$\mathcal{I}-$limit superior and limit inferior

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Abstract. *In this paper we extend concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to $\mathcal{I}-$limit superior and inferior and give some $\mathcal{I}-$analogue of properties of statistical limit superior and inferior for a sequence of real numbers. Also we extend the concept of the statistical core to $\mathcal{I}-$core for a complex number sequence and get necessary conditions for a summability matrix A to yield $\mathcal{I}-$core $\{Ax\} \subseteq \mathcal{I}-$core $\{x\}$ whenever x is a bounded complex number sequence.*

Key words: *statistical limit superior and inferior, statistical core of a sequence, $\mathcal{I}-$convergent sequence*

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1. Introduction

If K is a subset of natural numbers $\mathbb{N}$, K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ will denote the cardinality of K_n. Natural density of K [20], [13] is given by $\delta(K) := \lim_n \frac{1}{n} |K_n|$, if it exists. Fast introduced the definition of a statistical convergence using the natural density of a set. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero [7] ,[9] . Hence x is statistically convergent to L iff $(C_1\chi_{K(\varepsilon)})_n \to 0$, (as $n \to \infty$, for ever $\varepsilon > 0$), where C_1 is the Cesáro mean of order one and χ_K is the characteristic function of the set K. Properties of statistically convergent sequences have been studied in [1] , [2] , [9] , [18] , [21].

Statistical convergence can be generalized by using a nonnegative regular summability matrix A in place of C_1.

Following Freedman and Sember [8] , we say that a set $K \subseteq \mathbb{N}$ has $A-$density if $\delta_A(K) := \lim_n (A\chi_K)_n = \lim_n \sum_{k \in K} a_{nk}$ exists where $A = (a_{nk})$ is a nonnegative regular matrix.

The number sequence $x = (x_k)$ is $A-$statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon)$ has $A-$density zero[2], [8] , [18].

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Fridy [10] has introduced the notions of a statistical limit point and a cluster point. Fridy and Orhan [11] studied the idea of statistical limit superior and inferior. Connor and Kline [4] and Demirci [6] extended these concepts to $A-$statistical convergence using a nonnegative regular summability matrix A in place of C_1. Also Connor has introduced $\mu-$statistical analogue of these concepts using a finitely additive set function μ taking values in $[0,1]$ defined on a field Γ of subsets of $\mathbb{N}$ such that if $|A| < \infty$, then ; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; and $\mu(\mathbb{N}) = 1$ [3] , [5] .

The number sequence $x = (x_k)$ is $\mu-$statistically convergent to L provided that $\mu(\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$ [3], [5].

Kostyrko, Mačaj and Šalát [15] , [16] introduced the concepts of $\mathcal{I}-$convergence, $\mathcal{I}-$limit point and $\mathcal{I}-$cluster point of sequences of real numbers based on the notion of the ideal of subsets of $\mathbb{N}$.

In this paper we extend concepts of statistical limit superior and inferior to $\mathcal{I}-$limit superior and inferior and give some properties of $\mathcal{I}-$limit superior and inferior for a sequence of real numbers. We also extend the concept of a statistical core to $\mathcal{I}-$*core* for a complex number sequence and get necessary conditions for a summability matrix A to yield $\mathcal{I}-$*core* $\{Ax\} \subseteq \mathcal{I}-$*core* $\{x\}$ whenever x is a bounded complex number sequence.

2. Definition and notations

We first recall the concepts of an ideal and a filter of sets.

Definition 1. *Let $X \neq \phi$. A class $S \subseteq 2^X$ of subsets of X is said to be an ideal in X provided that S is additive and hereditary , i.e. if S satisfies the conditions:*

(i) $\phi \in S$,

(ii) $A, B \in S \Rightarrow A \cup B \in S$,

(iii) $A \in S,\ B \subseteq A \Rightarrow B \in S$

([17] ,*p.*34*).*

An ideal is called non-trivial if $X \notin S$.

Definition 2. *Let $X \neq \phi$. A non-empty class $\mathcal{F} \subseteq 2^X$ of subsets of X is said to be a filter in X provided that:*

(i) $\phi \in \mathcal{F}$,

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,

(iii) $\phi \in \mathcal{F},\ A \subseteq B \Rightarrow B \in \mathcal{F}$

([19] ,p.44).

The following proposition expresses a relation between the notions of an ideal and a filter:

Proposition 1. *Let S be non-trivial in X, $X \neq \phi$. Then the class*

$$\mathcal{F}(S) = \{M \subseteq X : \exists A \in S : M = X \setminus A\}$$

is a filter on X (we will call $\mathcal{F}(S)$ the filter associated with S) [15] .

Definition 3. *A non-trivial ideal S in X is called admissible if $\{x\} \in S$ for each $x \in X$* [15].

As usual, $\mathbb{R}$ will denote real numbers and $\mathbb{C}$ complex numbers.

Definition 4. *Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. Then*

(i) *A sequence $x = (x_n)$ of real numbers is said to be $\mathcal{I}-$convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [15]. In this case we write $\mathcal{I}-\lim x = L$.*

(ii) *An element $\xi \in \mathbb{R}$ is said to be $\mathcal{I}-$limit point of the real number sequence $x = (x_n)$ provided that there exists a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_k x_{m_k} = \xi$ [16].*

(iii) *An element $\xi \in \mathbb{R}$ is said to be $\mathcal{I}-$cluster point of the real number sequence $x = (x_n)$ iff for each $\varepsilon > 0$ we have $\{k : |x_k - \xi| < \varepsilon\} \notin \mathcal{I}$ [16].*

Note that the set of $\mathcal{I}-$cluster points of x is a closed points set in $\mathbb{R}$ where $\mathcal{I}$ is an admissible ideal [15].

Some results on $\mathcal{I}-$convergence, $\mathcal{I}-$limit point and $\mathcal{I}-$cluster point may be found in [15],[16].

Throughout the paper $\mathcal{I}$ will be an admissible ideal.

3. $\mathcal{I}-$limit superior and inferior

In this section we study the concepts of $\mathcal{I}-$limit superior and inferior for a real number sequence.

For a real number sequence $x = (x_k)$ let B_x denote the set

$$B_x := \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}.$$

Similarly,

$$A_x := \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}.$$

We begin with a definition.

Definition 5. *Let $\mathcal{I}$ be an admissible ideal and x a real number sequence. Then the $\mathcal{I}-$limit superior of x is given by*

$$\mathcal{I}-\limsup x := \begin{cases} \sup B_x, \text{ if } B_x \neq \phi, \\ -\infty, \text{ if } B_x = \phi. \end{cases}$$

Also, the $\mathcal{I}-$limit inferior of x is given by

$$\mathcal{I}-\liminf x := \begin{cases} \inf A_x, \text{ if } A_x \neq \phi, \\ +\infty, \text{ if } A_x = \phi. \end{cases}$$

Note that if we define $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$ and $\mathcal{I} = \{K \in \Gamma : \mu(K) = 0\}$ in *Definition 5*, then we get *Definition 1* of [6], *Definition 1* of [11] and Connor's definitions [5] of $\mu-$statistical superior and inferior, respectively. This observation suggests the following result which can be proved by a straightforward least upper bound argument.

Theorem 1. *If $\beta = \mathcal{I}-\limsup x$ is finite, then for every positive number ε*

$$\{k : x_k > \beta - \varepsilon\} \notin \mathcal{I} \text{ and } \{k : x_k > \beta + \varepsilon\} \in \mathcal{I}. \tag{1}$$

Conversely, if (1) holds for every positive ε, then $\beta = \mathcal{I}-\limsup x$.

The dual statement for $\mathcal{I}-\liminf x$ is as follows.

Theorem 2. *If $\alpha = \mathcal{I}-\liminf x$ is finite, then for every positive ε*

$$\{k \ : \ x_k < \alpha + \varepsilon\} \notin \mathcal{I} \text{ and } \{k \ : \ x_k < \alpha - \varepsilon\} \in \mathcal{I}. \tag{2}$$

Conversely, if (2) holds for every positive ε, then $\alpha = \mathcal{I}-\liminf x$.

Considering the definition of $\mathcal{I}-$cluster point in *Definition 4* we see that *Theorems 1* and *2* can be interpreted as saying that $\mathcal{I}-\limsup x$ and $\mathcal{I}-\liminf x$ are the greatest and the least $\mathcal{I}-$cluster points of x.

Now we have the following

Theorem 3. *For any real number sequence x,*

$$\mathcal{I}-\liminf x \leq \mathcal{I}-\limsup x.$$

Proof. First consider the case in which $\mathcal{I}-\limsup x = -\infty$. Hence we have $B_x = \phi$, so for every b in $\mathbb{R}$, $\{k : x_k > b\} \in \mathcal{I}$ which implies that $\{k \ : x_k \leq b\} \in F(\mathcal{I})$ so for every a in $\mathbb{R}$, $\{k \ : x_k \leq a\} \notin \mathcal{I}$. Hence $\mathcal{I}-\liminf x = -\infty$.

The case in which $\mathcal{I}-\limsup x = +\infty$ needs no proof, so we next assume that $\beta = \mathcal{I}-\limsup x$ is finite, and $\alpha := \mathcal{I}-\liminf x$. Given $\varepsilon > 0$ we show that $\beta + \varepsilon \in A_x$, so that $\alpha \leq \beta + \varepsilon$. By *Theorem 1*, $\{k \ : \ x_k > \beta + \varepsilon\} \in \mathcal{I}$ because $\beta = lub\ B_x$. This implies $\left\{k \ : \ x_k \leq \beta + \frac{\varepsilon}{2}\right\} \in F(\mathcal{I})$. Since $\left\{k \ : \ x_k \leq \beta + \frac{\varepsilon}{2}\right\} \subseteq \{k \ : \ x_k < \beta + \varepsilon\}$ and $F(\mathcal{I})$ is a filter on $\mathbb{N}$, $\{k \ : \ x_k < \beta + \varepsilon\} \in F(\mathcal{I})$. This implies $\{k \ : \ x_k < \beta + \varepsilon\} \notin \mathcal{I}$. Hence $\beta + \varepsilon \in A_x$. By definition $\alpha = \inf A_x$, so we conclude that $\alpha \leq \beta + \varepsilon$; and since ε is arbitrary this proves that $\alpha \leq \beta$. □

From *Theorem 3* and *Definition 5*, it is clear that

$$\liminf x \leq \mathcal{I}-\liminf x \leq \mathcal{I}-\limsup x \leq \limsup x$$

for any real number sequence x.

$\mathcal{I}-$limit point of a sequence x is defined in (ii) of *Definition 4* as the limit of a subsequence of x whose indices do not belong to $\mathcal{I}$. We cannot say that $\mathcal{I}-\limsup x$ is equal to the greatest $\mathcal{I}-$limit points of x. This can be seen from Example 4 in [11] where $\mathcal{I} = \{K \subseteq \mathbb{N} \ : \ \delta(K) = 0\}$.

Definition 6. *The real number sequence $x = (x_k)$ is said to be $\mathcal{I}-$bounded if there is a number B such that $\{k \ : \ |x_k| > B\} \in \mathcal{I}$.*

Note that $\mathcal{I}-$boundedness implies that $\mathcal{I}-\limsup$ and $\mathcal{I}-\liminf$ are finite, so properties (1) and (2) of *Theorems 1*and *2* hold.

Theorem 4. *The $\mathcal{I}-$bounded sequence x is $\mathcal{I}-$convergent if and only if*

$$\mathcal{I}-\liminf x = \mathcal{I}-\limsup x.$$

Proof. Let $\alpha := \mathcal{I}-\liminf x$ and $\beta := \mathcal{I}-\limsup x$. First suppose that $\mathcal{I}-\lim x = L$ and $\varepsilon > 0$. Then $\{k \ : \ |x_k - L| \geq \varepsilon\} \in \mathcal{I}$, so $\{k \ : \ x_k > L + \varepsilon\} \in \mathcal{I}$, which implies that $\beta \leq L$. We also have $\{k \ : \ x_k < L - \varepsilon\} \in \mathcal{I}$, which yields that $L \leq \alpha$. Therefore $\beta \leq \alpha$. Combining this with *Theorem 3* we conclude that $\alpha = \beta$.

Now assume $\alpha = \beta$ and define $L := \alpha$. If $\varepsilon > 0$ then (1) and (2) of *Theorem 1* and *2* imply $\left\{k \ : \ x_k > L + \frac{\varepsilon}{2}\right\} \in \mathcal{I}$ and $\left\{k \ : \ x_k < L - \frac{\varepsilon}{2}\right\} \in \mathcal{I}$. Hence $\mathcal{I}-\lim x = L$. □

4. $\mathcal{I}-$ core

In [11] Fridy and Orhan introduced the concept of the statistical core of a real number sequence, and proved the statistical core theorem. Those results have also been extended to the complex case too [12] . Using the same technique as in [12], we introduce the concept of $\mathcal{I}-$*core* of a complex sequence and get necessary conditions for a summability matrix A to yield $\mathcal{I}-core\{Ax\} \subseteq \mathcal{I}-core\{x\}$ whenever x is a bounded complex number sequence.

In this section x, y and z will denote complex number sequences and $A = (a_{nk})$ will denote an infinite matrix of complex entries which transforms a complex number sequence $x = (x_k)$ into the sequence Ax whose n-th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$.

In [14] the Knopp core of the sequence x is defined by

$$\mathcal{K}-core\{x\} := \cap_{n\in\mathbb{N}} C_n(x),$$

where $C_n(x)$ is the closed convex hull of $\{x_k\}_{k\geq n}$. In [22] it is shown that for every bounded x

$$\mathcal{K}-core\{x\} := \cap_{z\in\mathbb{C}} B_x^*(z),$$

where $B_x^*(z) := \{w \in \mathbb{C} : |w-z| \leq \limsup_k |x_k - z| \}$.

The next definition is an $\mathcal{I}-$analogue of statistical core [12] of a sequence.

Note that, if x and y are sequences such that $\{k \in N : x_k = y_k\} \notin \mathcal{I}$,then we write "$x_k = y_k$, for $\mathcal{I}-$ *a.a.* k".

Definition 7. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. For any complex sequence x let $H_{\mathcal{I}}(x)$ be the collection of all closed half-planes that contain x_k for $\mathcal{I}-$ a.a. k; i.e.,*

$$H_{\mathcal{I}}(x) := \{H : \text{is a closed half-plane } \{k \in \mathbb{N} : x_k \notin H\} \in \mathcal{I}\},$$

then the $\mathcal{I}-$core of x is given by

$$\mathcal{I}-core\{x\} := \cap_{H\in H_{\mathcal{I}}(x)} H.$$

It is clear that $\mathcal{I}-$core $\{x\} \subseteq \mathcal{K}-$core $\{x\}$ for all x. Also

$$\mathcal{I}-core\{x\} = [\mathcal{I}-\liminf x\ ,\ \mathcal{I}-\limsup x]$$

for any $\mathcal{I}-$bounded real number sequence.

The next theorem is an $\mathcal{I}-$analogue of the Lemma of [12].

Theorem 5. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and assume that x is an $\mathcal{I}-$bounded sequence; for each $z \in \mathbb{C}$ let*

$$B_x(z) := \left\{ w \in \mathbb{C} : |w-z| \leq \mathcal{I}-\limsup_k |x_k - z| \right\};$$

then $\mathcal{I}-core\{x\} := \cap_{z\in\mathbb{C}} B_x(z)$.

Proof. From the definition of $\mathcal{I}-\limsup x$ and *Theorem 1*, observe that the disk $B_x(z)$ is equal to the intersection of all closed disks centered at z that contain x_k

for $\mathcal{I}-$ *a.a.k*. First assume $w \notin \cap_{z\in\mathbb{C}} B_x(z)$, say $w \notin \cap_{z\in\mathbb{C}} B_x(z^*)$ for some z^*. Let H be the half-plane containing $B_x(z^*)$ whose boundary line is perpendicular to the line containing w and z^* and tangent to the circular boundary of $B_x(z^*)$. Since $B_x(z^*) \subset H$ and $B_x(z^*)$ contains x_k for $\mathcal{I}-$ *a.a.k*, it follows that $H \in H_{\mathcal{I}}(x)$. Since $w \notin H$, this implies $w \notin \cap_{H\in H_{\mathcal{I}}(x)} H$. Hence, $\mathcal{I}-$core$\{x\} \subseteq \cap_{z\in\mathbb{C}} B_x(z)$.

Conversely, $w \notin \cap_{H\in H_{\mathcal{I}}(x)} H$, let H be a plane in $H_{\mathcal{I}}(x)$ such that $w \notin H$. Let be the line through w that is perpendicular to the boundary of H and let p be the mid-point of the segment to L between w and H. Let z be a point of L such that $z \in H$ and consider the disk

$$B(z) := \{\xi \in \mathbb{C} : \ |\xi - z| \le |p - z| \ \}\,.$$

Since x is $\mathcal{I}-$bounded and $x_k \in H$ $\mathcal{I}-$ *a.a.k*, we can choose z sufficiently far from p so that $|p-z| = \mathcal{I}-\limsup_k |x_k - z|$. Thus $B(z)$ is one of the $B_x(z)$ disks, and since $w \notin B(z)$, we get that $w \notin \cap_{z\in\mathbb{C}} B_x(z)$. This establishes the proof. □

We note that *Theorem 5* is not necessarily valid if x is not $\mathcal{I}-$bounded. This can be seen from Remark in [12] where $\mathcal{I} = \{K \subseteq \mathbb{N} \ : \ \delta(K) = 0\}$.

Throughout the remainder of this paper the set of bounded complex sequences will be denoted by ℓ^∞.

Now we give necessary conditions on matrix A so that the Knopp core of Ax is contained in the $\mathcal{I}-$core of x for every bounded complex number sequence.

Theorem 6. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. If matrix A satisfies* $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ *and the following conditions*

(i) A regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$ *whenever $E \in \mathcal{I}$;*

(ii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1$,

then $\mathcal{K}-$core $\{Ax\} \subseteq \mathcal{I}-$core $\{x\}$ for every $x \in \ell^\infty$.

Proof. Assume (i) and (ii) and let $w \in \mathcal{K}-$*core* $\{Ax\}$. For any $z \in \mathbb{C}$ we have

$$\begin{aligned} |w-z| &\le \limsup_n |z - (Ax)_n| \\ &= \limsup_n \left| z - \sum_{k=1}^{\infty} a_{nk} x_k \right| \\ &= \limsup_n \left| \sum_{k=1}^{\infty} a_{nk}\ (z - x_k) + z\ (1 - \sum_{k=1}^{\infty} a_{nk}) \right| \\ &\le \limsup_n \left| \sum_{k=1}^{\infty} a_{nk}\ (z - x_k) \right| + \limsup_n |z| \ \left| 1 - \sum_{k=1}^{\infty} a_{nk} \right| \\ &= \limsup_n \left| \sum_{k=1}^{\infty} a_{nk}\ (z - x_k) \right|. \end{aligned} \tag{3}$$

Let $r = \mathcal{I}-\limsup_n |x_n - z|$, suppose $\varepsilon > 0$, and let $E := \{k \ : \ \ |z_k - L| > r + \varepsilon\}$. Then $E \in \mathcal{I}$, and we have

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_{nk}\,(z-x_k)\right| &= \left|\sum_{k\in E} a_{nk}\,(z-x_k)+\sum_{k\notin E} a_{nk}\,(z-x_k)\right|\\ &\le \sum_{k\in E}|a_{nk}|\,|z-x_k|+\sum_{k\notin E}|a_{nk}|\,|z-x_k|\\ &\le \sup_k|z-x_k|\sum_{k\in E}|a_{nk}|+(r+\varepsilon)\sum_{k\notin E}|a_{nk}|\,.\end{aligned}$$

Now (i) and (ii) imply that

$$\limsup_n = \left|\sum_{k=1}^{\infty} a_{nk}\,(z-x_k)\right| \le r+\varepsilon.$$

It follows from (3) that $|w-z| \le r+\varepsilon$; and since ε is arbitrary, this yields $|w-z| \le r$. Hence,$w \notin B_x(z)$ so by the *Theorem 5* we get $w \in \mathcal{I}-core\,\{x\}$. Hence the proof is completed. □

Since $\mathcal{I}-core\,\{x\} \subseteq \mathcal{K}-core\,\{x\}$, we have the following corollary.

Corollary 1. *If matrix A satisfies* $\sup_n \sum_{k=1}^{\infty}|a_{nk}| < \infty$ *and properties* (i) *and* (ii) *of* Theorem 6*, then*

$$\mathcal{I}-core\,\{Ax\} \subseteq \mathcal{I}-core\,\{x\}$$

for every $x \in \ell^\infty$.

Note that the converse of *Corollary 1* does not hold. This can be seen from Example in [12] where $\mathcal{I} = \{K \subseteq \mathbb{N} \,:\, \delta(K) = 0\}$.

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