A generalization of the butterfly theorem from circles to conics

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Abstract. *This paper presents a generalization of the Butterfly Theorem that is true for all conic curves.*

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The original Butterfly Theorem claims that whenever chords AB and CD of a circle intersect at the midpoint S of the third chord PQ, then S is also the midpoint of the segment formed by the intersections X and Y of the chords AD and BC with the line PQ (see *Figure 1*). Moreover, S is also the midpoint of the segment UV, where U and V are intersections of the lines AC and BD with the line PQ.

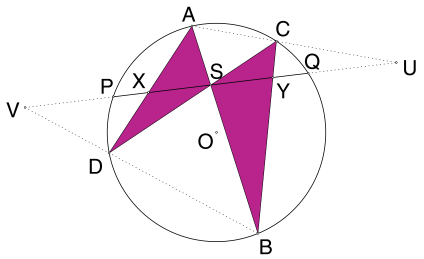

Figure 1. *The point S is the body and the triangles ADS and BCS are the wings of the butterfly. Another butterfly with the same body has the triangles ACS and BDS as wings.*

The above theorem dates back to at least 1815 according to [3]. It appears as an exercise in Coxeter's book [1] and it has been the subject of several papers that either give new proofs or propose improvements in various directions.

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The following generalization of the Butterfly Theorem has recently been presented in [6].

Butterfly Theorem for Circles. *Let A, B, C, D be four points on a circle k with the centre O and let S be the orthogonal projection of the point O onto the given line w. If S is the midpoint of points $H = w \cap AB$ and $K = w \cap CD$, then S is the midpoint of the points $U = w \cap AC$ and $V = w \cap BD$ and the midpoint of the points $X = w \cap AD$ and $Y = w \cap BC$.*

The aim of this paper is to present the proof of the following improvement of the Butterfly Theorem for Circles that could be named the Butterfly Theorem for Conics. The idea in this generalization is to replace the circle by a conic (i. e., either an ellipse, a hyperbola, or a parabola), to take for w any line which is perpendicular to an axis z of the conic, and to set $S = w \cap z$.

Butterfly Theorem for Conics. *Let S be a point on an axis z of a conic k and let w denote the line through S perpendicular to z. Let A, B, C, and D be different points on k and let H, K, U, V, X, and Y denote the intersections of the line w with the lines AB, CD, AC, BD, AD, and BC, respectively. If S is the midpoint of one among the segments HK, UV, and XY, then it is the midpoint of all three segments.*

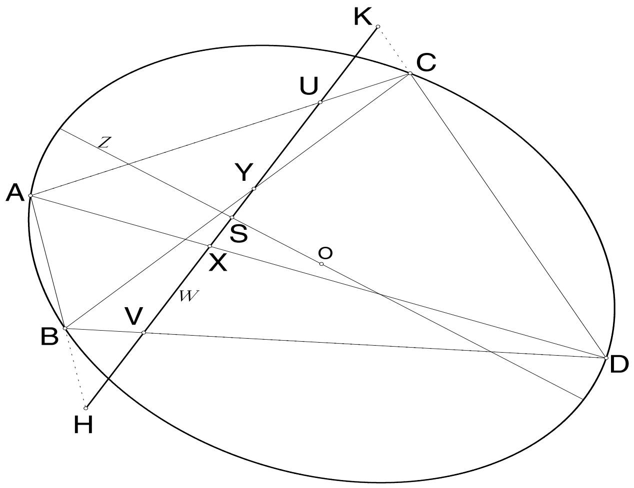

Figure 2. *The Butterfly Theorem for the Ellipse.*

Of course, the Butterfly Theorem for Circles is a special case of our Butterfly Theorem for Conics because the line z can be any line through the centre of the circle.

In order to prove the Butterfly Theorem for Conics, recall that if we take a focus of k as the pole (the origin) and the main axis m as the polar axis of the polar coordinate system, then the conic k has the equation $\varrho = p/(1 + \varepsilon \cos\vartheta)$, where ϱ is the polar radius, ϑ is the polar angle, and p and ε are nonnegative real numbers. Hence, in the associated rectangular coordinate system the points A, B, C, and

D have coordinates $(p\cos\vartheta/(1+\varepsilon\cos\vartheta), p\sin\vartheta/(1+\varepsilon\cos\vartheta))$, where ϑ is α, β, γ, and δ. We could continue using trigonometric functions but it is easier at this point to employ universal trigonometric substitution to write

$$\cos\alpha = \frac{1-a^2}{1+a^2}, \qquad \sin\alpha = \frac{2\,a}{1+a^2},$$

and similarly for the remaining three points (and their corresponding letters). We conclude that the points A, B, C, and D have coordinates

$$\left(\frac{p\,(1-t^2)}{\varepsilon\,(1-t^2)+t^2+1}, \frac{2\,p\,t}{\varepsilon\,(1-t^2)+t^2+1}\right)$$

for t equal to a, b, c, and d.

Let us first prove the above theorem in the case when the line z is the main axis m of k. The point S then has the coordinates $(s, 0)$ for some real number s and the line w has the equation $x = s$. The line AB has the equation

$$(a\,b\,(\varepsilon-1)+\varepsilon+1)\,x + (a+b)\,y - p\,(a\,b+1) = 0,$$

while the other lines CD, AD, BC, BD, and AC have analogous equations. The intersection H of the lines w and AB has the first coordinate s and the second coordinate

$$\frac{s\,(a\,b-1)+(p-\varepsilon\,s)\,(a\,b+1)}{a+b}.$$

We obtain this value by substituting s for x in the above equation of AB and solving for y. The other intersections K, U, V, X, and Y have the same first coordinate and similar second coordinates. The midpoints of the segments HK, UV, and XY have the second coordinates equal to

$$\frac{\mathcal{P}\,(p-\varepsilon\,s)-s\,\mathcal{R}}{2\,(a+b)(c+d)}, \qquad \frac{\mathcal{P}\,(p-\varepsilon\,s)-s\,\mathcal{R}}{2\,(a+c)(b+d)}, \qquad \frac{\mathcal{P}\,(p-\varepsilon\,s)-s\,\mathcal{R}}{2\,(a+d)(b+c)},$$

(the arithmetic means of the second coordinates of its endpoints), where

$$\mathcal{P} = a+b+c+d+a\,b\,c+a\,b\,d+a\,c\,d+b\,c\,d,$$

$$\mathcal{R} = a+b+c+d-a\,b\,c-a\,b\,d-a\,c\,d-b\,c\,d.$$

The conclusion in the theorem for the case when $z = m$ now follows obviously.

If k is a parabola, then the proof is complete because it has only one axis which is its main axis.

When k is either an ellipse or a hyperbola, we must consider also the secondary axis (the perpendicular n to m at the centre O of k) as the second (and the last, if k is not a circle) possibility for the line z. Hence, when $z = n$, the point S has the coordinates $(p\,\varepsilon/(\varepsilon^2-1), s)$ for some real number s and the line w has the equation $y = s$. The intersection H of the lines w and AB has the second coordinate s and the first coordinate

$$\frac{p\,(a\,b+1)-s\,(a+b)}{a\,b\,(\varepsilon-1)+\varepsilon+1}.$$

We obtain this value by substituting s for y in the above equation of AB and solving for y. The other intersections K, U, V, X, and Y have the same second coordinate and similar first coordinates. The numerator of the difference of the first coordinates of the midpoints of the segments HK, UV, and XY and the number $p\,\varepsilon/(\varepsilon^2-1)$ (the first coordinate of the point S) is equal to

$$2\,[(a\,b\,c\,d-1)(\varepsilon^2+1)-2\,(a\,b\,c\,d+1)\,\varepsilon]\,p-(\varepsilon^2-1)(\mathcal{P}\,\varepsilon+\mathcal{R})\,s.$$

The conclusion in the theorem for this case now follows immediately.

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