Some properties of a function studied by De Rham, Carlitz and Dijkstra and its relation to the (Eisenstein–)Stern's diatomic sequence

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Abstract. *We present a novel approach to a remarkable function $D:\mathbb{N}_0\to\mathbb{N}_0$ defined by $D(0)=0, D(1)=1, D(2n)=D(n), D(2n+1)=D(n)+D(n+1)$, studied independently by well known researchers in different areas of mathematics and computer science. Besides some known properties we add some new ones (including a relation to the (Eisenstein–)Stern's diatomic sequence). Some historical remarks are added at the end of this paper.*

Key words: *recurrences, reduced fractions, continuants, (hyper) binary representation, Stern's diatomic sequence, 2-adic order, Stern-Brocot tree, Jacobsthal's numbers*

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1. Introduction

Several properties of the function $D:\mathbb{N}_0\to\mathbb{N}_0$ defined by

$$D(0)=0, D(1)=1, D(2n)=D(n), D(2n+1)=D(n)+D(n+1)$$

are proved together with brief historical remarks about its occurrence.

Following the definition and some obvious consequences we present several properties concerning the evaluation of $D(n)$ in terms of the binary representation of n. One of the remarkable properties is the following (ignoring possible leading zeroes):

$$D((b_r b_{r-1}\cdots b_2 b_1 b_0)_2)=D((b_0 b_1 b_2\ldots b_{r-1} b_r)_2)\,.$$

Some left to right maxima properties of the sequence $(D(n))_{n\in\mathbb{N}_0}$ are given ending with its asymptotic upper bound.

An explicit formula for $D(n)$ in terms of continuants is given which enables elegant proofs of many properties. By means of a three term recursion (which

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seems to be new) we give a new proof of another remarkable property: for every pair of relatively prime natural numbers a and b there exists a unique natural number n such that $a = D(n)$ and $b = D(n+1)$. This implies that the associated sequence of successive quotients $(D(n)/D(n+1))_{n\in\mathbb{N}}$ enumerates all nonnegative reduced fractions, each fraction occurring only once.

Next we treat some other properties of the sequence, including the recursions and properties of $\diamondsuit(m) = \diamondsuit_m := \sum_{k=1}^{m-1} \frac{1}{D(k)D(k+1)}$ and $\mathcal{D}(n) = \mathcal{D}_n := \sum_{0\le k<n} D(k)$.

We also single out the subsequence $(D(2n+1))_{n\in\mathbb{N}_0}$ of the sequence $(D(n))_{n\in\mathbb{N}_0}$ which plays the crucial role (because D does not depend on the even part of its argument). Some of its properties are given in *Section 2.* ('*The sequence* $(D(2n+1))_{n\in\mathbb{N}_0}$').

Section 3. presents a historical overview of the sequence $(D(n))_{n\in\mathbb{N}_0}$.

1.1. The function $D : \mathbb{N}_0 \to \mathbb{N}_0$

In this section we recall some (and prove some new) properties of a remarkable function $D : \mathbb{N}_0 \to \mathbb{N}_0$ defined recursively by (see [D1] and [D2]):

Definition 1.

$$\begin{aligned} &D(0) = 0, D(1) = 1,\\ &D(2n) = D(n),\\ &D(2n+1) = D(n) + D(n+1). \end{aligned} \tag{1}$$

Lemma 1. $\forall m, k \in \mathbb{N}_0 \quad D(2^k \cdot m) = D(m)$.

Proof. Immediate from *Definition 1.* □

Since each number can be written uniquely as $n = 2^\varepsilon(2b+1)$ we can combine the two lines of the definition of D into one line:

$$D(0) = 0, D(1) = 1, D(2^\varepsilon(2b+1)) = D(b) + D(b+1),$$

which clearly shows that D depends only on the odd part of its argument.

The function D is well defined, and to find its value for some given natural number, say 21, one can proceed as follows:

$$\begin{aligned} D(21) &= D(10) + D(11) = D(5) + (D(5) + D(6)) =\\ &= 2D(5) + D(3) = 2(D(2) + D(3)) + D(3) =\\ &= 2D(1) + 3D(3) = 2 + 3(D(1) + D(2)) =\\ &= 2 + 3\cdot 2 = 8 \end{aligned}$$

Remark 1. *Observe that for all* $m \in \mathbb{N}$ *one has* $D(m) > 0$ *and* $D(2m) = D(m) < D(m)+D(m+1) = D(2m+1)$ *and* $D(2m+2) = D(m+1) < D(m)+D(m+1) = D(2m+1)$*.This means that D is an 'up-down' sequence (i.e. it is 'seesawlike'), i.e.* $D(2) < D(3) > D(4) < \cdots > D(2k) < D(2k+1) > D(2k+2) < \cdots$.

Remark 2. *The defining property* $D(2m) = D(m)$ *implies that if we skip every odd indexed member of the sequence* $(D(i))_{i\in\mathbb{N}}$*, then we will end up with the original sequence. Equivalently, if we pick up every second (hence every fourth, every eighth, every* 2^k*–th) member, we will again obtain the original sequence. This means that the sequence has some kind of self-similarity ('fractal–like') property. This also implies that the odd indexed members of the sequence can be used to define the* *whole sequence. Some properties of that 'core' subsequence are given in* Section 2. *('The sequence $(D(2n+1))_{n\in\mathbb{N}_0}$').*

Proposition 1. $3|m \Leftrightarrow 2|D(m), \ \ \forall m \in \mathbb{N}$.

Proof. By induction on $k := \lfloor \frac{m+3}{6} \rfloor$. The induction base $k = 0$ follows directly $(1 = D(1) = D(2))$. Suppose true up to $k-1$. Then for k the following six cases can occur:

Case $m = 6k$**:** $D(m) = D(6k) = D(3k) =$ even,

Cases $m = 6k \pm 1$**:** $D(m) = D(6k \pm 1) = D(3k) + D(3k \pm 1) =$ even + odd = odd,

Cases $m = 6k \pm 2$**:** $D(m) = D(6k \pm 2) = D(3k \pm 1) =$ odd,

Case $m = 6k + 3$**:** $D(m) = D(6k+3) = D(3k+1) + D(3k+2) =$ odd + odd = even.

(The property in Proposition 1 is stated in [D2].) □

Lemma 2. *For all $a, m \in \mathbb{N}_0$ we have*

$$D(2^a m + 1) = aD(m) + D(m+1).$$

Proof. By induction on a and m arbitrary. Case $m = 0$ is obvious. The induction base $a = 0$ and $m > 0$ is obvious too. Suppose true for $a - 1$. Then

$$\begin{aligned} D(2^a m + 1) = D(2 \cdot 2^{a-1} m + 1) &= D(2^{a-1} m) + D(2^{a-1} m + 1) \\ &= D(m) + ((a-1)D(m) + D(m+1)) \\ &= aD(m) + D(m+1) \end{aligned}$$

finishes the induction step. □

Proposition 2. *For all $a, b, m \in \mathbb{N}_0$, $b \le a$, we have*

$$D(2^a m + 2^b - 1) = (1 + b(a-b))D(m) + bD(m+1).$$

Proof. By induction on a we prove that

$$D(2^a m + 2^b - 1) = bD(2^{a-b} m + 1) + D(m), \text{ for } b \in [0..a].$$

Induction base $a = 0$ is clearly valid. Suppose true for $a - 1$. Then, if $b > 0$ (case $b = 0$ coincides with Lemma 1)

$$\begin{aligned} D(2^a m + 2^b - 1) &= D(2(2^{a-1} m + 2^{b-1} - 1) + 1) \\ &= D(2^{a-1} m + 2^{b-1} - 1) + D(2^{a-1} m + 2^{b-1}) \\ &= ((b-1)D(2^{a-b} m + 1) + D(m)) + D(2^{a-1-(b-1)} m + 1) \\ &= bD(2^{a-b} m + 1) + D(m) \\ &= b((a-b)D(m) + D(m+1)) + D(m) \text{ (by } \textit{Lemma 2}) \\ &= (1 + b(a-b))D(m) + bD(m+1). \end{aligned}$$ □

Corollary 1. *For each $k \in \mathbb{N}_0$ we have*

$$D(2^k - 1) = k, \ D(2^k) = 1, \ D(2^k + 1) = k + 1.$$

It turns out that the sequence $(D(n))_{n\in\mathbb{N}}$ is an universal sequence in the sense that it contains every sequence of positive integers as its subsequence. To prove this assertion we need to show that for given $a \in \mathbb{N}$ and arbitrary k there exist $m > k$ such that $D(m) = a$. By *Corollary 1* for any given a there exists j such that $D(j) = a$. For any k there exists n such that $m = 2^n j > k$ and by *Lemma 1* $D(m) = D(j) = a$.

From now on, we shall also consider sequences of ones and zeroes (or 0–1 sequences), interpreting them as binary representations of numbers. In that way, instead of writing $45 = 2^5 + 2^3 + 2^2 + 1 = (101101)_2$, we shall write $45 = \mathbf{101101}$. Moreover, to simplify the notation we abbreviate $\underbrace{\mathbf{00\ldots0}}_{k}$ and $\underbrace{\mathbf{11\ldots1}}_{k}$ by $\mathbf{0}^k$ and $\mathbf{1}^k$ respectively. E.g. we abbreviate $\mathbf{110001111}$ by $\mathbf{1}^2\mathbf{0}^3\mathbf{1}^4$.

We shall freely mix both notations on purpose, using a more appropriate one when needed (slightly abusing notation when other possibilities lead to a more cumbersome writing). For example, instead of writing $288 = 2^8 + 2^5 = (2^8 + 2^4 + 2^3 + 2^1 + 1) + 5$, we may write $288 = \mathbf{100100000} = \mathbf{10}^2\mathbf{10}^5 = \mathbf{100011011} + \mathbf{101} = \mathbf{100011011} + 5 = \mathbf{10}^3\mathbf{1}^2\mathbf{0}\mathbf{1}^2 + 5 = 283 + \mathbf{101}$, also $2^k = (\underbrace{11\ldots1}_{k})_2 + 1 = \mathbf{1}^k + 1$.

An empty sequence will be denoted by ε. Also, $\mathbf{1}^0 = \mathbf{0}^0 = \varepsilon$.

If $\mathbf{m}$ is a finite 0–1 sequence, then $|\mathbf{m}|$ will denote its length. By definition $|\varepsilon| = 0$.

Note that if $\mathbf{m}$ is a binary representation of the number n, then $\mathbf{m0}$ represents $2n$ and $\mathbf{m1}$ represents $2n+1$. Sometimes, a sequence representing number $m \in \mathbb{N}$ will be denoted by $\mathbf{m}$ and vice versa (so expressions $\mathbf{m1} = 2m+1$ or $\mathbf{m0}^3\mathbf{10}^4 = 2^8m+2^4$ are meaningful).

Leading zeroes can be safely ignored or added at the beginning of a sequence if necessary, since the number represented remains the same, (i.e. $5 = \mathbf{101} = \mathbf{0101} = \mathbf{00101} = \ldots$).

Now *Definition 1* can be rephrased as follows:

Definition 2. *The function D can be defined on the set of all finite 0–1 sequences by*

$$\begin{aligned} &D(\mathbf{1}) = 1, D(\mathbf{0}) = 0,\\ &D(\mathbf{m0}) = D(\mathbf{0m}) = D(\mathbf{m}),\\ &D(\mathbf{m1}) = D(\mathbf{m}) + D(\mathbf{m}{+}\mathbf{1}). \end{aligned} \tag{2}$$

Due to $D(0) = 0$, it may be added that $D(\varepsilon) = 0$.

Now *Corollary 1* can be restated as follows

Corollary 2. *For each $k \in \mathbb{N}_0$ we have*

$$D(\mathbf{1}^k) = k,\ D(\mathbf{10}^k) = 1,\ D(\mathbf{10}^{k-1}\mathbf{1}) = k + 1\ (k > 0).$$

Proposition 2 can now be rewritten as

Proposition 3. *For every finite 0–1 sequence $\mathbf{m}$ and for all $s, r \in \mathbb{N}_0$ we have*

$$D(\mathbf{m0}^r\mathbf{1}^s) = (1 + rs)D(\mathbf{m}) + sD(\mathbf{m} + \mathbf{1}) = (1 + rs - s)D(\mathbf{m}) + sD(\mathbf{m1}). \tag{3}$$

Proof. By *Proposition 2* we firstly get $D(\mathbf{m0}^r\mathbf{1}^s) = (1 + rs)D(\mathbf{m}) + sD(\mathbf{m} + \mathbf{1})$ Then by *Definition 2* $D(\mathbf{m1}) = D(\mathbf{m}) + D(\mathbf{m} + \mathbf{1})$, and the proof follows. □

Let us now restate a recursion for $D(n)$ involving the leading block of 1's in the binary representation of n.

Lemma 3. *For every finite* 0–1 *sequence* $\mathbf{m}$ *and for every* $k \in \mathbb{N}$ *we have*

$$D(\mathbf{1}^k\mathbf{0m}) = D(\mathbf{1}^{k-1}\mathbf{0m}) + D(\mathbf{1m}), \tag{4}$$

or in standard notation

$$D((2^k - 1) \cdot 2^r + m) = D((2^{k-1} - 1) \cdot 2^r + m) + D(2^{r-1} + m) \quad (m < 2^{r-1}).$$

Proof. By induction on the length $|\mathbf{m}|$ of the sequence $\mathbf{m}$ (k arbitrary).
Induction base: $\mathbf{m}=\varepsilon$, i.e. $|\mathbf{m}| = 0$

$$\begin{aligned} &D(\mathbf{1}^k\mathbf{0m}) = \\ &\quad = D(\mathbf{1}^k\mathbf{0}\varepsilon) = D(\mathbf{1}^k\mathbf{0}) = D(\mathbf{1}^k) \text{ (by Definition 2)} \\ &\quad = k \text{ (by Corollary 2)} \\ &\quad = D(\mathbf{1}^{k-1}\mathbf{0}) + D(\mathbf{1}) = D(\mathbf{1}^{k-1}\mathbf{0}\varepsilon) + D(\mathbf{1}\varepsilon) = D(\mathbf{1}^{k-1}\mathbf{0m}) + D(\mathbf{1m}). \end{aligned}$$

Induction step: Suppose that $D(\mathbf{1}^k\mathbf{0m}) = D(\mathbf{1}^{k-1}\mathbf{0m}) + D(\mathbf{1m})$ is valid for every $k \in \mathbb{N}$ and every sequence $\mathbf{m}$ of length $< d$. Let $\mathbf{n}$ be a 0–1 sequence of length d. The case $\mathbf{n} = \mathbf{m0}$ is trivial.
In case $\mathbf{n} = \mathbf{m1}$ we have $|m| = d - 1$. Then by *Definition 2* we have

$$\begin{aligned} D(\mathbf{1}^k\mathbf{0m1}) &= D(\mathbf{1}^k\mathbf{0m}) + D(\mathbf{1}^k\mathbf{0m} + 1) \\ &= D(\mathbf{1}^k\mathbf{0m}) + \begin{cases} D(\mathbf{1}^k\mathbf{0m}'), \text{ if } \mathbf{m}' = \mathbf{m} + \mathbf{1} < 2^{d-1} \\ D(\mathbf{1}^k\mathbf{10}^s)\text{ , if } \mathbf{m} + \mathbf{1} = 2^{d-1} \end{cases} \end{aligned} \tag{5}$$

Subcase ($\mathbf{m}' = \mathbf{m} + \mathbf{1} < 2^{d-1}$): Here we have

$$\begin{aligned} &D(\mathbf{1}^k\mathbf{0n}) = D(\mathbf{1}^k\mathbf{0m1}) = D(\mathbf{1}^k\mathbf{0m}) + D(\mathbf{1}^k\mathbf{0m}') \\ &= D(\mathbf{1}^{k-1}\mathbf{0m}) + D(\mathbf{1m}) + D(\mathbf{1}^{k-1}\mathbf{0m}') + D(\mathbf{1m}') \text{ (by induction assumption)} \\ &= (D(\mathbf{1}^{k-1}\mathbf{0m}) + D(\mathbf{1}^{k-1}\mathbf{0m}')) + (D(\mathbf{1m}) + D(\mathbf{1m}')) \\ &= D(\mathbf{1}^{k-1}\mathbf{0m1}) + D(\mathbf{1m1}) \text{ (by Definition 2)} \\ &= D(\mathbf{1}^{k-1}\mathbf{0n}) + D(\mathbf{1n}). \end{aligned}$$

Subcase ($\mathbf{m} + \mathbf{1} = 2^{d-1}$): Here $\mathbf{m} = \mathbf{1}^{d-1}$ and the l.h.s. of (5) equals

$$\begin{aligned} D(\mathbf{1}^k\mathbf{0m1}) &= D(\mathbf{1}^k\mathbf{0}^1\mathbf{1}^d) \\ &= (1 + 1 \cdot d)D(\mathbf{1}^k) + dD(\mathbf{10}^k) \text{ (by Proposition 3)} \\ &= (d + 1)D(\mathbf{1}^k) + dD(\mathbf{1}) \\ &= (d + 1)k + d \cdot 1 = k + d + kd \text{ (by Corollary 2).} \end{aligned}$$

The r.h.s. of (5) for this subcase equals:

$$\begin{aligned} &D(\mathbf{1}^k\mathbf{01}^{d-1}) + D(\mathbf{1}^k\mathbf{10}^{d-1}) = \\ &\quad = (1 + 1 \cdot (d - 1))D(\mathbf{1}^k) + (d - 1)D(\mathbf{1}^k + 1) + D(\mathbf{1}^{k+1}) \text{ (by Proposition 3)} \\ &\quad = dk + (d - 1) \cdot 1 + (k + 1) = k + d + kd \text{ (by Corollary 2).} \end{aligned}$$

□

Lemma 4. *For every finite* 0–1 *sequence* $\mathbf{m}$ *and for every* $k \in \mathbb{N}_0$ *we have*

$$D(\mathbf{1}^k\mathbf{0m}) = D(\mathbf{m}) + kD(\mathbf{1m}).$$

Proof. By induction on k using *Lemma 3.* $\square$

Lemma 5. *Let $l \in \mathbb{N}_0$. Then for any* 0–1 *sequence* $\mathbf{n}$ *we have*

$$D(\mathbf{10}^l\mathbf{n}) = lD(\mathbf{n}) + D(\mathbf{1n}).$$

Proof. Case $l = 0$ is clear. Suppose $l > 0$: $D(\mathbf{10}^l\mathbf{n}) = D(\mathbf{100}^{l-1}\mathbf{n}) = D(\mathbf{0}^{l-1}\mathbf{n}) + D(\mathbf{10}^{l-1}\mathbf{n})$ (by *Lemma 3*) $= D(\mathbf{n}) + D(\mathbf{10}^{l-1}\mathbf{n}) = D(\mathbf{n}) + (l-1)D(\mathbf{n}) + D(\mathbf{1n})$ (by the induction hypothesis) $= lD(\mathbf{n}) + D(\mathbf{1n})$. $\square$

Proposition 4. *For $r, s \in \mathbb{N}$ we have*

$$D(\mathbf{1}^r\mathbf{0}^s\mathbf{m}) = (rs - r + 1)D(\mathbf{m}) + rD(\mathbf{1m}).$$

Proof. $D(\mathbf{1}^r\mathbf{0}^s\mathbf{m}) = D(\mathbf{m}) + rD(\mathbf{10}^{s-1}\mathbf{m})$ (by *Lemma 4*) $= D(\mathbf{m}) + r((s-1)D(\mathbf{m}) + D(\mathbf{1m}))$ (by *Lemma 5*) $= (rs - r + 1)D(\mathbf{m}) + rD(\mathbf{1m})$. $\square$

Let $\overleftarrow{\mathbf{m}}$ denotes a sequence obtained by reading the sequence $\mathbf{m}$ from right to left, i.e. $\overleftarrow{\mathbf{b}_k\mathbf{b}_{k-1}\cdots\mathbf{b}_1\mathbf{b}_0} = \mathbf{b}_0\mathbf{b}_1\cdots\mathbf{b}_{k-1}\mathbf{b}_k$, where $\mathbf{b}_i \in \{\mathbf{0}, \mathbf{1}\}$, $i = 1, 2, \ldots, k$; e.g. $\overleftarrow{\mathbf{11101}} = \mathbf{10111}$.

Proposition 5. *For all finite* 0–1 *sequences* $\mathbf{m}$ *we have the following mirror property:*

$$D(\mathbf{m}) = D(\overleftarrow{\mathbf{m}}).$$

Proof. By induction on the length of $\mathbf{m}$. Induction base is obvious and the induction step follows from *Proposition 4* and *Proposition 3.* $\square$

Now we turn our attention to finding maxima of the function D restricted to numbers with a given number of bits.

Lemma 6. *For $n \in \mathbb{N}$ let α_n be the following numbers:*

$$\alpha_n = 2^{n-1} - 2^{n-2} + \cdots + (-1)^{n-2}2^1 + (-1)^{n-1} \quad (= (2^n - (-1)^n)/3)\,,$$

and let $\alpha_n^{\pm} = \alpha_n \pm (-1)^n$ be the neighbours of α_n. Then

1. $\alpha_n = 2\alpha_{n-1} + (-1)^{n-1} = \alpha_{n-1}^{+} + \alpha_{n-1}$,
2. $\alpha_n^{+} = 2\alpha_{n-1}$,
3. $\alpha_n^{-} = 4\alpha_{n-2}$.

(The sequence $(\alpha_n)_{n\in\mathbb{N}}$ is called the Jacobsthal sequence (cf. [Sl], A001045).)

Proof. The first two identities are obvious. For the third we use the recursion 1. twice:
$\alpha_n^{-} = \alpha_n - (-1)^n = 2\alpha_{n-1} + (-1)^{n-1} - (-1)^n = 2(2\alpha_{n-2} + (-1)^{n-2}) + (-1)^{n-1} - (-1)^n = 4\alpha_{n-2} + 2(-1)^{n-2} + 2(-1)^{n-1} = 4\alpha_{n-2}$. $\square$

Lemma 7. *For every $n \in \mathbb{N}$ we have*

$$D(\alpha_n^{-}) = F_{n-2}, \quad D(\alpha_n) = F_n, \quad D(\alpha_n^{+}) = F_{n-1},$$

where F_n is the n–th Fibonacci number ($F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n > 1$).

Proof. By induction on n we prove that for every $n \in \mathbb{N}$ we have $D(\alpha_n) = F_n$. Induction base:

$D(\alpha_1) = D(1) = 1 = F_1$ and $D(\alpha_2) = D(1) = 1 = F_2$.
Induction step: Suppose true for $n-2$ and $n-1$.

$$\begin{aligned} D(\alpha_n) &= D(\alpha_{n-1}^+ + \alpha_{n-1}) = D(\alpha_{n-1}^+) + D(\alpha_{n-1}) \text{ (by \textit{Definition 1})} \\ &= D(2\alpha_{n-2}) + D(\alpha_{n-1}) = D(\alpha_{n-2}) + D(\alpha_{n-1}) = F_{n-2} + F_{n-1} = F_n \end{aligned}$$

The rest follows from *Lemma 6*:
$D(\alpha_n^+) = D\left(2\alpha_{n-1}\right) = D\left(\alpha_{n-1}\right) = F_{n-1}$,
$D(\alpha_n^-) = D(4\alpha_{n-2}) = D(\alpha_{n-2}) = F_{n-2}$. □

Lemma 8. *For all $k \in [2^{n-1}..2^n]$ we have*

$$D(k) \leq F_{n+1}.$$

Proof. By induction on n.
Induction base: Trivial by direct check ($n \in \{0, 1\}$).
Induction step: Suppose true up to $n-1$. Let $k \in [2^n..2^{n+1}]$. Two cases can occur:

Case $k = 2j$:
$D(k) = D(2j) = D(j) \leq F_{n+1}$ by induction assumption, so $D(k) \leq F_{n+2}$.

Case $k = 4j + 1$:
$D(k) = D(2j) + D(2j+1) = D(j) + D(2j+1) \leq F_{n-1} + F_n = F_{n+1}$.

Case $k = 4j + 3$:
$D(k) = D(2j+1) + D(2j+2) = D(2j+1) + D(j+1) \leq F_n + F_{n-1} = F_{n+1}$. □

Theorem 1. *For every $n \in \mathbb{N}$*

$$\max\{D(k) | k \in [0..2^n]\} = \max\{D(k) | k \in [2^{n-1}..2^n]\} = F_{n+1}.$$

Proof. By *Lemma 8* F_{n+1} is an upper bound. But by *Lemma 7* this upper bound is achieved at $\alpha_{n+1} \in [2^{n-1}..2^n]$. □

Corollary 3. *$D(n) = O(n^{\lg \phi})$, where ϕ is the golden ratio, i.e. $\phi = \frac{1+\sqrt{5}}{2}$.* Since $\lg \phi \approx 0.694$, it follows $D(n) = O\left(n^{0.7}\right)$.

Next we shall explain several explicit formulas for $D(n)$. According to *Exercise 6.50* in [GKP], p. 314 and its solution on p. 553, $D(n)$ can be written explicitly, in terms of the binary representation of $n = (1^{a_1}0^{a_2} \ldots 1^{a_{m-1}}0^{a_m})_2$, as follows (cf. [R]):

$$D(n) = K_{m-1}(a_1, a_2 \ldots, a_{m-1}), \tag{6}$$

where $K_n(x_1, x_2, \ldots, x_n)$ is the continuant polynomial, or simply continuant, defined by the following recurrence (see [GKP], (6.136) and (6.131)):

$$\begin{aligned} &K_0() = 1, \\ &K_1(x_1) = x_1, \\ &K_m(x_1, \ldots, x_m) = K_{m-1}(x_1, \ldots, x_{m-1})x_m + K_{m-2}(x_1, \ldots, x_{m-2}). \end{aligned} \tag{7}$$

We have $D(13) = K_3(2, 1, 1)$ $(= 5)$ because $13 = (1^2 0^1 1^1 0^0)_2$, and $D(0) = 0 = K(0)$ since $0 = (1^0 0^1)_2$.

In [SU] yet another formula for $D(n)$ was obtained: if $n-1 = (1^{a'_1}0^{a'_2}\dots 1^{a'_{m'-1}}0^{a'_{m'}})_2$ then

$$D(n) = K_{m'-1}(a'_1, a'_2 \dots, a'_{m'})$$

(e.g. $D(13) = K_2(2,2)$, since $13-1 = (1^2 0^2)_2$).

One basic property of continuants is their mirror symmetry property ([GKP], (6.131), or [BSQ], p. 100):

$$K_n(a_1, a_2, \dots, a_{n-1}, a_n) = K_n(a_n, a_{n-1}, \dots, a_2, a_1). \tag{8}$$

We shall also need the following identity: $K(1, a, b, c, \dots, z) = K(a+1, b, c, \dots, z)$ which easily follows from the recurrence given in (7).

For an amazing property of the function D (posed as a problem by Dijkstra, see [D1], [D3], [ZES]), we give below a short solution (shorter than the solution given by *Proposition 5*) in the following

Corollary 4. *Let $(b_r b_{r-1} \cdots b_2 b_1 b_0)_2$ be a binary representation of the natural number n. Then for the number $\overleftarrow{n} := \overleftarrow{(b_r b_{r-1} \cdots b_2 b_1 b_0)}_2 = (b_0 b_1 b_2 \dots b_{r-1} b_r)_2$ obtained from n by reversing its binary representation (and ignoring possible leading zeroes) we have*

$$D(n) = D(\overleftarrow{n}).$$

Proof. Let $n = (1^{a_1}0^{a_2}1^{a_3}0^{a_4} \cdots 1^{a_{m-1}}0^{a_m})_2$. If we reverse the binary representation of n (and ignore all leading zeroes), we get a binary representation $(1^{a_{m-1}}0^{a_{m-2}}1^{a_{m-3}} \cdots 0^{a_2}1^{a_1}0^0)_2$ of the number $\overleftarrow{n}$. Then by (6) and mirror symmetry (8) we get $D(\overleftarrow{n}) = K(a_{m-1}, a_{m-2}, \dots, a_2, a_1) = K(a_1, a_2, \dots, a_{m-2}, a_{m-1}) = D(n)$ (see [D1], [D3], [ZES]). $\square$

Remark 3. *From* Definition 1 *of D it is evident that $D(n+1) = b(n)$, $n \in \mathbb{N}$, where $b : \mathbb{N} \to \mathbb{N}$ is the hyperbinary partition function defined in [CW]. So one can also write a formula for $b(n)$.*

Corollary 5. *For all $n \in \mathbb{N}$ and for all $r \in [0..2^n]$ we have $D(2^n + r) = D(2^{n+1} - r)$.*

Proof. For $r = 2^n$ there is nothing to prove. So suppose that $r < 2^n$. Then $2^n + r = (10^{a_2}1^{a_3}0^{a_4} \dots 1^{a_{m-1}}0^{a_m})_2$ and $2^{n+1} - r = (1^{a_2+1}0^{a_3}1^{a_4} \dots 0^{a_{m-1}-1}10^{a_m})_2$ (or $2^{n+1} - r = (1^{a_2+1}0^{a_3}1^{a_4} \dots 0^{a_{m-2}+1}10^{a_m})_2$ in case $a_{m-1} = 1$). By (6) and basic properties of continuants we have $D(2^n + r) = K(1, a_2, a_3, a_4, \dots, a_{m-1}) = K(a_2 + 1, a_3, a_4, \dots, a_{m-1} - 1, 1)$ (or $= K(a_2 + 1, a_3, a_4, \dots, a_{m-2} + 1)$ in case $a_{m-1} = 1$) $= D(2^{n+1} - r)$. $\square$

Remark 4. *Yet another (in fact more direct) proof of* Corollary 5 *can be given as follows:*

> *Thanks to the property $D(2m) = D(m)$ it is sufficient to consider the case r odd, i.e. $r = 2s+1$. Now, the left-hand side is, by* Definition 1 *of D, equal to $D(2^{n-1} + s) + D(2^{n-1} + s + 1)$ and similarly the right-hand side equals $D(2^n - s) + D(2^n - s - 1)$. Thus the proof immediately follows by induction.* $\square$

Remark 5. Corollary 5 *implies that, for each k, the sequence $D(2^k)$, $D(2^k+1)$, $D(2^k+2)$, …, $D(2^k+2^k-1)$, $D(2^{k+1})$ is palindromic.*

Lemma 9. *The sum of all members of the sequence* $D(2^k)$, $D(2^k+1)$, $D(2^k+2)$, $\ldots$, $D(2^k+2^k-1)$, $D(2^{k+1})$ *is equal to* 3^k+1, *i.e.* $\sum_{r=2^k}^{2^{k+1}} D(r) = 3^k+1$.

Proof. By induction on k.

Induction base: Trivial, since for $k=0$, we get $\sum_{r=2^0}^{2^{0+1}} D(r) = D(1)+D(2) = 2 = 3^0+1$

Induction step: We have:

$$\begin{aligned}\sum_{r=2^{k+1}}^{2^{k+2}} D(r) &= \sum_{r=2^{k+1},r \text{ odd}}^{2^{k+2}} D(r) + \sum_{r=2^{k+1},r \text{ even}}^{2^{k+2}} D(r) = \\ &= \left(\sum_{r=2^k}^{2^{k+1}-1} D(r) + \sum_{r=2^k+1}^{2^{k+1}} D(r)\right) + \sum_{r=2^k}^{2^{k+1}} D(r) = \\ &= (3^k+3^k)+3^k+1 = \\ &= 3^{k+1}+1 \qquad \square\end{aligned}$$

Corollary 6. *If* $\mathbf{x}+\mathbf{y}=\mathbf{1}^k=2^k-1$, *then* $D(\mathbf{1x1})=D(\mathbf{1y1})$, *i.e. if all inner digits of two odd numbers* m *and* n *are complementary, then* $D(m)=D(n)$. *(See [D1].)*

Proof. If $\mathbf{x}+\mathbf{y}=\mathbf{1}^k=2^k-1$, then $\mathbf{1y1}=2^{k+1}+2y+1=2^{k+1}+2^{k+1}-2x-1$ and $D(\mathbf{1y1})=D(2^{k+2}-2x-1)=$ (by *Corollary 5*) $=D(2^{k+1}+2x+1)=D(\mathbf{1x1})$. $\square$

Now we derive a three term recursion (which seems to be new) for $D(n)$. First we recall the 2–adic order of an integer n, denoted by $\varepsilon_2(n)$, as the greatest power of 2 that divides n, i.e. $n=2^{\varepsilon_2(n)}(2b+1)$.

Proposition 6. *Let* $\chi(n)=2\varepsilon_2(n)+1$. *Then*

$$D(n+1)+D(n-1)=\chi(n)D(n). \tag{9}$$

In particular, for n *odd we have* $D(n+1)+D(n-1)=D(n)$, *and for each* n

$$D(n) \mid D(n-1)+D(n+1).$$

(See [Sl], A028415.)

Proof. Let $D(n+1)+D(n-1)=\chi(n)D(n)$. Clearly $\chi(1)=1$. For n odd, $n=2m+1$, we have

$$\chi(2m+1)D(2m+1)=D(2m+2)+D(2m)=D(m+1)+D(m)=D(2m+1).$$

Thus

$$\chi(2m+1)=1. \tag{10}$$

From

$$\begin{aligned}\chi(2n)D(2n) &= D(2n+1)+D(2n-1) = (\text{by } (1)) = \\ &= (D(n+1)+D(n))+(D(n)+D(n-1)) = \\ &= 2D(n)+\chi(n)D(n)=(2+\chi(n))D(n)\end{aligned}$$

we get a recursion $\chi(2n)=\chi(n)+2$, which for $n=2^{\varepsilon_2(n)}(2m+1)$ gives

$$\chi(n)=2\varepsilon_2(n)+\chi(2m+1)=(\text{by } (10))=2\varepsilon_2(n)+1. \qquad \square$$

Remark 6. *In the reference [St] Stern has derived a formula almost identical to the formula (9). He (and Eisenstein, according to [Sl], sequences A064881–A064885) defined arrays of numbers starting with an array consisting of two non-negative integers m and n. Each array is generated from the previous one by inserting between every pair of adjacent numbers their sum. The second array is thus m, $m+n$, n, the third one is m, $2m+n$, $m+n$, $m+2n$, n etc. It is now easy to check that for $m=n=1$ its p^{th} row coincides with the sequence $D(2^p), D(2^p+1), D(2^p+2), \dots, D(2^{p+1})$.*

(The same rows of number occur, somewhat unexpectedly, in connection with a family of tangent circles (Ford's circles) described in [WB].)

Property 4 ([St], p. 198) states that if a, b, c are three consecutive numbers occurring in the p^{th} row, then $a+c=(2k+1)b$, where $k=(a+c-b)/2b \in \mathbb{N}_0$. The number b occurs in the $(p-k)^{th}$ row and it is equal to a sum of two adjacent numbers from the $(p-k-1)^{th}$ row.

Remark 7. *In the reference [R] De Rham starts with two linearly independent vectors $\overrightarrow{i}$, $\overrightarrow{j}$ in a plane. By the geometrical construction he obtained vectors $\overrightarrow{i}$, $\overrightarrow{i}+\overrightarrow{j}$, $\overrightarrow{j}$. In the next step he got vectors $\overrightarrow{i}$, $2\overrightarrow{i}+\overrightarrow{j}$, $\overrightarrow{i}+\overrightarrow{j}$, $\overrightarrow{i}+2\overrightarrow{j}$, $\overrightarrow{j}$. Each sequence of vectors is obtained from the previous one by inserting their sum (see* Remark 6*) between every pair of adjacent vectors, thus generating the Stern's diatomic sequence.*

Lemma 10. *For each $n \in \mathbb{N}_0$ we have $\gcd(D(m), D(m+1)) = 1$.*

Proof. Suppose that a number d divides both $D(m)$ and $D(m+1)$. *Proposition 6* implies that d also divides $D(m+2), D(m+3), \dots$, i.e. for every $k > m$ $d | D(k)$ so d divides $D(2^m) = 1$, because $2^m > m$. Therefore $d = 1$ and for every $m \in \mathbb{N}$ the numbers $D(m)$ and $D(m+1)$ are relatively prime. □

Remark 8. *Now we prove one other property discovered by Dijkstra: If $m+n=2^k$ then $D(m)$ and $D(n)$ are relatively prime (cf. [D1]). If $m=n$, then $m=n=2^{k-1}$ and $D(m)=D(n)=1$. Suppose $m<n$ (this implies $m<2^{k-1}$). Now we have $D(n)=D(2^k-m)=$ (by* Corollary 5*) $=D(2^{k-1}+m)=D(\mathbf{10}^s\mathbf{m})=$ (for some $s \geq 0$) $=D(\overleftarrow{\mathbf{10}^s\mathbf{m}})=$ (by* Corollary 4*) $=D(\overleftarrow{\mathbf{m}}\mathbf{0}^s\mathbf{1})=$ (by* Lemma 3*) $=(1+s)D(m)+D(m+1)$. So, if d divides $D(m)$ and $D(n)$, then it also divides $D(m+1)$. Now* Lemma 10 *implies that $D(m)$ and $D(n)$ are relatively prime.*

Remark 9. Corollary 4, Definition 1 *and* Lemma 1, *imply:*

$$\begin{aligned} a_k^{\mathbf{m}} = D(\mathbf{m1}^k) &= D(\mathbf{m1}^{k-1}+1) + D(\mathbf{m1}^{k-1}) = \\ &= D(\mathbf{m}+1) + D(\mathbf{m1}^{k-1}) = \cdots = \\ &= kD(\mathbf{m}+1) + D(\mathbf{m}). \end{aligned}$$

and

$$\begin{aligned} b_k^{\mathbf{m}} = D(\mathbf{10}^k\mathbf{m}) &= D(\overleftarrow{\mathbf{m}}\mathbf{0}^k\mathbf{1}) = D(\overleftarrow{\mathbf{m}}\mathbf{0}^{k-1}) + D(\overleftarrow{\mathbf{m}}\mathbf{0}^{k-1}\mathbf{1}) = \\ &= D(\overleftarrow{\mathbf{m}}) + D(\overleftarrow{\mathbf{m}}\mathbf{0}^{k-1}\mathbf{1}) = \cdots = \\ &= kD(\mathbf{m}) + D(\overleftarrow{\mathbf{m}}\mathbf{1}) = kD(\mathbf{m}) + D(\mathbf{1m}) \end{aligned}$$

Thus $(a_k^{\mathbf{m}})_{k\in\mathbb{N}}$ and $(b_k^{\mathbf{m}})_{k\in\mathbb{N}}$ are both linear progressions. Since $D(\mathbf{1m}) = D(\overleftarrow{\mathbf{m}}\mathbf{1}) = D(\overleftarrow{\mathbf{m}}) + D(\overleftarrow{\mathbf{m}}+1)$, by Lemma 10 *it follows that $D(\mathbf{m})$ $(= D(\overleftarrow{\mathbf{m}}))$ and $D(\mathbf{1m})$ are relatively prime. So, we have linear progressions, each with the first element and difference relatively prime. A famous theorem of Dirichlet (1837) says that there are infinitely many primes in every such linear progression. Where are those* *sequences? In the* j^{th} *row and the* k^{th} *column in a table below we put the number* $D(2^j + k)$*. Note that the* j^{th} *row corresponds to the same row (and the values of a function* s_j*) of a table in [Li]. Each column (except, of course, the* 0^{th}*) corresponds to one of the sequences* $(b_k^{\mathbf{m}})_{k\in\mathbb{N}}$*.*

$D(2^j+k)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 . . .
0	1	1															
1	1	2	1														
2	1	3	2	3	1												
3	1	4	3	5	2	5	3	4	1								
4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
5	1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2. . .
6	1	7	6	11	5	14	9	13	4	15	11	18	7	17	10	13	3. . .
7	1	8	7	13	6	17	11	16	5	19	14	23	9	22	13	17	4. . .
8	1	9	8	15	7	20	13	19	6	23	17	28	11	27	16	21	5. . .
9	1	10	9	17	8	23	15	22	7	27	20	33	13	32	19	25	6. . .
10	1	11	10	19	9	26	17	25	8	31	23	38	15	37	22	29	7. . .
11	1	12	11	21	10	29	19	28	9	35	26	43	17	42	25	33	8. . .
12	1	13	12	23	11	32	21	31	10	39	29	48	19	47	28	37	9. . .
13	1	14	13	25	12	35	23	34	11	43	32	53	21	52	31	41	10. . .
14	1	15	14	27	13	38	25	37	12	47	35	58	23	57	34	45	11. . .
15	1	16	15	29	14	41	27	40	13	51	38	63	25	62	37	49	12. . .
16	1	17	16	31	15	44	29	43	14	55	41	68	27	67	40	53	13. . .
17	1	18	17	33	16	47	31	46	15	59	44	73	29	72	43	57	14. . .
18	1	19	18	35	17	50	33	49	16	63	47	78	31	77	46	61	15. . .
19	1	20	19	37	18	53	35	52	17	67	50	83	33	82	49	65	16. . .
20	1	21	20	39	19	56	37	55	18	71	53	88	35	87	52	69	17. . .
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Sequences $(a_k^{\mathbf{m}})_{k\in\mathbb{N}}$ *can't be seen so easily in the above table, but they actually coincide with the sequences* $(b_k^{\mathbf{m}})_{k\in\mathbb{N}}$ *by* Corollary 5*. Let* $\mathbf{m}$ *be a finite* 0–1 *sequence beginning with* $\mathbf{1}$ *and let* $\mathbf{n}$ *be the shortest sequence such that for some* $r \in \mathbb{N}$ $\mathbf{m}+\mathbf{n} = \mathbf{1}^r$*, i.e.* $\mathbf{m}+\mathbf{n} = 2^r - 1$*. We have:* $D(\mathbf{1}^k\mathbf{m}) = D(\mathbf{1}^k\mathbf{m}+\mathbf{n}-\mathbf{n}) = D(\mathbf{1}^{k+r}-\mathbf{n}) = D(2^{k+r}-\mathbf{n}) = D(2^{k+r-1}+\mathbf{n})$*, if* $\mathbf{n} < 2^{k+r-1}$*, which is true, because* $\mathbf{n}$ *is shorter than* $\mathbf{m}$*.*

Lemma 11. $D(m) = 1 \Leftrightarrow \exists k \in \mathbb{N}_0 \quad m = 2^k$*.*

Proof. It follows from *Corollary 1* and the fact that for $m > 1$ $D(m) \geq 1$ and $D(2m+1) = D(m) + D(m+1) \geq 2$. □

Theorem 2. *If* $\frac{D(m)}{D(m+1)} = \frac{D(n)}{D(n+1)}$*, then* $m = n$*.*

Proof. Obviously, if $D(m)$ or $D(n)$ equals 0, then $m = n = 0$. Let m and n, $m < n$, be the smallest pair (using, say, a lexicographical order on $\mathbb{N}\times\mathbb{N}$) such that $\frac{D(m)}{D(m+1)} = \frac{D(n)}{D(n+1)}$. Then $D(n+1) = \frac{D(n)D(m+1)}{D(m)}$. It follows that $D(m)|D(n)$ because $\gcd(D(m), D(m+1)) = 1$ by *Lemma 10*. Similarly we get $D(n)|D(m)$ so $D(m) = D(n)$ and $D(m+1) = D(n+1)$. Numbers m and n cannot be of different parity as due to *Remark 1* for $k, l \geq 0$ is $\frac{D(2k)}{D(2k+1)} < 1 \leq \frac{D(2l+1)}{D(2l+2)}$.

Case $m = 2k$**,** $n = 2l$**:** $D(m) = D(k)$, $D(m+1) = D(k) + D(k+1)$, $D(n) = D(l)$, $D(n+1) = D(l) + D(l+1)$.

Case $m = 2k+1$**,** $n = 2l+1$**:** $D(m) = D(k) + D(k+1)$, $D(m+1) = D(2k+2) = D(k+1)$, $D(n) = D(l) + D(l+1)$, $D(n+1) = D(2l+2) = D(l+1)$.

In both cases we arrive at a pair of numbers k and l, $k < l$, such that $D(k) = D(l)$ and $D(k+1) = D(l+1)$, where $k < m$ and $l < n$. Therefore, we got a pair smaller than the smallest pair with such property – a contradiction. The conclusion is that $m = n$ and the proof is finished. □

Theorem 3. *For any two numbers* $a, b \in \mathbb{N}$ *such that* $\gcd(a, b) = 1$ *there exists an odd number* $i \in \mathbb{N}$ *such that* $D(i-1) = a$*,* $D(i) = a + b$ *and* $D(i+1) = b$*.*

Proof. By induction on $m = a + b$.
Induction base: Trivial.
Induction step: Suppose that the statement is true for every $m < M$. Let $a < b$ be such that $\gcd(a, b) = 1$ and $a + b = M$. Then for $a_1 = a$ and $b_1 = b - a$ we have: $a_1 + b_1 = b < M$, $\gcd(a_1, b_1) = \gcd(a, b-a) = 1$ and by induction assumption there exists an odd $i \in \mathbb{N}$ such that $D(i-1) = a_1 = a$, $D(i) = a_1 + b_1 = b$ and $D(i+1) = b_1 = b - a$. Now we have: $D(2i-2) = D(i-1) = a$, $D(2i-1) = D(i-1) + D(i) = a + b$ and $D(2i) = D(i) = b$, and the proof in case $a < b$ is done.

Note that, by *Corollary 5*, there exists an odd $j \in \mathbb{N}$ so that $D(j-1) = b$, $D(j) = a + b$ and $D(j+1) = a$, what finishes the proof in case $a > b$. □

Corollary 7. *The sequence* $\left(\frac{D(n)}{D(n+1)}\right)_{n\in\mathbb{N}_0}$ *is a sequence of all nonnegative reduced fractions with each fraction occurring only once.*

Proof. Directly from *Theorem 2* and *Theorem 3*. □

Now we prove one more recursion for D, different from the one in *Proposition 6*.

Proposition 7. *For each* $n \in \mathbb{N}$ *we have*

$$D(n) = D\left(\frac{n - 2^{\varepsilon_2(n)}}{2^{1+\varepsilon_2(n)}}\right) + D\left(\frac{n + 2^{\varepsilon_2(n)}}{2^{1+\varepsilon_2(n)}}\right).$$

Proof. For $n = 2^k(2r+1) = 2^{k+1}r + 2^k$ we have $\varepsilon_2(n) = k$, so

$$D(n) = D(2r+1) = D(r) + D(r+1).$$

Note that $r = (n - 2^k)/2^{k+1}$, and $r + 1 = (n + 2^k)/2^{k+1}$, with $k = \varepsilon_2(n)$ implies the proof. □

Definition 3. *Let* $\diamondsuit(m) := \sum_{k=1}^{m-1} \frac{1}{D(k)D(k+1)} = \sum\nolimits_1^m \frac{1}{D(k)D(k+1)}\ \delta k$ *(written as a definite sum, [GKP], 2.48). We abbreviate* $\diamondsuit(m)$ *by* $\diamondsuit_m$.

Proposition 8. *For* $m \in \mathbb{N}$ *we have*

$$\Gamma_m = \diamondsuit_{2m} - \diamondsuit_m = \sum_{m \le k < 2m} \frac{1}{D(k)D(k+1)} = 1.$$

Proof. By induction on m we prove that the above sum reduces to a shorter sum of the same form. Induction base is obviously valid for $m = 1$. Suppose that the statement is true up to $m - 1$. We have:

$$\begin{aligned}
\Gamma_m &= \sum_{m \le k < 2m} \frac{1}{D(k)D(k+1)} \\
&= \sum_{m \le 2k < 2m} \frac{1}{D(2k)D(2k+1)} + \sum_{m \le 2k+1 < 2m} \frac{1}{D(2k+1)D(2k+2)} \\
&= \sum_{\lceil \frac{m}{2} \rceil \le k < m} \frac{1}{D(2k)D(2k+1)} + \sum_{\lceil \frac{m-1}{2} \rceil \le k < m} \frac{1}{D(2k+1)D(2k+2)} \\
&= \sum_{\lceil \frac{m}{2} \rceil \le k < m} \frac{1}{D(2k)D(2k+1)} + \sum_{\lceil \frac{m}{2} \rceil \le k < m} \frac{1}{D(2k+1)D(2k+2)} + \frac{[m \text{ odd}]}{D(m)D(m+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lceil \frac{m}{2} \rceil \le k < m} \frac{D(2k+2)+D(2k)}{D(2k)D(2k+1)D(2k+2)} + \frac{[m \text{ odd}]}{D(m)D(m+1)} \\
&= \sum_{\lceil \frac{m}{2} \rceil \le k < m} \frac{1}{D(k)D(k+1)} + \frac{[m \text{ odd}]}{D(m)D(m+1)} \quad \text{(by } \textit{Definition 1}\text{)} \\
&= \sum_{\lceil \frac{m}{2} \rceil \le k < 2\lceil \frac{m}{2} \rceil} \frac{1}{D(k)D(k+1)} = \Gamma_{\lceil \frac{m}{2} \rceil} = 1. \qquad \square
\end{aligned}
$$

Corollary 8. *The function $\diamondsuit$ satisfies the following properties:*

1. $\diamondsuit_{2m} = \diamondsuit_m + [m > 0]$,
2. $\diamondsuit_{2m+1} = \diamondsuit_m + 1 + \frac{[m>0]}{D(2m)D(2m+1)}$,
3. $\diamondsuit_m = \diamondsuit_{\lfloor \frac{m}{2} \rfloor} + 1 + \frac{[m \text{ odd}]}{D(m-1)D(m)}$,
4. $\diamondsuit_{2^\varepsilon(2b+1)} = \diamondsuit_b + \varepsilon + 1 + \frac{[b>0]}{D(2b)D(2b+1)}$.

Remark 10. *As a special case* ($m = 2^n$) *of* Proposition 8 *we obtain another proof that n^{th}–level sum of simplicities of fractions in the tree of fractions (see [CW]) and therefore in Stern–Brocot tree equals 1 (cf. Bogomolny, Lamothe [B]).* We conclude this section by stating some properties of the function $\mathcal{D}$ (a primitive or an antidifference of D) defined by

Definition 4. *For $n \in \mathbb{N}$ we set* $\mathcal{D}(n) = \mathcal{D}_n := \sum_{0 \le k < n} D(k) = \sum_0^n D(k)\delta k$.

Proposition 9. *Function $\mathcal{D}$ satisfies the following recurrence relations:*

1. $\mathcal{D}_1 = 0$, $\mathcal{D}_2 = 1$,

2. $\mathcal{D}_n = \mathcal{D}_{\lceil \frac{n+1}{2} \rceil} + \mathcal{D}_{\lceil \frac{n}{2} \rceil} + \mathcal{D}_{\lceil \frac{n-1}{2} \rceil}$ ($\Leftrightarrow \mathcal{D}_{2n+1} = \mathcal{D}_n + 2\mathcal{D}_{n+1}$, $\mathcal{D}_{2n} = 2\mathcal{D}_n + \mathcal{D}_{n+1}$).

Proof. It follows from the definition of the function D.

$$
\begin{aligned}
\mathcal{D}_n &= \sum_{0 \le k < n} D_k = \sum_{0 \le 2k < n} D_{2k} + \sum_{0 \le 2k+1 < n} D_{2k+1} \\
&= \sum_{0 \le k < \lceil \frac{n}{2} \rceil} D_k + \sum_{0 \le k < \lceil \frac{n-1}{2} \rceil} (D_k + D(k+1)) \\
&= \mathcal{D}_{\lceil \frac{n+1}{2} \rceil} + \mathcal{D}_{\lceil \frac{n}{2} \rceil} + \mathcal{D}_{\lceil \frac{n-1}{2} \rceil}.
\end{aligned}
$$

Other relations follow directly from the one we just proved. $\square$

Corollary 9. *We have* $\mathcal{D}_{2n} + \mathcal{D}_{2n+1} = 3(\mathcal{D}_n + \mathcal{D}_{n+1})$. *In particular, for the level sums of D we have (by an easy induction)*

$$\mathcal{D}_{2^{n+1}} - \mathcal{D}_{2^n} = (2\mathcal{D}_{2^n} + \mathcal{D}_{2^n+1}) - \mathcal{D}_{2^n} = \mathcal{D}_{2^n} + \mathcal{D}_{2^n+1} = 3(\mathcal{D}_{2^{n-1}} + \mathcal{D}_{2^{n-1}+1}) = 3^n.$$

2. The sequence $(D(2n+1))_{n\in\mathbb{N}_0}$

Special attention should be given to a subsequence of $D(n)$'s with n odd which explicates all so-called Lehmer's dyads (see [Le]), i.e. Stern's *Summengliedern* (see [St]).

Definition 5. *Let* $E(n) := D(2n+1)$ *for* $n \in \mathbb{N}_0$.

The sequence $(E(n))$ appears in [H] and in [Sh] but without reference to D. In [H] it appears under the name *Farey's sequence*. In [Sh] it appears as the direct limit of numerators of n^{th}–level fractions of the Stern–Brocot tree.

Many properties of $E(n)$ are easier to prove using the properties of its parent sequence $D(n)$.

The sequence $(E(n))_{n\in\mathbb{N}_0}$ is interesting since the sequence $(D(n))_{n\in\mathbb{N}_0}$ can be defined by $D(0)=0, D(2m)=D(m), D(2m+1)=E(m)$.

All results concerning the sequence $(E(n))_{n\in\mathbb{N}_0}$ in this section are proved in [SU].

By (6) we can write an explicit formula for $E(n)$. If the binary representation of n is $(1^{a_1}0^{a_2}\dots 1^{a_{m-1}}0^{a_m})_2$ then $2n+1=(1^{a_1}0^{a_2}\dots 1^{a_{m-1}}0^{a_m}1^1 0^0)_2$ and by (6) we obtain

$$E(n) = D(2n+1) = K(a_1,a_2,\dots,a_{m-1},a_m,1) = K(a_1,a_2,\dots,a_{m-1},a_m+1).$$

The numerator of the n–th fraction of the Stern–Brocot tree (in the usual "book–like" reading of the nodes in the tree) equals $D(J(n)) = E(n-2^{\lfloor \log_2 n\rfloor})$, where the function J is the Josephus' function defined in [GKP], 1.10.

The sequence of numerators of the n–th level of the Stern–Brocot tree is an initial segment of length 2^n of the sequence $(E(n))_{n\in\mathbb{N}}$ (see the sequence A007305 in [Sl]). A level in the tree consists of all nodes that are at the same distance (depth) from the root downwards. Each level is considered to be ordered from left to right.

For every $n\in\mathbb{N}$ and every $r\in[0..2^n]$ we have

$$E(2^n+r) = E(2^{n+1}-r-1),$$

or, in the binary notation, $E(1b)=E(1\overline{b})$, where the sequence of binary digits $\overline{b}$ is obtained from b by complementing each digit $(0 \leftrightarrow 1)$.

For every $n\in\mathbb{N}$ and $r\in[1..2^{n-1})$ we have (cf. Maple code SternBrocotTreeNum given on the web page of a sequence A007305 in [Sl])

$$E(2^n+r) = E(2^n-r-1)+E(r) = E(2^{n-1}+r)+E(r).$$

It turns out that the recurrence $E(0)=1$, $E(2^n+r)=E(2^{n-1}+r)+E(r)$ is equivalent to a recurrence defining the Farey sequence in [H].

Function E satisfies the following relations $(E(0)=1)$:

$$\begin{aligned} E(m) &= E\left(\left\lfloor \tfrac{m}{2}\right\rfloor\right) + D\left(\left\lceil \tfrac{m}{2}\right\rceil\right)\\ &= E\left(\left\lfloor \tfrac{m}{2}\right\rfloor\right) + E\left(\frac{\beta\left(\left\lceil \frac{m}{2}\right\rceil\right)-1}{2}\right)\\ &= (1+\varepsilon_2(m))D(\beta(m)) + D(\beta(m)+1). \end{aligned}$$

where $\beta:\mathbb{N}\to\mathbb{N}$ is a function defined by $\beta(2m)=\beta(m)$ and $\beta(2b+1)=2b+1$, i.e. $\beta(m)=m/2^{\varepsilon_2(m)}$ or $\beta(2^{\varepsilon}(2b+1))=2b+1$, where $\varepsilon_2(n)$ is the greatest power of 2 that divides n.

The function E satisfies the following properties (which were originally derived from the generating function of E):

$$\sum_{j=1}^{2m-1}(-1)^{j+1}E(j) = E(m) \quad \text{and} \quad \sum_{j=1}^{2m}(-1)^{j}E(j) = D(m). \tag{11}$$

By setting $m = 2^k$ in the second of (11) we get

$$\sum_{j=1}^{2^{k+1}}(-1)^{j}E(j) = D(2^k) = 1 \text{ and } \sum_{j=2^k+1}^{2^{k+1}}(-1)^{j}E(j) = 0.$$

Finally we state and prove some additional properties of the sequence E:
Proposition 10. *For all $m \in \mathbb{N}$ we have*

1. $E(2m+1) + E(2m) = 3E(m)$, $(m \in \mathbb{N}_0)$
2. $E(2m) + E(2m-2) = 2E(m-1) + E(m)$,
3. $E(2m) - E(2m-1) = E(m) - E(m-1)$.

Proof.

1. $E(2m+1) + E(2m) = D(4m+3) + D(4m+1) = \chi(4m+2)D(4m+2) = \chi(2(2m+1))D(2m+1) = 3E(m)$,
2. $E(2m) + E(2m-2) = D(4m+1) + D(4m-3) = D(2m) + D(2m+1) + D(2m-2) + D(2m-1) = D(2m-1) + \chi(2m-1)D(2m-1) + D(2m+1) = 2D(2m-1) + D(2m+1) = 2E(m-1) + E(m)$,
3. $E(2m) - E(2m-1) = D(4m+1) - D(4m-1) = D(2m) + D(2m+1) - D(2m-1) - D(2m) = E(m) - E(m-1)$. □

In a similar fashion one can easily prove (for $n \in \mathbb{N}_0$), using recursions for D again, the following two identities:

$$E(4n) = 2E(2n) - E(n), \quad E(4n+2) = 4E(n) - E(2n),$$

which, together with the first property in *Proposition 10*, show that the sequence E is a 2–regular sequence (cf. Shallit [Sh]).

3. Historical remarks

Eisenstein started it all. On page 356 of his work "Über ein einfaches Mittel zur Auffindung der höheren Reciprocitätsgesetze und der mit ihnen zu verbindenden Ergänzungssätze" ([E1]), he defined auxiliary function $\chi_{\mu,\nu}$ whose properties he needed while working on laws of reciprocity. According to [Sl] (sequences A064881–A064885, submitted by Wolfdieter Lang, see [E2]), Eisenstein studied number sequences which are known as Stern's diatomic series today.

His proofs of those properties were complicated and he (according to Stern) asked his friend M. A. Stern to help him find simpler proofs. Stern apparently did not have enough time to work on it since he finished his work [St] 1855, three years after Eisenstein's death. (See *Remark 6*) Stern proved most properties which lead to enumeration of rational numbers, but he did not state it explicitly, since he was primarily interested in Eisenstein's auxiliary function.

D. H. Lehmer studied Stern's diatomic sequence (see [Le]) and had found many of its properties.

Function D, which enumerates numerators (and denominators, too!) of Calkin–Wilf tree, has many faces indeed, although its twin brother, $b(m) = D(m+1)$ is encountered more often.

L. Carlitz "discovered" both functions and named them $\theta_0 = b$ and $\theta_1 = D$. He defined them as follows: If $c_{n,r} = S(n+1, r+1)$ (Stirling number of the second kind), then for fixed n, $\theta_0(n)$ denotes the number of odd $c_{n,2r}$, $2r < n$, and $\theta_1(n)$ denotes the number of odd $c_{n,2r+1}$, $2r+1 \le n$ (see [C1] and [C2], for generalization see [C3]). Carlitz also states that $\theta_0(n) =$ number of odd binomials $\begin{pmatrix} n-k \\ k \end{pmatrix}$, $0 \le 2k \le n$. Functions θ_0 and θ_1 satisfy "dual" recurrence relations:

$$\begin{array}{ll} \theta_0(0) = 1, & \theta_1(0) = 0, \\ \theta_0(2m) = \theta_0(m) + \theta_0(m+1), m > 0, & \theta_1(2m) = \theta_1(m), \\ \theta_0(2m+1) = \theta_0(m), & \theta_1(2m+1) = \theta_1(m) + \theta_1(m+1). \end{array}$$

Generating function $G_b(x)$ of the sequence $(b(k))_{k\in\mathbb{N}}$ was found by Carlitz (see [C1]):

$$G_b(x) = \prod_{k=1}^{\infty}(1 + x^{2^{k-1}} + x^{2^k}) = (1 + x + x^2)G_b(x^2) = \sum_{r=0}^{\infty} b(r)x^r.$$

The generating function $\mathbf{D}$ of sequence $(D(k))_{k\in\mathbb{N}}$ is then, of course, $\mathbf{D}(x) = xG_b(x)$.

D. A. Lind gathered (see [Li]) all known properties of Stern's diatomic sequence, added some new ones and corrected some mistakes.

Dijkstra "rediscovered" function D in [D3] where he gave *Corollary 4* as a problem. Several solutions were given in [ZES]. Dijkstra named this function $fusc$ in [D1] and [D2] where he gave an instructive derivation of the iterative algorithm for calculating $D(m)$.

Algorithm 1. *Dijkstra derived the following elegant iterative algorithm for computing $D(n)$ in [D2]:*

```
n := N; a := 1; b := 0;
WHILE n>0 DO
    IF ODD(n) THEN b := a+b ELSE a := a+b; ENDIF
    n := ⌊n/2⌋;
ENDWHILE
```

The final value of variable `b` *equals $A(N)$.* He also proved some of its remarkable properties by investigating the behavior of algorithm during its execution.

Calkin and Wilf in [CW] mentioned that $\theta_0(n) =$ number of *hyperbinary* representations of the integer n, i.e. the number of ways of writing the integer n as a sum of powers of 2, each power being used at most twice.

Sloane's excellent On–Line Encyclopedia of Integer Sequences ([Sl]) is a good starting point for searching information about these two sequences (actually, for any imaginable sequence), although sequence $(D(k))_{k\in\mathbb{N}}$ was declared dead in favour of the sequence $(b(k))_{k\in\mathbb{N}}$. The label of sequence $(D(k))_{k\in\mathbb{N}}$ was A028415 and the label of sequence $(b(k))_{k\in\mathbb{N}}$ was A002487. It can be found that for the function χ defined in *Proposition 6*, there holds $\chi(n) = 2\left\lfloor \frac{D(n)}{D(n+1)} \right\rfloor + 1$ (by David Newman, see *Remark 6*).

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