{2\mathrm{Par}}\left(x_1\right)=3/(4x_1)+x_1$. (13)

From (12), x_1 can be expressed as a function of f_1:

$$x_1=f\left(f_1\right)=\frac{1}{6}\left(3f_1-\sqrt{-48+9f_1^2}\right), \quad (14)$$

and incorporating (14) into Eq. (13) gives $f_2=f(f_1)$:

$$f_2=f\left(f_1\right)=\frac{7-6f_1^2+2f_1\sqrt{-48+9f_1^2}}{-6f_1^2+2\sqrt{-48+9f_1^2}}. \quad (15)$$

Function $f_2=f(f_1)$ according to (15) is shown in design objective space, Fig. 8, with verification points and feasible space according to Fig. 7.

The Pareto frontier function plot [10], as shown in Fig. 8, can also be shown by the parametric plotting of functions (12) and (13), with coordinates x_1 as parameters.

The calculation procedure as described in section 1.2 is conducted. Based on the known points position A(2.3094, 1.80422) and B(2.40563, 1.73205), Fig. 8, the reference point R(2.41, 1.81) is chosen. The angle $\varphi_A\le\varphi\le\varphi_B$ is distributed to 50 equal sections, and the calculated Pareto frontier line is shown in Fig. 9. The results correlate strongly with the results in Fig. 8.

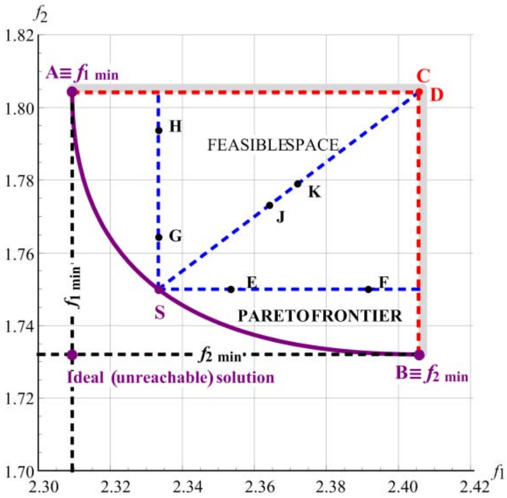

Fig 8 Pareto frontier in the design objective space including constraints and verification points.

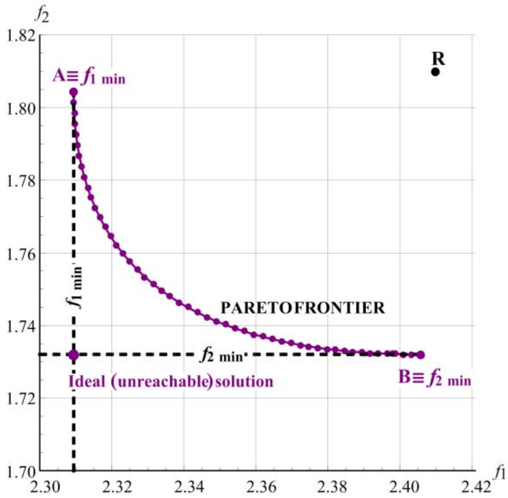

Fig. 9 Pareto frontier calculated according to section 1.2

3. Example B

The goal is to find the Pareto frontier for objective functions (8) and (9), with the constraints:

$$g_1 \rightarrow x_1 = h_1/a \geq 0.9\,, \qquad g_2 \rightarrow x_2 = h_2/a \geq 0.9 \tag{16}$$

and the objective function constraints according to (11).

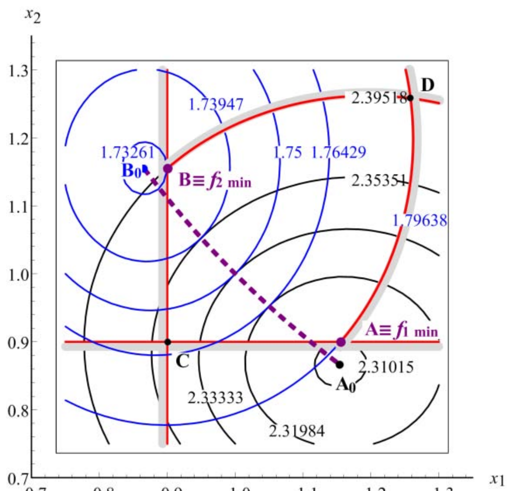

Fig. 10 Design variable space

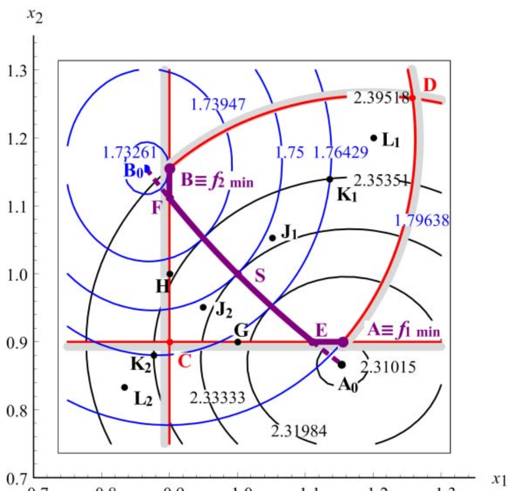

Fig. 11 Pareto frontier A-E-S-F-B

Fig. 10 shows the design variable space including given constraints, contour lines in contact, the Genuine Pareto frontier without constraints (dashed line A_0–B_0), which partially passes through feasible design variable space defined by bordering points A–C–B–D–A, and with individual minima of objective functions at point A(1.15508, 0.9), with $f_{1\min}$=2.31015, and at point B(0.9,1.15508), with $f_{2\min}$=1.73261. The Genuine Pareto frontier A_0–B_0 remains valid along the length which is in the feasible design variable space, while in other sections, the Pareto frontier follows active constraints $x_1 \geq 0.9$ and $x_2 \geq 0.9$, Fig. 11. The Pareto frontier and the feasible design objective space, with verification points according to Fig. 11, are shown in Fig. 12. In order to verify the results by the Modified Goal attainment method, the reference point is chosen with the position R(f_{1R}=1.2 f_{1B}, f_{2R}=1.2 f_{2A}). Angle ARB, Fig. 5a, is divided into 50 equal sections. The results correlate strongly with the results in Fig. 12.

4. Example C

The goal is to find the Pareto frontier for objective functions (8) and (9), with the constraints:

$$g_1 \to 0.9 \le x_1 = h_1/a \le 1.1\,,$$
$$g_2 \to 0.9 \le x_2 = h_2/a \le 1.1$$

Fig. 13 shows the design variable space including given constraints, with individual minima of objective functions at point A(1.1, 0.866596), with $f_{1,A}$=2.31092 as the function minimum, and also at point B(0.866596, 1.1), with $f_{2,B}$=1.73319 as the function minimum.

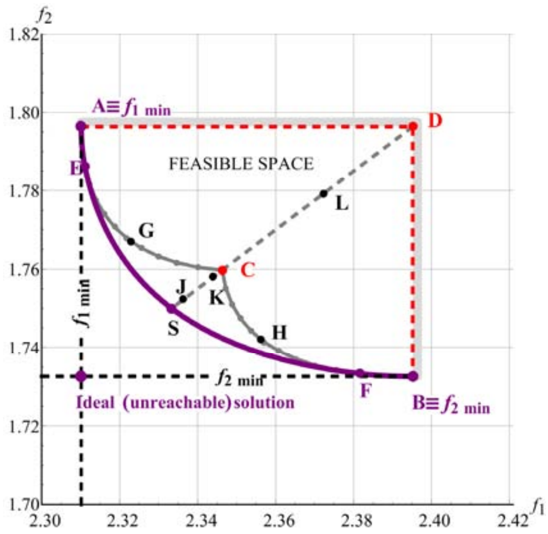

Fig. 12 Pareto frontier A-E-S-F-B, design objective space, example B

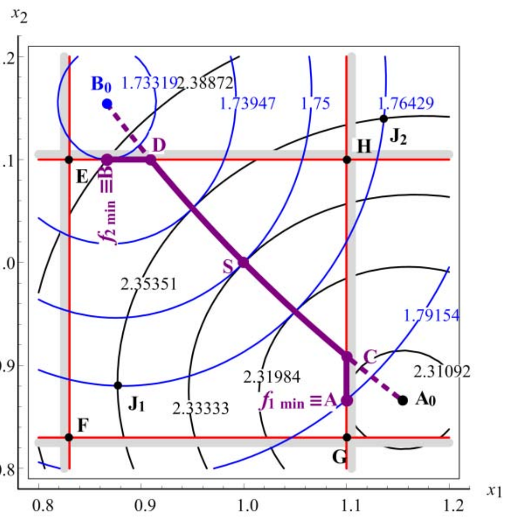

Fig. 13 Pareto frontier A-C-S-D-B in the design variable space

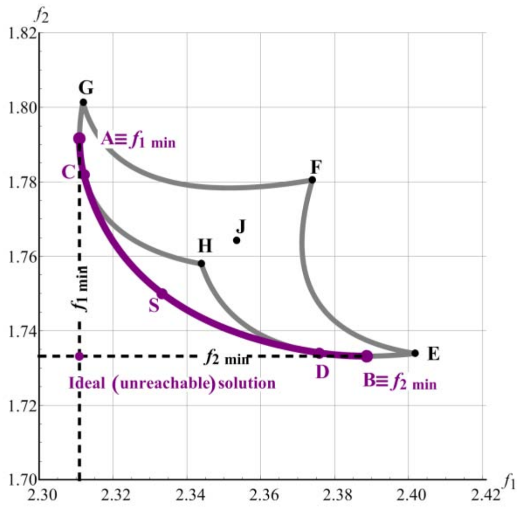

Fig. 14 Pareto frontier A-C-S-D-B in the design objective space

The Pareto frontier stretches from point A to point C along the active constraint $x_1 \le 1.1$, from C to D along the Genuine Pareto frontier A_0–B_0 and then from D to B along the active constraint $x_2 \le 1.1$. The Pareto frontier including constraints and verification points, transferred from the design variable space into the design objective space, is shown in Fig. 14. The values of objective functions along the requirement edges are shown by lines A–G–F–E–B and B–D–H–C–A. Result verification is conducted by the Modified Goal attainment method with the reference point at position R(f_{1R}=0.75f_{1min}, f_{2R}=0.75f_{2min}), Fig 5b. The verification results correlate strongly with the results in Figs 13 and 14.

5. Example D

Consider the two-dimensional, multi-objective optimization problem with two conflicting objectives [7, page 247]:

$$f_1(x) = x_1^2 + x_2\,, \quad f_2(x) = x_1 + x_2^2\,, \tag{17}$$

subjected to constraints: $-10 \le x_1\,,\; x_2 \le 10$.

The goal is to find the Pareto frontier in the design variable space and in the design objective space. By solving the requirement (1) for the system of equations (17), the Pareto frontier equation is obtained in the design variable space

$$x_2 = 1/(4x_1)\,. \tag{18}$$

Individual minima of objective functions, for the given variable value range, are $f_{1\min}$ = -10 for function $f_1(x)$ at point A(0,-10), and $f_{2\min}$=-10 for function $f_2(x)$ at point B(-10, 0),.

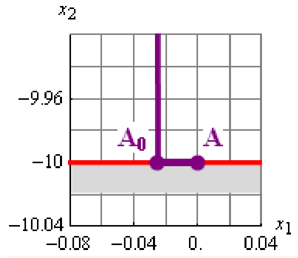

Fig. 15 Detail at point A

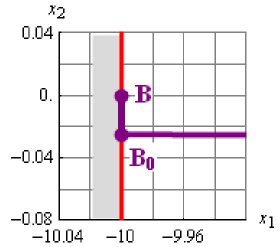

Fig. 16 Detail at point B

The Pareto frontier in the design variable space is shown in Fig. 17. Due to variable constraints ($-10 \le x_1,\ x_2 \le 10.$), the Genuine Pareto frontier is stretched over equation (18) from point A_0(-1/40, -10), Fig. 15, to point B_0(-10, -1/40), Fig. 16, while for A_0 to A and from B_0 to B it is stretched along the constraint edge. At point S(-0.5, 0.5), Fig. 17, both goal functions have the same value: $f_{1S} = f_{2S}$ = -0.25. The design variable space, transferred to the design objective space is shown in Fig. 18. Points A_0 and B_0 are not shown in this figure since they are practically at the same position as points A and B. The values of objective functions along the boundary edges (connection lines between points A-L-M-G-N-H-B-C-A in the design variable space, Fig. 17) are shown by curves connecting appropriate points A-L-M-G-N-H-B-C-A in the design objective space, Fig. 18. Introducing (18) into (17) generates the objective function equations along the Genuine Pareto frontier from A_0 to B_0, depending on the parameter x_1:

$$f_{1\text{P}} = \frac{1}{4x_1} + x_1^2\,, \qquad f_{2\text{P}} = \frac{1}{16x_1^2} + x_1^2\,, \tag{19}$$

in such a manner that the Pareto frontier in Fig. 18 can be drawn in a parametric manner or in a way described in section 2.3.

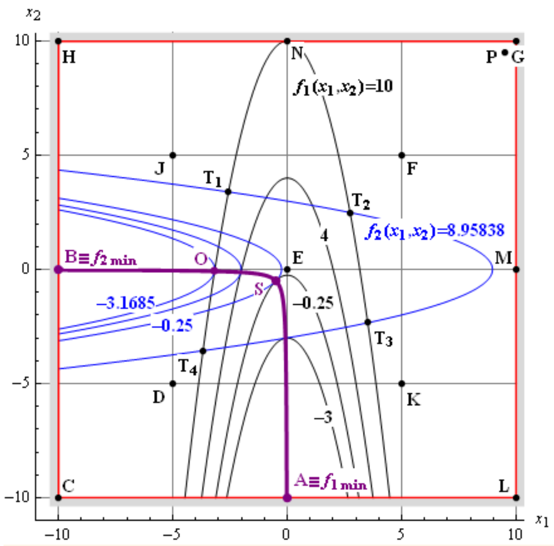

Fig. 17 Design variable space, including Pareto frontier and verification points.

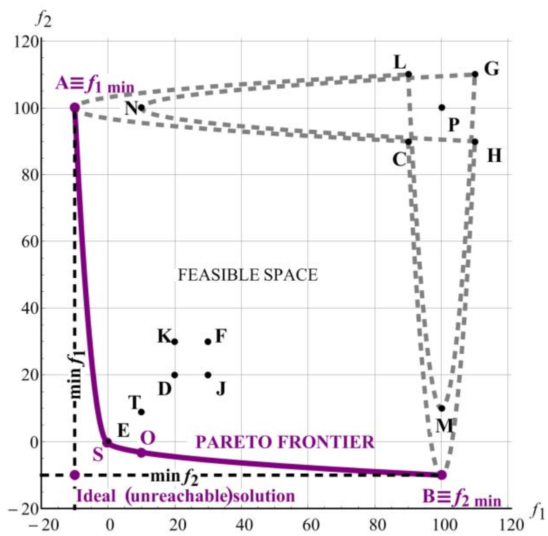

Fig. 18 Pareto frontier in the design objective space including constraints and verification points.

The Pareto frontier points in the design objective space, e.g., points O and S in Fig. 18, correspond uniquely to the Pareto frontier points O and S in the design variable space. In this example, since the Genuine Pareto frontier is addressed, the coordinates of points O and S in the design objective space are contour line values which are in contact on the Pareto frontier in the design variable space. For instance, the design variable values T_1, T_2, T_3 and T_4, Fig. 17, correspond to the point T(10, 8.95838) in the design objective space, because the contour lines $f_1(x) = 10$ and $f_2(x) = 8.95838$ cross each other at these points. Both functions have a zero value at point E(0,0). Result verification is conducted by the Modified Goal attainment method with the reference point at position $R(1.2f_{1B}, 1.2f_{2A})$ and by dividing the angle $\varphi_B - \varphi_A$ into 60 equal sections according to Fig. 5a. The verification results correlate completely with the results in Fig. 18.

6. Example E

The goal is to find the Pareto frontier in the design variable space and in the design objective space and to perform the analysis of two-dimensional problem with two objective functions [2, Table 5]

$$f_1 = (x_1 - 2)^2 + (x_2 - 1)^2 + 2, \quad f_2 = 9x_1 - (x_2 - 1)^2, \tag{20}$$

with constraints $g_1 = 225 - x_1^2 - x_2^2 \geq 0$; $g_2 = 3x_2 - x_1 - 10 \geq 0$.

Solving the requirement (1) for equation system (20) derives the Genuine Pareto frontier equation in the design variable space

$$x_1 = -5/2. \tag{21}$$

The design variable space, including lines in contact, constraints, and the Pareto frontier, is shown in Fig. 19. Individual minima of objective functions, for the given range of variable values, are: $f_{1\min} = 10.1$ for function $f_1(x)$ at point A(1.1, 3.7), and $f_{2\min} = -217.739$ for function $f_2(x)$ at point B(-4.84093, 14.1974).

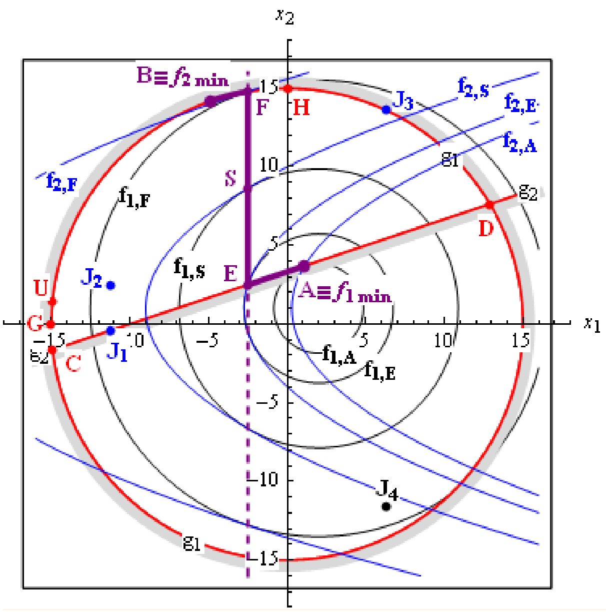

Fig. 19 Design variable space, including the Pareto frontier, touching contour lines and points $J_1, \ldots, J_4$

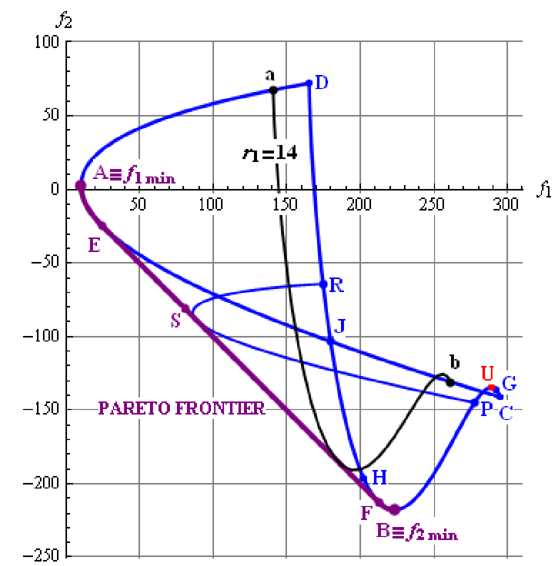

Fig. 20 Pareto frontier in the design objective space, including objective function values along the g_1 and g_2 constraints

True Pareto frontier lies on the line $x_1 = -5/2$, which is the line EF in the feasible space. The Pareto frontier stretches further along the constraint edges to the individual objective function minima; along the boundary g_2 edge (line EA) up to the point A and along the boundary g_1 (arc FB) up to the point B. The design objectives space is shown in Fig. 20, including the Genuine Pareto frontier and the objective function values along the constraints and verification points.

The point J is the crossing point of function $f_2 = f(f_1)$ along the boundary g_2 (curve D-A-E-C) with the corresponding curve along the boundary g_1 (curve D-H-F-B-G-C) in the design objective space. The objective function values at point J are $f_{1,\mathrm{J}} = 179.172$ and $f_{2,\mathrm{J}} = -103.309$. In the design variable space, Fig. 19, the contour lines cross each other (i.e. have the same values) in 4 points; J1 to J4. It should be noted that the points J_1 and J_3 are located at the boundary edges, the point J_2 is within the feasible space and the point J_4 lies outside the feasible space.

The point U is the local maximum of $f_2 = f(f_1)$ function along the boundary edge g_1 (curve D-H-F-B-G-C in Fig. 20). In the design variable space, Fig. 19, it is situated at the location where the function f_2 contour line touches the boundary g_1 from within, i.e., the circle with radius of 15. If the objective function values along the circle with a different radius, e.g., r_1=14, inside the feasible design variable space are examined, $f_2 = f(f_1)$ function will also have a similar maximum, Fig. 20, curve a-b. Geometrical location of local maximum points in the design variable space can be calculated according to (1), as with the Pareto frontier equation calculation, with the difference that the functions in question are $f_2 = f(x_1, x_2)$ and $f_3 = x_1^2 + x_2^2$.

Curve P-R, Fig. 20, shows the objective function values along the line x_2=10+x_1/3 in the feasible design variable space (line parallel to line C-D, Fig. 19).

As in the previous examples, it is shown that the design objective space is important for establishing the position of individual combinations of objective function values with respect to the Pareto frontier. However, for a final decision it is critical to establish the positions of these combinations in the design variable space.

Result verification is conducted by the Modified Goal attainment method with the reference point at position R($1.2f_{1\mathrm{B}}$, $1.2f_{2\mathrm{A}}$) and by dividing the angle $\varphi_\mathrm{B} - \varphi_\mathrm{A}$ into 75 equal sections according to Fig. 5a. The verification results correlate completely with the results in Figs 19 and 20.

In the case that the constraint g_2 from (20) is defined as $g_2 = x_2 \geq 0$, the design variable space would look as shown in Fig. 3. Namely, solving the system of equations (1) generates 2 solutions: x_1 = -5/2 and x_2 = 1. Since the line x_2 = 1 is in this case within the feasible design variable space, the Genuine Pareto frontier stretches along A-E-F, while from point F to B, the Pareto frontier follows the constraint g_1 border, Fig. 3.

7. Example F

An illustrative engineering example for which the Genuine Pareto frontier, as defined in section 1.1 does not exist, can be found in [1, page 19]. The goal is to determine the diameter $d = x_1$ and the length $l = x_2$, Fig. 21, for a cantilever beam with a round cross section, with two conflicting objectives: minimize the cantilever beam weight and deflection with constraints regarding the stress, deflection and objective parameter values. Optimization problem is defined as follows:

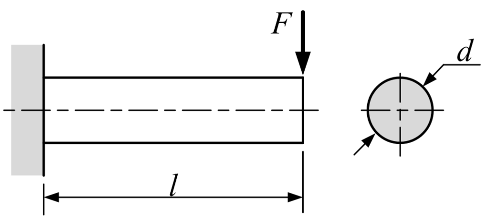

Fig. 21 Cantilever beam, schematics

$$\text{Minimize}\left(f_1(x_1,x_2) = \rho\frac{\pi}{4}x_1^2 \cdot x_2,\ f_2(x_1,x_2) = \delta = \frac{64F}{3E\pi}\cdot\frac{x_2^3}{x_1^4}\right),\ \text{with}\ \sigma_{\max} = \frac{32F}{\pi}\frac{x_2}{x_1^3},$$

subject to $g_1 = S_\mathrm{y} - \sigma_{\max} \geq 0$; $g_2 = \delta_{\max} - \delta \geq 0$; $g_3 = x_1 - 1 \geq 0$; $g_4 = 5 - x_1 \geq 0$;
$g_5 = x_2 - 20 \geq 0$; $g_6 = 100 - x_2 \geq 0$.

The following parameter values are used:

$\rho = 7800\ \text{kg/m}^3$, $F = 1$ kN, $E = 207$ GPa, $S_y = 300$ MPa, $\delta_{max} = 5$ mm.

The design variable space is limited by variable values $1 \le d \le 5$ cm and $20 \le l \le 100$ cm. The problem formulation leads to a conclusion that the console mass $f_1(d, l)$ will be proportionally smaller, depending on the smaller values of d and l, under the condition of respecting the constraints. In the same manner, deflection of the cantilever end $f_2(d, l)$ will be proportionally smaller, depending on the larger stiffness, i.e., for the minimum allowable length and the maximum allowable diameter.

According to criteria (1), for the objective functions f_1 and f_2 no solution exists for the Genuine Pareto frontier equation. This can be seen from contour lines, Fig. 22, which cannot touch in the design variable space. The feasible design variable space is A-B-C-D-A.

The point A(1.89366, 20) is the objective mass function individual minimum, $f_{1\min} = 0.43936$ kg, determined by the crossing point of constraints edges and $g_5 = x_2 - 20 = 0$.

The point B(5, 20) is the objective deflection function individual minimum, $f_{2\min} = 0.004199$ cm, defined by the constraints $g_5 = x_2 - 20 = 0$ and $g_4 = 5 - x_1 = 0$ crossing point. The point C(5, 98.3944) is the crossing point of constraints $g_4 = 5 - x_1 = 0$ and $g_2 = 0$, and finally, the point D(2.26534, 34.2391) is the crossing point of constraints $g_1 = 0$ and $g_2 = 0$.

All points within the feasible design variable space A-B-C-D-A (e.g., the points E and F) are dominated points, as determined by the definition, section 1.1, except the points located on the line segment A-B along which the increase in x_1 increases the cantilever mass and decreases the deflection. Thus, in this example, the Pareto frontier in the design variable space, according to definitions 1 and 3, section 1.1, is formed by the line segment A-B, Fig. 22.

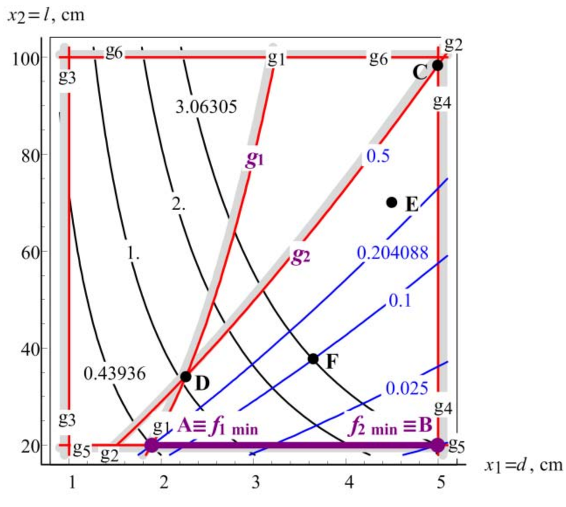

Fig. 22 Feasible design space A-B-C-D-A and the Pareto frontier A-B

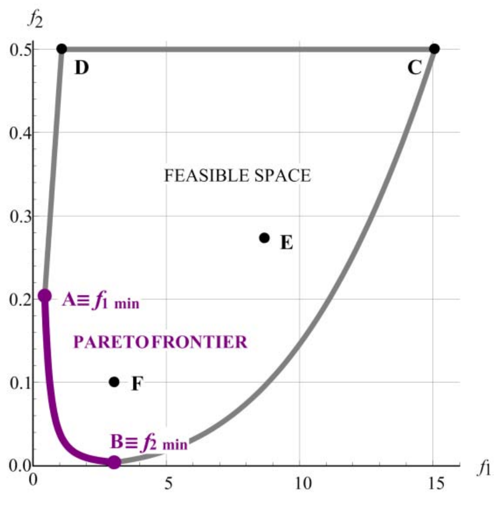

Fig. 23 Feasible design objective space A-B-C-D-A, the Pareto frontier A-B

The feasible design space in Fig. 22, transferred to the design objective space is shown in Fig. 23.Contrary to the previous examples, the transfer from one space into the other in this example is uniform, i.e., every point in the feasible design objective space is related to a single point in the feasible design variable space. This is caused by the objective function contour crossing at a single crossing point in the design variable space.

Result verification is conducted by the Modified Goal attainment method with the reference point at position R(10, 0.5) and by dividing the angle $\varphi_B - \varphi_A$ into 50 equal sections according to Fig. 5a. The verification results correlate completely with the results in Fig. 23.

8. Conclusion

In this article we have presented the determination method, the meaning, and characteristics of the Pareto frontier using several examples. The results are presented graphically, in the design variable space and in the design objective space. All considered problems are two dimensional in order to enable an insightful visual presentation of results. It should be noted that the presented approach can be used also for solving more complex multi-dimensional problems. Each example was solved in depth, including explanations and discussions. Thus, an educational value [1)] is attributed to this article.

Three main contributions presented in this article are as follows. Firstly, it was established that the multi-criteria optimization decisions should be made only after a thorough determination of characteristics in both design spaces. Thus, it is important to present results both in the design variable and in the design objective space. Secondly, an analytical (explicit) approach to determining the Pareto frontier equation in the design space is adopted. A prerequisite for this approach is that the objective functions are differentiable. Thirdly, the Pareto frontier for some classes of multi-optimization problems, both with and without constraints, has been efficiently determined using a suitable modification of the Goal Attainment method.

[1)] For full *Mathematica* code please contact authors at mhoic@fsb.hr.

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Submitted: 11.3.2013
Accepted: 16.5.2013

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