On weak convergence of functionals on smooth random functions*

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Abstract. *The numbers of level crossings and extremes for random processes and fields play an important role in reliability theory and many engineering applications. In many cases for Gaussian processes the Poisson approximation for their asymptotic distributions is used. This paper extends an approach proposed in Rusakov and Seleznjev (1988) for smooth random processes on a finite interval. It turns out that a number of functionals (including some integervalued ones) become continuous on the space of smooth functions and weak convergence results for the sequences of such continuous functionals are applicable. Examples of such functionals for smooth random processes on infinite intervals and for random fields are studied.*

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1. Introduction

The asymptotic behaviour of the number of level crossings or extremes for a sequence of random functions has been studied in detail mostly for the Gaussian case, and is not very much known for non-Gaussian processes. The mainly used approach exploited Poisson approximation for the corresponding distributions (see for references, e.g., Leadbetter, Lindgren, and Rootzén (1983), Piterbarg (1996), Hüsler, Piterbarg, and Seleznjev (2000)). For such problems, Rusakov and Seleznjev (1988) proposed an approach based on continuity properties of some functionals of interest for the sequence of vector random processes on a bounded interval. Narrowing the class of continuous functions $C(T)$ to the class of continuously differentiable functions $C^k(T)$ allows us to investigate the weak convergence of a wider class of

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functionals defined on the space of smooth functions. In particular, some integer valued functionals are continuous in this case in the corresponding metric and the standard weak convergence technique (Continuity Theorem) is applicable. The aim of the present paper is to develop this method for infinite intervals and random fields, and for some new functionals on random functions.

First we consider some auxiliary results about weak convergence of random elements with realizations in spaces of smooth functions. In short communication Wilson (1986), weak convergence of probability measures on spaces smooth functions was considered. The weak convergence results in *Theorems 1* and *2* were obtained by authors independently and they generalize the corresponding results in Rusakov and Seleznjev (1988). These results are of technical character and the proofs directly follow from the well known results for the space of continuous functions (see e.g., Billingsley (1968), Witt (1970)). We state the following results for completeness of exposition and convenience of references.

Consider at first the space of continuously differentiable functions with continuous k-th derivative $C^k[0,\infty)$ with the norm for $x(\cdot) \in C^k[0,\infty)$,

$$||x|| = \sum_{i=0}^{k-1} |x^{(i)}(0)| + ||x^{(k)}||_C,$$

where for any $y(\cdot) \in C[0,\infty)$,

$$||y||_C = \sum_{j=1}^{\infty} 2^{-j}||y||_j/(1+||y||_j), \qquad ||y||_j = \sup_{[0,j]} |y(t)|, \quad j \geq 1.$$

Convergence in the space $C^k[0,\infty)$ is equivalent to convergence for every $j \geq 1$ in the space $C^k[0,j)$ of continuously differentiable functions on the interval $[0,j]$ with continuous k-th derivative in the standard norm $||\cdot||_j$ for the interval $[0,j]$. For any set $t_1,\dots,t_p,\ t_i \in [0,\infty), i = 1,\dots,p$, denote by

$$\phi_k(x,t_1,\dots,t_p) := (x(0),\dots,x^{(k-1)}(0), x^{(k)}(t_1),\dots,x^{(k)}(t_p))\,.$$

Denote by "$\Rightarrow$" weak convergence of probability measures or random elements. Let $w_j(y,\delta)$ be a modules of continuity for a function $y(\cdot) \in C[0,j], 0 \leq \delta \leq j$,

$$w_j(y,\delta) := \sup_{|t-s|<\delta} \{|y(t)-y(s)|,\ t,s \in [0,j]\}.$$

Let X, X_n be $C^k[0,\infty)$-valued random elements defined on the probability spaces $(\Omega,\mathcal{F},\mathbf{P})$ and $(\Omega_n,\mathcal{F}_n,\mathbf{P}_n)$, respectively, $n \geq 1$.

Theorem 1. *$X_n \Rightarrow X$ as $n \to \infty$ in $C^k[0,\infty)$ iff*

(i) for any set $\{t_i \in [0,\infty), i = 1,\dots,p\}$, $\quad \phi_k(X_n,t_1,\dots,t_p) \Rightarrow \phi_k(X,t_1,\dots,t_p)$ and

(ii) for any $\varepsilon > 0$ and for every $j \geq 1$, $\lim_{\delta\to 0}\sup_n \mathbf{P}_n\{w_j(X_n^{(k)},\delta) > \varepsilon\} = 0$.

Remark 1. Theorem 1 (ii) *holds if for every* $j \geq 1$ *there exist* $\alpha_j, \beta_j, C_j > 0$ *such that*

$$E|X^{(k)}(t) - X^{(k)}(s)|^{\alpha_j} \leq C_j|t-s|^{\beta_j+1}, \quad t,s \in [0,j]. \tag{1}$$

For a Gaussian sequence $X_n, n \geq 1$, *it is sufficient to verify (1) for* $\alpha_j = 2, \beta_j > 0$, *for every* $j \geq 1$. *The assertion follows directly from Theorem 12.3, Billingsley (1968), in the space* $C[0,j]$ *for every* $j \geq 1$.

Now we consider the space of continuously differentiable functions defined on an m-dimensional unit cube $T = [0,1]^m$. Denote this space by $C^k(T),\ k \geq 1$, with the norm for any $x(\cdot) \in C^k(T)$,

$$||x|| = \sum_{|l| \leq k} \sup_T \mid x^{(l)}(t) \mid,$$

where $l = (l_1, ... l_m),\ \mid l \mid = l_1 + \cdots + l_m,\ l_i \geq 0,\ i = 1, ..., m$. The convergence in that norm means the uniform convergence of functions and derivatives up to order k on T and the space $C^k(T)$ is a complete separable normed space (see Kufner, John, and Fučik (1977), p. 26, 31). Denote by $w(x, \delta)$ the module of continuity of a continuous function $x(\cdot) \in C(T),\ 0 < \delta < m^{1/2},\ \mid t \mid_m = (\sum_{i=1}^m t_i^2)^{1/2}$,

$$w(x,\delta) = \sup_{|t-s|_m \leq \delta} \{\mid x(t) - x(s) \mid, \quad t,s \in T\}.$$

We order the vectors l according to some rule. For any function $x(\cdot) \in C^k(T)$ and set $t_i \in T,\ i = 1, ..., p$, denote the vector of derivatives ordered according to this rule by

$$\psi_k(x, t_1, ..., t_p) = (x^{(l)}(t_1), \ldots, x^{(l)}(t_p), \quad \mid l \mid \leq k).$$

Let X, X_n be $C^k(T)$-valued random elements defined on probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$, respectively, $n \geq 1$.

Theorem 2. $X_n \Rightarrow X$ *as* $n \to \infty$ *in* $C^k(T)$ *iff*

(i) for any set $\{t_i \in T,\ i = 1, ..., p\}$, $\psi_k(X_n, t_1, ..., t_p) \Rightarrow \psi_k(X, t_1, ..., t_p)$ *and*

(ii) for any $\varepsilon_{(l)} > 0,\ \mid l \mid \leq k$, $\lim_{\delta \to 0} \sup_n \mathbf{P}_n\{w(X^{(l)}, \delta) > \varepsilon_{(l)}\} = 0$.

Remark 2. Theorem 2(ii) *holds if there exist* $C, \alpha_{(l)}, \beta_{(l)} > 0,\ n_0 \in N$, *such that*

$$E\{\mid X_n^{(l)}(t) - X_n^{(l)}(s) \mid^{\alpha_{(l)}}\} \leq C \mid t-s \mid_m^{m+\beta_{(l)}}$$

for any $n \geq n_0,\ t, s \in T,\ \mid t-s \mid_m \leq h_0 < m^{-1/2},\ \mid l \mid \leq k$. *The assertion follows directly from Corollary 12.3.1, Billingsley (1968).*

Remark 3. *The assertions of* Theorems 1 *and* 2 *are directly generalized for vector random processes and fields by using the corresponding arguments in Rusakov and Seleznjev (1988) and Witt (1970).*

This paper is organized as follows. In *Section 2*, we consider several examples of continuous functionals on sequences of smooth random processes on an infinite interval (e.g., the ith crossing moment, the number of crossings) and apply the corresponding weak convergence results for investigation of asymptotic distributions. In *Section 3* some examples of weak convergent sequences of smooth random fields are given. We apply the weak convergence technique in order to study the asymptotic distribution of the number of local extremes of sequences of random fields. *Section 4* contains the proofs of the statements from previous sections.

2. Weak convergence of functionals on a sequence of smooth random processes on $[0, \infty)$

Consider several examples of continuous functionals in $C^k[0, \infty), k \geq 1$. Let $T_i(x)$ be the ith zerocrossing moment of function $x(\cdot) \in C^k$. Let $\mathcal{T}_i$ be the set of functions $x(\cdot) \in C^k$ such that $T_i(x) < \infty$, and in a moment of a zerocrossing τ, $x(\tau) = 0, x^{(1)}(\tau) \neq 0$. We will call such zerocrossing moments *nondegenerate*. Denote also by $\mathcal{N}_j$ the set of functions $x(\cdot) \in C^k$ with a finite number of nondegenerate zerocrossings on $[0, j]$, $N(x, [0, j]) = N_j(x), j \geq 1$.

Proposition 1. *Functionals $T_i(x), i = 1, \ldots, m$, and $N_j(x)$ are continuous on sets $\mathcal{T}_m$ and $\mathcal{N}_j$, respectively, $j \geq 1$.* Note that there are simple examples of functions in $C[0, \infty)$ for which these functionals are discontinuous in the space only continuous functions (see e.g., Rychlik(1987)).

Let $X, X_n, n \geq 1$, be C^k-valued random elements such that almost all sample paths of $X(\cdot)$ belong to $\mathcal{T}_m \cap \mathcal{N}$. For example, if the random process $X(t), t \in [0, \infty)$, is a Gaussian process or a function of a Gaussian process, or a Slepian model process, or a Fourier-, Karhunen-Loève expansion, or the sum of a Gaussian process and any independent continuously differentiable random process, i.e. any so-called decomposable random process (see Rychlik (1990)), then for finiteness of the functional $N(x) = N(x, [0, \infty))$ the following condition is sufficient (see Rychlik (1990)),

$$\int_0^\infty \int_{-\infty}^\infty |z| p_t(z, 0) dz\, dt < \infty, \tag{2}$$

where $p_t(z, v)$ is the joint density of $y^{(1)}(0)$ and $y(t)$. Further, if the conditional expectation

$$\mathrm{E}\{\prod_{i=1}^m |X^{(1)}(t_i)| / X(t_1), \ldots, X(t_m)\} < \infty, \tag{3}$$

then the random variables $T_1(X), \ldots, T_m(X)$ have the joint density. For non-degeneracy of zerocrossings, the following condition is sufficient:

For every $j \geq 1$, the density of $X(t)$ at the point x,

$$p_t(x) \leq C_j,\ t \in [0, j], x \in R, 0 < C_j < \infty. \tag{4}$$

Condition (4) is a simple consequence of Bulinskaya's Theorem (see Bulinskaya (1961)).

Theorem 3. *Let $X_n \Rightarrow X$ as $n \to \infty$ in $C^k[0, \infty)$ for $k \geq 1$ and suppose the random element X satisfies conditions (2)-(4). Then*

$$T_i(X_n) \Rightarrow T_i(X), i = 1, \ldots, m, \quad \textit{and for every } j \geq 1, N_j(X_n) \Rightarrow N_j(X) \textit{ as } n \to \infty.$$

Example 1. *Let $X(t), t \in [0, \infty)$, be a zero mean stationary ergodic Gaussian process with continuously differentiable sample paths and a covariance function $r(t)$, $r(0) = 1$. Consider the Slepian model process corresponding to the behaviour of the* *process $X(t)$ after a moment of crossing of a level u (see Leadbetter, Lindgren, and Rootzén (1983), p. 198, Wilson (1988)),*

$$X_u(t) := ur(t) - \zeta r^{(1)}(t)/\lambda_2 + \kappa(t),$$

where $\kappa(t)$ is a zero mean Gaussian process with the covariance function r_κ,

$$r_\kappa(t.s) := r(t-s) - r(t)r(s) - r(t)^{(1)} r^{(1)}(s)/\lambda_2,$$

the second spectral moment $\lambda_2 = -r^{(2)}(0)$, and ζ is a random value with Rayleigh distribution and density

$$p(z) := z/\lambda_2 \exp\{-z^2/(2\lambda_2)\}.$$

ζ is independent of $\kappa(t)$. The random process $\kappa(t)$ has also continuously differentiable realizations, i.e. X_u is a C^1-valued random element. Let convergence "as $u \to \infty$" mean convergence for any countable subsequence $u_n \to \infty$ as $n \to \infty$.

Proposition 2. *Let $\eta(t) := u(X_u(t/u) - u)$ be the normed model process. Then $\eta_u \Rightarrow \eta$ as $u \to \infty$ in $C^1[0,\infty)$, where $\eta(t) := -\lambda_2 t^2/2t + \zeta t$, and moreover, $\eta_u \to \eta$ as $n \to \infty$ a.s. in $C^1[0,\infty)$, for every $j \geq 1$,*

$$T_2(\eta_u) \Rightarrow T_2(\eta) = 2\zeta/\lambda_2 \text{ and } N_j(\eta_u) \Rightarrow N_j(\eta) \text{ as } u \to \infty. \tag{5}$$

3. Weak convergence of functionals on a sequence of smooth random fields

Consider first some examples of weakly convergent sequences of random elements in the space $C^k(T)$.

Example 2. *Let $X(t), X_n(t),\ t \in T$, be zero mean Gaussian homogeneous fields with covariance functions $r(t), r_n(t)$, respectively, $n \in N$. Let the spectral function $F(\lambda)$ of the random field $X(t)$ be such that for some $\alpha, 0 < \alpha < 2$, and integer $k \geq 1$,*

$$\int_{R^m} |\lambda|_m^{2k+\alpha}\, dF(\lambda) < C,$$

and

$$r_n(t) = \int_{|\lambda|_m \leq n} e^{i\lambda' t} dF(\lambda).$$

Then X, X_n are $C^k(T)$-valued random elements, $n \in N$. We show that $X_n \Rightarrow X$ in $C^k(T)$. In fact, the convergence of finite-dimensional distributions of $X_n^{(l)}(t)$, $|l| \leq k$, follows from the pointwise convergence of the covariance functions and its derivatives up to the order $2k$. Then the weak convergence of the sequence $X_n(t),\ n \in N$, follows from normality of $X_n(t)$ and Remark 2, *since for any $l, |l| \leq k$,*

$$E(X_n^{(l)}(t) - X_n^{(l)}(0))^2 \leq 2^{2-\alpha} \int_{|\lambda|_m \leq n} |\lambda|_m^{2|l|} |\lambda' t|^\alpha\, dF(\lambda) \leq C_1 |t|_m^\alpha .$$

Example 3. *Let $X_n(t),\ t \in T$, be a zero mean Gaussian homogeneous field with covariance function*

$$r_n(t) = \int_{R_+^m} \cos(\lambda' t) f_n(\lambda) d\lambda, \quad R_+^m = [0,\infty)^m, \quad n \in N.$$

Let the following conditions for the spectral density $f_n(\lambda),\ n \in N$, hold

1. *there exists $\kappa \in R_+^m$ such that for any $\delta > 0$ and $U_\delta = \{\lambda \in R_+^m : \mid \lambda - \kappa \mid_m < \delta\}$*

$$\lim_{n\to\infty} \int_{U_\delta} f_n(\lambda) d\lambda = 1; \tag{6}$$

2. *there exists a function $f(\lambda)$ such that $f_n(\lambda) \le f(\lambda)$ for every $\lambda \in R_+^m \setminus U_{\lambda_0}$, $\lambda_0 > 0$ and for some $0 < \alpha < 2,\ k \in N$*

$$\int_{R_+^m} \mid \lambda \mid_m^{2k+\alpha} f(\lambda) d\lambda < C. \tag{7}$$

Then $X_n \Rightarrow X$ in $C^k(T)$, where $X(t)$ is a zero mean Gaussian homogeneous field with the covariance function $r(t) = \cos(\kappa t)$. The convergence of finite-dimensional distributions follows as in Example 2 *from the convergence of the corresponding sequences of the covariance functions and its derivatives up to the order $2k$. In fact, if we write $R_\delta = R_+^m \setminus U_\delta,\ t^l = \prod_{i=1}^m t_i^{l_i}$ for $t \in T$, then*

$$r_n^{(2l)}(t) = (-1)^{|l|} \int_{R_+^m} \lambda^{2l} \cos(\lambda' t) f_n(\lambda) d\lambda, \qquad r^{(2l)}(t) = (-1)^{|l|} \kappa^{2l} \cos(\kappa' t).$$

So, for any $\delta > 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} \mid \int_{R_\delta} \lambda^{2l} \cos(\lambda' t) f_n(\lambda) d\lambda \mid &\le \int_{R_+^m} \mid \lambda \mid_m^{2|l|} f_n(\lambda) d\lambda \\ = \ & \int_{R_\delta \cap \{|\lambda|_m \le \lambda_1\}} \mid \lambda \mid_m^{2l} \cos(\lambda' t) f_n(\lambda) d\lambda + \int_{R_\delta \cap \{|\lambda|_m > \lambda_1\}} \lambda^{2l} \cos(\lambda' t) f_n(\lambda) d\lambda \\ \le \ & \lambda_1^{2l} \varepsilon_n + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

by (6) and (7) for sufficiently large and λ_1. It also follows from (6) that

$$\int_{U_\delta} \lambda^{2l} \cos(\lambda' t) f_n(\lambda) d\lambda \to \kappa^{2l} \cos(\kappa' t) \quad \text{as } n \to \infty$$

and thus, $r_n^{(2l)}(t) \to r^{(2l)}(t)$ as $n \to \infty$. Now (2) implies for every l, $\mid l \mid \le k$, that

$$\begin{aligned} E(X_n^{(l)}(t) - X_n^{(l)}(0))^2 &= 2 \int_{R_+^m} \lambda^{2l} (1 - \cos(\lambda' t)) f_n(\lambda) d\lambda \\ &\le C_1 \lambda_1^{2k+2} \mid t \mid_m^2 + C_2 \mid t \mid_m^\alpha \le C_3 \mid t \mid_m^\alpha, \end{aligned}$$

and as in Example 2 *the assertion follows from* Remark 2.

Consider now one of the examples of the continuous integer valued functionals on smooth random fields and apply the weak convergence technique.

Denote by

$$\nabla x(t) := (\partial x(t)/\partial t_1, ..., \partial x(t)/\partial t_m), \quad G(x,t) := \det(\partial^2 x(t)/\partial t_i \partial t_j)$$

and define the functional

$$N(x) := \#\{t \in (0,1)^m : \nabla x(t) = 0\},$$

i.e., the number of stationary points of function $x(\cdot) \in C^k(T),\ k \geq 2$. Let $D \subset C^k(T),\ k \geq 2$ be a set of functions with the finite number of non-degenerate stationary points τ, i.e. $\nabla x(t) = 0,\ G(x,t) \neq 0$.

Proposition 3. *The functional $N(x)$ is continuous on the set of functions $D \subset C^k(T),\ k \geq 2$.*

Let $\zeta(t),\ t \in T$, be a zero mean homogeneous random field with sample paths from the space $C^k(T),\ k \geq 2,\ 0 < p_t(z) < C < \infty$, where $p_t(z)$ is the density of distribution for the random vector $\nabla\zeta(t)$ at the point $z, z \in R^m$, and suggest that

$$P\{G(\zeta,t) = 0 \mid \nabla\zeta(t) = z\} = 0. \tag{8}$$

Then $N(\zeta)$ is a.s. finite for this random field and if τ is a stationary point, then $G(\zeta,\tau) \neq 0$ a.s., i.e., τ is non-degenerate (see Belyaev (1967)). For the case of a Gaussian homogeneous field, (8) holds if e.g., the joint distribution of partial derivatives $\partial^2\zeta(t)/\partial t_i \partial t_j,\ i,j = 1,...,m$, is non-degenerate (see Belayev (1967)).

Theorem 4. *Let $X(t), X_n(t)$ be zero mean homogeneous random fields with sample paths from the space $C^k(T),\ k \geq 2$, satisfying regularity condition (8), $n \in N$. Let $X_n \Rightarrow X$ in $C^k(T)$. Then $N(X_n) \Rightarrow N(X)$ as $n \to \infty$.*

In particular, *Theorem 4* is valid for the corresponding sequences of Gaussian homogeneous fields considered in *Examples 2* and *3*.

4. Proofs

A) Introduction

Proof of Theorem 1. Let $S^k[0,\infty)$ be the space $R^k \times C[0,\infty)$ with the topology of direct product, $S^k = S^k[0,\infty)$. Then the mapping

$$f : C^k \to S^k, \quad x(\cdot) \in C^k, \qquad f(x) := (x(0), \dots, x^{(k-1)}(0), x^{(k)}(\cdot)),$$

is a homeomorphism of the spaces C^k and S^k and furthermore, the space $C^k[0,\infty)$ is a complete separable normed function space. The following argument is similar to that in Rusakov and Seleznjev (1988) for the space $C^k[0,1]$. Let $\mathcal{C}^k$ and $\mathcal{S}^k$ be the Borel σ-algebras on the spaces C^k and S^k, respectively, $P, P_n, n \geq 1$, probability measures on $(C^k, \mathcal{C}^k)$ (or equivalently, defined on $(S^k, \mathcal{S}^k)$ by using the homeomorphism f for the spaces $C^k[0,\infty)$ and $S^k[0,\infty)$). Since probability measures coincide on $C[0,\infty)$ if their finite dimensional distributions coincide (see Witt (1970)), the probability measures P and Q on $(C^k, \mathcal{C}^k)$ coincide if $P(A) = Q(A)$ for every set A such that

$$A = \{x(\cdot) \in C^k : \phi_k(x, t_1, \dots, t_p) \in H\},$$

where

$$\phi_k(x, t_1, \dots, t_p) := (x(0), \dots, x^{(k-1)}(0), x^{(k)}(t_1), \dots, x^{(k)}(t_p)),$$

and a set $H \in \mathcal{R}^{k+p}$, $\mathcal{R}^{k+p}$ is the Borel σ-algebra for the space R^{k+p}. Denote the class of such sets A by $\mathcal{M}$. Then $\mathcal{M}$ is a determining class (see Billingsley (1968), p. 15) for probability measures defined on $(C^k, \mathcal{C}^k)$. □

Proposition 4. *$P_n \Rightarrow P$ as $n \to \infty$ in $C^k[0, \infty)$ iff*

(i) for any $A \in \mathcal{M}$, $\quad P_n(A) \to P(A)$ as $n \to \infty$ and

(ii) the family of measures $\{P_n, n \geq 1\}$ is tight in $C^k[0, \infty)$.

Proof. The proof is straightforward. One can apply the argument used for $C[0, \infty)$ (see Witt (1970)) due to the beforehand given remark and the structure of the space $C^k[0, \infty)$. □

Let P, P_n be probability measures induced by random elements X, X_n, respectively, $n \geq 1$. Then *Theorem 1 (i)* is equivalent to *Proposition 4 (i)*. The tightness of the family $\{P_n\}$ in $C^k[0, \infty)$ is equivalent to the tightness of the families of marginal distributions on R^k and $C[0, \infty)$, respectively (cf., Rusakov and Seleznjev (1988)). The necessity of (ii) follows as in Theorem 6, Witt (1970). Using (i) together with (ii) we also obtain that the family $\{P_n\}$ is tight. Now the assertion of *Theorem 1* follows by *Proposition 4*.

Proof of Theorem 2. The following arguments are analogous to the corresponding results in Rusakov and Seleznjev (1988). We formulate the following proposition which corresponds to the Arzela-Ascoli theorem for the space $C^k(T)$, (cf., Theorem 1.5.4, Kufner, John, and Fučik (1977), and Billingsley (1968), p. 221).

Proposition 5. *A set $K \subset C^k(T)$ is relatively compact iff*

(i) $\sup_{x \in K}\{\mid x^{(l)}(0) \mid, \quad \mid l \mid \leq k,\} < \infty$ *and*

(ii) for any l $\mid l \mid \leq k$, $\quad \lim_{\delta \to 0} \sup_{x \in K} w(x^{(l)}, \delta) = 0.$

Denote by $q = \#\{l : \mid l \mid \leq k\} = (m+k)!/(m!k!)$. Let for any set $t_i \in T,\ i = 1, ..., p$, $\mathcal{R}^{pq}$ and $\mathcal{C}^k$ be the Borel σ-algebras of the space R^{pq} and $C^k(T)$, respectively. Then, considering any sets $\{t_i \in T,\ i = 1, ...p\}$, and sets $H \in \mathcal{R}^{pq}$ we obtain the class of sets $\mathcal{M} = \{A\}$,

$$A = \{x(\cdot) :\ \phi_k(x, t_1, ..., t_p) \in H\} \in C^k.$$

A ball in the space $C^k(T)$ is a limit of a sequence of sets of the class $\mathcal{M}$ and consequently, the class $\mathcal{M}$ is a determining class (see Billingsley (1968), p. 15) for probability measures on $(C^k(T), \mathcal{C}^k)$.

Proposition 6. *Let $P, P_n, n \in N$, be probability measures on the space $C^k(T)$. Then the family $\{P_n\}$ is tight and $P_n(A) \to P(A)$ as $n \to \infty$ for any set $A \in \mathcal{M}$ iff $P_n \Rightarrow P$ as $n \to \infty$.*

Proof. The assertion follows from the Prohorov's theorem for a complete separable metric space (see Billingsley (1968), p. 37). □

Thus for investigation of weak convergence of a sequence of probability measures $\{P_n\}$ we need a criterion of tightness of the family $\{P_n\}$ in the space $C^k(T)$.

Proposition 7. *A family of probability measures $\{P_n\}$ is tight in the space $C^k(T)$ iff*

(i) for any $\eta > 0$ there exists $a_{(l)} > 0$ such that $P_n\{x(\cdot) : | x^{(l)}(0) |> a_{(l)}\} < \eta$, $| l |\leq k$ and

(ii) for any $\varepsilon_{(l)}, \eta_{(l)} > 0$, there exist $n_0 \in N$, $\delta_{(l)} > 0$ such that

$$P_n\{x(\cdot) : w(x^{(l)}, \delta_{(l)}) |> \varepsilon_{(l)}\} < \eta_{(l)}, \quad | l |\leq k, \;\; n \geq n_0 \,.$$

Proof. One can apply arguments used for $C[0,1]$ with *Proposition 5* instead of the Arzela-Ascoli Theorem (cf., Theorem 8.2. Billingsley (1968)). □

Let P, P_n be probability measures on the space $(C^k(T), \mathcal{C}^k)$ induced by the random elements $X, X_n,\ n \in N$. Now we return to the proof of *Theorem 2*. We have, if $X_n \Rightarrow X$ then by *Proposition 6*, $P_n \to P(A)$ as $n \to \infty$ for any $A \in \mathcal{M}$, i.e., *Theorem 2 (i)* holds. Further, *Theorem 2 (ii)* follows from the tightness of $\{P_n\}$ and *Proposition 7 (ii)*. Conversely, *Theorem 2(i)* and *(ii)* together imply that $\{P_n\}$ is tight and $P_n(A) \to P(A)$ as $n \to \infty$ for any $A \in \mathcal{M}$, i.e., $X_n \Rightarrow X$ by *Proposition 6.*

B) Weak convergence of functionals on smooth random processes on $[0, \infty)$.

Proof of Proposition 1. Consider the mapping $F : C^k \times [0,\infty) \to R,\ F(y,t) := y(t)$, in some neighbourhood $U = U_x \times U_\tau$ of a point of zerocrossing (x,τ) in the topology of direct product. By nondegeneracy of the point τ we can always find a neighbourhood U_τ such that there are no other zerocrossings in U_τ. Then, for the mapping F, the conditions of the Implicit Function Theorem (see e.g., Lusternik and Sobolev (1961), p. 194) hold: $F(x,\tau) = 0, F^{(1)}(y,t) = y^{(1)}(t)$ is continuous in the neighbourhood U, $F^{(1)}(x,\tau) \neq 0$. Consequently, for any $\varepsilon > 0$ there exist $\delta > 0$ and a function $g(\cdot)$,

$$g(y) = t = \tau_y, \tag{9}$$

which is defined in the δ-neighbourhood of $||y - x|| < \delta$ and such that if (9) holds for a pair (y, τ_y) then it satisfies the relation

$$F(y, \tau_y) = 0. \tag{10}$$

Conversely, if (10) holds for some pair (y, τ_y) and $||y - x|| < \delta, |\tau_y - \tau| < \varepsilon$, then for this pair (9) holds and the function $g(y)$ is continuous in some neighbourhood of $x(\cdot)$. Thus, there is only one point τ_y in some ε-neighbourhood of the point τ for every function $y(\cdot)$ belonging to the δ-neighbourhood of the function $x(\cdot)$. Now, for the function $x(\cdot) \in \mathcal{T}_m$, let $V_i(x)$ be a corresponding neighbourhood of the point $T_i(x), i = 1, \dots, m, \delta = \min(\delta_i, i = l, \dots, m)$. Let $T' := [0, T_m(x)] \setminus \cup_{i=1}^m V_i(x)$. Then

$$\min_{T'} |y(t)| \geq \min_{T'} |x(t)| - \min_{T'} |x(t) - y(t)| > 0$$

for sufficiently small δ, i.e., there exist no other zerocrossings for the function $y(\cdot)$ and we obtain the continuity of the functionals $T_i(x), i = 1, \dots, m$. For the functional $N_j(x), j \geq 1, x(\cdot) \in \mathcal{N}_j$, the argument is analogous and we can find such a δ-neighbourhood $V_j(x)$ that for any function $y(\cdot) \in V_j(x), N_j(y) = N_j(x)$, i.e. $N_j(\cdot)$ is continuous. □

Proof of Theorem 3. Let P be the measure induced by the random element X in $C^k[0,\infty)$. If conditions (2)-(4) hold, then $P(\mathcal{T}_m \cap \mathcal{N}_j) = 1$ for every $j \geq 1$. So the assertion follows by the Continuity Theorem (Theorem 5.1, Billingsley (1968)) and *Proposition 1*. □

Proof of Proposition 2. There is no need to use *Theorem 1* since weak convergence $\eta \Rightarrow \eta$ follows from the evident convergence of almost all sample paths of $\eta_u(t)$ in $C^1[0,\infty)$. In fact, $\eta_u(0) = \eta(0) = 0, \quad \eta_u^{(1)}(0) = \eta^{(1)}(0) = \zeta$ and

$$\begin{aligned}\eta_u^{(1)}(t) - \eta^{(1)}(t) &= ur^{(1)}(t/u) - \zeta r^{(2)}(t/u)/\lambda_2 + \kappa^{(1)}(t/u) + t\lambda_2 - \zeta \\ &= -t\lambda_2 + \zeta + \mathrm{o}(1) + t\lambda_2 - \zeta = \mathrm{o}(1) \text{ as } u \to \infty\end{aligned}$$

uniformly in $t \in [0,j]$ for every $j \geq 1$, $\kappa(0) = \kappa^{(1)}(0) = 0$. Now (5) follows directly from *Propositions 3* and *2*. □

C) Weak convergence of functionals on a sequence of smooth random fields.

Proof of Proposition 3. For any given function $x(\cdot) \in D$ we shall find a δ-neighborhood such that for any function $y(\cdot)$, with $\| x - y \| < \delta$, $N(y) = N(x)$.

Let τ be a nongenerate stationary point of the function $x(t)$. We shall find $\delta < 0$ such that every function $y(\cdot)$ from the δ-neighborhood of $x(\cdot)$ will have in some ε-neighborhood of the point τ, the nondegenerate stationary point τ_y and only one such point.

Consider the mapping $F(y,t) = \nabla y(t),\ F:\ C^k(T) \times T \to R$, in some neighborhood U of the point (x,t) in the space $C^k(T) \times T$ (in the topology of direct product) such that there no other stationary points of the function $x(\cdot)$ in U. This neighborhood exists since the point τ is nondegenerate. Then the Implicit Function Theorem is valid for the mapping F (see e.g. Lusternik and Sobolev (1961), p. 194). In fact, the function $F(y,t)$ is continuous at the point (x,τ) by definition, $F(x,\tau) = 0$, the partial derivative operator, $F_t^{(1)}(y,t) = (\partial^2 y(t)/\partial t_i \partial t_j)$ exists and is continuous in U and the operator $F_t^{(1)}(y,t)$ is invertible since the point τ is nondegenerate. Then there exist $\varepsilon, \delta > 0$ and a mapping $f(\cdot)$,

$$t = f(y),\ f:\ C^k(T) \to T, \tag{11}$$

defined for $\| y - x \| < \delta$, such that $f(y)$ is continuous at the point x, and

$$F(y,t) = \nabla y(t) = 0, \tag{12}$$

and conversely, every pair (y,t), for which relation (12) and the conditions $\| y - x \| < \delta$, $| t - \tau | < \varepsilon$ hold, satisfies relation (7), i.e. $t = \tau_y$ and whence τ_y is nondegenerate also for sufficiently small δ and ε by continuity of functions $\partial^2 y(t)/\partial t_i \partial t_j$.

Thus, in some neighbourhood of every nondegenerate stationary point τ, every function $y(\cdot)$, $\| y - x \| < \delta$, has a nondegenerate stationary point τ_y. We shall show that there are no other stationary points for the function $y(\cdot)$. Really, for the remaining part of the set T, $T_\varepsilon = T \setminus \cup_{i=1}^{q} O_{\varepsilon_i}(\tau_i)$, where $\tau_i,\ i = 1, ..., q$, are stationary points for the function $x(\cdot)$,

$$\mid \nabla y(t) \mid_m = \mid \nabla x(t) + (\nabla y(t) - \nabla x(t)) \mid_m > \min_{T_\varepsilon} \mid \nabla x(t) \mid_m - \delta > 0$$

for sufficiently small δ, i.e. for this δ we have $N(y) = N(x)$. □

Proof of Theorem 4. It follows by (8) for X and X_n that the conditions of *Proposition 3* hold. Therefore, we obtain that the functional $N(\cdot)$ is continuous on the set D, $P(D) = P_n(D) = 1$, where P, P_n are probability measures induced by the random elements X, X_n on $C^k(T)$, respectively. So the assertion of *Theorem 4* follows directly from the Continuity Theorem (Theorem 5.1, Billingsley (1968)). □

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