Hyperspaces which are products or cones

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Abstract. *Let $C(X)$ be the hyperspace of all subcontinua of a metric continuum X. Alejandro Illanes has proved that $C(X)$ is a finite-dimensional Cartesian product if and only if X is an arc or a circle. In this paper we shall prove, using the inverse systems and limits, that if X is a non-metric rim-metrizable continuum and $C(X)$ is a finite-dimensional Cartesian product, then X is a generalized arc or a generalized circle.*

It is also proved that if X is a non-metric continuum such that $dim(X) < \infty$ and such that X has the cone = hyperspace property, then X is a generalized arc, a generalized circle, or an indecomposable continuum such that each nondegenerate proper subcontinuum of X is a generalized arc.

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1. Introduction

All spaces in this paper are compact Hausdorff spaces and all mappings are continuous mappings. The symbol "$\approx$" means "is homeomorphic to".The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by card(A). We shall use the notion of an inverse system as in [1, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$. We say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a well-ordered inverse system if A is a well-ordered set. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system; an element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [1, p. 135].

In the sequel we shall use the following results.

Lemma 1. *[1, Corollary 2.5.7]. Any closed subspace Y of the limit X of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is the limit of the inverse system $\mathbf{X}_Y = \{Cl(p_a(Y)), p_{ab}|Cl(p_b(Y)), A\}$.*

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Lemma 2. *[1, Corollary 2.5.11]. Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be an inverse system and* B *a subset cofinal in* A. *The mapping consisting in restriction all threads from* $X = \lim \mathbf{X}$ *to* B *is a homeomorphism of* X *onto the space* $\lim\{X_b,\ p_{bc},\ B\}$.

A *generalized arc* is a Hausdorff continuum with exactly two non - separating points a and b. The points a and b are *end-points.* A generalized arc with end-points a and b will be denoted by $[a, b]$. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

A *generalized closed curve* J is the union of two generalized arcs L_1 and L_2 with end-points a and b such that $L_1 \cap L_2 = \{a, b\}$.

Lemma 3. *[2, Theorem 5]. Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-directed inverse system of (generalized) arcs. Then* $X = \lim \mathbf{X}$ *is a generalized arc.*

Lemma 4. *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-directed inverse system of (generalized) circles and monotone bonding mappings. Then* $X = \lim \mathbf{X}$ *is a generalized circle.*

Proof. Let x, y, z be distinct points of X. There exists an $a \in A$ such that $p_b(x)$, $p_b(y)$, $p_b(z)$ are distinct points of X_b for every $b \geq a$. For every $b \in A$, let $L_b \subset X_b$ be the arc with end-points $p_b(x)$ and $p_b(y)$ which contain $p_b(z)$. Let u_a be a point of $X_a \setminus L_a$. There exists a point $u \in X$ such that $p_a(u) = u_a$. For every $b \geq a$ let M_b be the arc with end-points $p_b(x)$ and $p_b(y)$ which contain $p_b(u)$. If $c \geq b$, then $p_{bc}^{-1}(L_b)$ is a continuum (since p_{bc} is monotone) containing $p_c(x)$, $p_c(y)$ and $p_c(z)$. This means that $L_c \subset p_{bc}^{-1}(L_b)$. Hence $p_{bc}(L_c) \subset L_b$. Similarly, we have $p_{bc}(M_c) \subset M_b$. It follows that $\{L_b,\ p_{bc}|L_c\ ,\ a \leq b \leq c\}$ and $\{M_b,\ p_{bc}|M_c\ ,\ A\}$ are inverse systems of arcs and monotone bonding mappings whose limits L and M are generalized arcs (*Lemma 3*). It is clear that $L \cup M = X$ and $L \cap M = \{x, y\}$. □

Let X be a compact space. By 2^X we denote the set of all nonempty closed subsets of X, by $C(X)$ the set of all nonempty closed connected subsets of X and by $X(n)$, where n is a positive integer, the set of all nonempty subsets consisting of at most n points [5]. We consider $C(X)$ and $X(n)$ as a subset of 2^X. The topology on 2^X is the Vietoris topology and $C(X)$, $X(n)$ are subspaces of 2^X.

Let X and Y be compact spaces and let $f : X \rightarrow Y$ be a continuous map. Define $2^f : 2^X \rightarrow 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [9, 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be an inverse system of compact spaces with natural projections $p_a : \lim X \rightarrow X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(X) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n),\ 2^{p_{ab}}|\ X_b(n),\ A\}$ are inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$, $p_a(F) \subseteq X_a$ is closed and compact. Therefore, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \rightarrow 2^{X_a}$ induced by p_a, for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \rightarrow lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Note that $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$. Mapping M is continuous and 1-1. It is also an onto mapping since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Therefore, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \rightarrow 2^{X_a}, a \in A$, are the corresponding projections, then $P_a M = 2^{p_a}$. Identifying F with $M(F)$, we have $P_a = 2^{p_a}$.

Lemma 5. *[5, Lemma 2.]. Let* $X = \lim \boldsymbol{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\boldsymbol{X})$ *and* $X(n) = \lim \boldsymbol{X}(n)$.

2. Inverse σ-systems and hyper-onto representations of continua

This section contains some special features of inverse systems which are needed in the next sections.

2.1. σ-complete inverse systems

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, \dots, a_k, \dots$ of members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Theorem 1. *[6, Theorem 1.1] Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f{:}X{\rightarrow}Y$ there exists an $a{\in}A$ such that for each $b{\geq}a$ there is a mapping $g_b : X_b \rightarrow Y$ such that $f = g_b p_b$.*

Let τ be an infinite cardinal number. We say that a directed set A is *τ-complete* if for each transfinite sequence $a_1 \leq a_2 \leq ... \leq a_\alpha \leq, ..., \alpha < \tau, a_\alpha \in A$, there exists $\sup a_\alpha \in A$.

We say that a well-ordered inverse system $\{X_a, p_{ab}, A\}$ is *continuous* if for each limit ordinal γ, $0 < \gamma < w(X)$, the maps $p_{\alpha\gamma} : X_\gamma \rightarrow X_\alpha$ induce a homeomorphism of the spaces X_γ and $\lim\{X_\alpha, p_{\alpha\beta}, \alpha \leq \beta < \gamma\}$. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *continuous* if for each chain $B \subset A$ with $supB = \gamma$ the maps $p_{\alpha\gamma} : X_\gamma \rightarrow X_\alpha$ induce a homeomorphism of the spaces X_γ and $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\{X_a, p_{ab}, A\}$ is said to be an inverse *τ-complete* system if $\{X_a, p_{ab}, A\}$ is continuous and A is τ-complete. An inverse system is said to be an *inverse τ-system* if it is τ-complete and $w(X_a) \ \leq \tau, a \in A$ [14, p. 9]. A directed set A is *σ-complete* if A is $\aleph_0$-complete. An inverse system is said to be an *inverse σ-system* if it is σ-complete and $w(X_a) \ \leq \aleph_0, a \in A$.

Theorem 2. *For each Tychonoff cube I^m, $m \geq \aleph_1$, there exists an inverse σ-system $\mathbf{I} = \{I^a, P_{ab}, A\}$ of Hilbert cubes I^a such that I^m is homeomorphic to* $\lim \mathbf{I}$.

Proof. a) Let us recall that the *Tychonoff cube* I^m is the Cartesian product $\prod\{I_s : s \in S\}$, $\text{card}(S) = m$, $I_s = [0, 1]$ [1, p. 114]. If $\text{card}(S) = \aleph_0$, the Tychonoff cube I^m is called the *Hilbert cube*. Let A be the set of all countable subsets of S ordered by inclusion. If $a \subseteq b$, then we write $a \leq b$. It is clear that A is σ-directed. For each $a \in A$ there exists a Hilbert cube I^a. If $a, b \in A$ and $a \leq b$, then there exists the projection $P_{ab} : I^b \rightarrow I^a$. Finally, we have the system $\mathbf{I} = \{I^a, P_{ab}, A\}$.

b) Let us prove that $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system. It is clear that A is σ-directed. Moreover, A is σ-complete. Namely, if $a_1 \leq a_2 \leq ... \leq a_n, ...$ is a countable chain in A, then we have a countable chain $a_1 \subseteq a_2 \subseteq ... \subseteq a_n, ...$ of countable subsets of S. It is clear that $a = \bigcup\{a_n : n \in \mathbb{N}\}$ is a countable subset of S and $a = \sup a_n$. It remains to prove that $\mathbf{I} = \{I^a, P_{ab}, A\}$ is continuous. Let $B = a_1 \leq a_2 \leq ... \leq a_\alpha, ..., \alpha < \tau, a_\alpha \in A$, be a chain with $\sup a_\alpha = \gamma \in A$. We have a transfinite inverse sequence $\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Let us prove that a mappings $P_{a_\alpha\gamma}, \alpha < \tau$ induce a homeomorphism of the spaces I^γ and $\lim\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Let $x \in I^\gamma$. It is clear that $P_{a_\alpha\gamma}(x) = x_{a_\alpha}$ is a point of I^{a_α} and that $P_{a_\alpha a_\beta}(x_{a_\beta}) = x_{a_\alpha}$ if $a_\alpha \leq a_\beta$. This means that (x_{a_α}) is a thread in $\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. Set $H(x) =$

(x_{a_α}). We have the mapping $H : I^\gamma \to \lim\{I^{a_\alpha}, P_{a_\alpha a_\beta}, B\}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. Finally, $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system since $w(I^a) \leq \aleph_0$.

c) Let us prove that I^m is homeomorphic to $\lim \mathbf{I}$. Let $x \in I^m$. It is clear that $P_{am}(x) = x_a$ is a point of I^a and that $P_{ab}(x_b) = x_a$ if $a \leq b$. This means that (x_a) is a thread in $\mathbf{I} = \{I^a, P_{ab}, A\}$. Set $H(x) = (x_a)$. We have the mapping $H : I^m \to \lim \mathbf{I}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. □

By a similar method of proof we have the following theorem.

Theorem 3. *For each uncountable Cartesian product* $\prod\{X_a : a \in A\}$ *of continua* X_a *there exists an inverse* σ*-system* $\mathbf{X} = \{X^b, P_{ab}, B\}$ *of countable infinite Cartesian products* X^b *and monotone mappings* P_{ab} *such that* $\prod\{X_a : a \in A\}$ *is homeomorphic to* $\lim \mathbf{X}$.

Theorem 4. *Let* X *be a compact Hausdorff space such that* $w(X) \geq \aleph_1$. *There exists an inverse* σ*-system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of compact metric spaces* X_a *such that* X *is homeomorphic to* $\lim \boldsymbol{X}$.

Proof. By [1, Theorem 2.3.23.] the space X is embeddable in $I^{w(X)}$. From *Theorem 2* it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$, where every I^a is the Hilbert cube. Now, X is a closed subspace of $\lim \mathbf{I}$. Let $X_a = P_m(X)$, where $P_m : I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the Hilbert cube I^a. Let p_{ab} be the restriction of P_{ab} to X_b. We have an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$. By virtue of *Lemma 1* X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is an inverse σ-system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system. □

Lemma 6. *Let* B *be an infinite subset of a directed set* A. *There exists a directed subset* $F_\infty(B)$ *of* A *such that* $B \subseteq F_\infty(B)$ *and* $card(F_\infty(B)) = card(B)$.

Proof. If B is directed, then we let $F_\infty(B) = B$. Suppose that B is not directed. By B_{fin} we shall denote the set all finite subsets of B. Let ν be any finite subset of A. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup\{\delta(\nu) : \nu \in B_{fin}\}$. Put

$$F_{n+1} = F_1(F_n(B), \tag{1}$$

and

$$F_\infty(B) = \bigcup\{F_n(B) : n \in \mathbb{N}\}. \tag{2}$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq ... \subseteq F_n(B) \subseteq ... \tag{3}$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_{n+1}(B) \subset F_\infty(B)$. From $card(B) \geq \aleph_0$, it follows $card(\{\delta(\nu) : \nu \in B\}) \leq card(B)\aleph_0$. We infer that $card(F_1(B)) \leq card(B)\aleph_0$. Similarly, $card(F_n(B)) \leq card(B)\aleph_0$. This means that $card(F_\infty(B)) \leq card(B)\aleph_0$. Thus

$$card(F_\infty(B)) \leq card(B)\aleph_0. \tag{4}$$

We infer that $card(F_\infty(B)) = card(B)$. □

Let $X = \{X_a, p_{ab}, A\}$ be a usual inverse system of compact spaces and let $\tau < card(A)$ be an infinite cardinal. Consider the set A_τ of all $F_\infty(B), B \subseteq A, card(B) =$

τ, ordered by inclusion. It is clear that A_τ is τ-directed. Each element α of A_τ is some $F_\infty(B)$. We define X_α as the limit of $\{X_a, p_{ab}, F_\infty(B)\}$. Let $\alpha = F_\infty(B)$ and $\beta = F_\infty(C)$. If $\alpha \subseteq \beta$, then there exists the natural projection $q_{\alpha\beta} : X_\beta = lim\{X_a, p_{ab}, F_\infty(C)\} \to X_\alpha = lim\{X_a, p_{ab}, F_\infty(B)\}$. It is clear that $q_{\alpha\gamma} = q_{\alpha\beta}q_{\beta\gamma}$ if $\alpha \leq \beta \leq \gamma$. It follows that $\{X_\alpha, q_{\alpha\beta}, A_\tau\}$ is an inverse system.

Theorem 5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with limit X. For each infinite cardinal $\tau < card(A)$ there exists a τ-complete inverse system $\mathbf{X}_\tau = \{X_\alpha, q_{\alpha\beta}, A_\tau\}$ such that X is homeomorphic to* $\lim\{X_\alpha, q_{\alpha\beta}, A_\tau\}$.

Proof. The proof of the fact that X is homeomorphic to $lim\mathbf{X}_\tau$is the same as the proof of Theorem 9.4. of [13]. It remains to prove that $\mathbf{X}_\tau$ is τ-complete. Let C be a chain of A_τ of the cardinality $\leq \tau$. Every $c \in C$ is some $F_\infty(B_c)$. Consider the union $\bigcup\{F_\infty(B_c) : c \in C\}$. It is clear that it is directed and has the cardinality $\leq \tau$. Hence, $\bigcup\{F_\infty(B_c) : c \in C\}$ is a member d of A_τ. Moreover, $\bigcup\{F_\infty(B_c) : c \in C\} \supseteq F_\infty(B_c)$ for each $c \in C$. This means that $d \geq c$ (in the ordering of A_τ) for each $c \in C$. Clearly, if $e \geq c$ for every $c \in C$, then $e \geq d$ since d is defined as the union $\bigcup\{F_\infty(B_c) : c \in C\}$. □

Theorem 6. *Let X be a compact space of finite dimension* $\dim X$*. There exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$ *and* $\dim X_a \leq \dim X$.

Proof. By virtue of [7, Theorem 1.] every compact space X is homeomorphic to the inverse limit of an inverse system of metrizable compacta $\{Q_a, q_{ab}, B\}$ with $\dim Q_a \leq \dim X$ and $card(B) \leq w(X)$. From *Theorem 5* it follows that $\mathbf{Q}_\sigma = \{Q_\Delta, q_{\Delta\Gamma}, A_\sigma\}$ is a σ-system such that $\lim \mathbf{X}$ and $\lim \mathbf{X}_\sigma$ are homeomorphic. Every Q_Δ is metrizable as the inverse limit of an inverse system over a countable directed set. Moreover, by [1, p. 504, Exercise 7.3.I.], $\dim Q_\Delta \leq \dim X$. Denote A_σ by A and Q_Δ by X_a. We obtain the desired inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$ and $\dim X_a \leq \dim X$. □

2.2. Factorizable inverse systems

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *factorizable* [14, p. 24] if for each continuous real-valued function $f : \lim X \to I = [0, 1]$ there exists an $a \in A$ such that for $b \geq a$ there exists a continuous function $f_b : X_b \to I$ such that $f = f_b p_b$.

By virtue of *Theorem 1* we have the following lemma.

Lemma 7. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then $\mathbf{X}$ is factorizable.*

Theorem 7. *[14, Theorem 40.]. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ are factorizable inverse τ-systems of compact spaces with surjective bonding mappings, then for each mapping $f : \lim X \to \lim Y$ there exists a cofinal subset $B(f)$ of A and the mappings $f_b : X_b \to Y_b, b \in B(f)$, such that each diagram*

$$\begin{array}{ccc} X_b & \xleftarrow{p_{bc}} & X_c \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \xleftarrow{q_{bc}} & Y_c \end{array} \qquad (5)$$

commutes and the mapping f is induced by the collections $\{f_b : b \in B(f)\}$*, i.e., each diagram*

$$\begin{array}{ccc} X_b & \overset{p_b}{\longleftarrow} & \lim \mathbf{X} \\ \downarrow f_b & & \downarrow f \\ Y_b & \overset{q_b}{\longleftarrow} & \lim \mathbf{Y} \end{array} \tag{6}$$

commutes. If $f : \lim X \to \lim Y$ *is a homeomorphism, then each* f_b *is a homeomorphism.*

Proof. For the sake of the completeness we give the proof. Let us prove that there exists a cofinal subset $B(f)$ of A such that every diagram (5) commutes. Let $a \in A$ be any member of A. Set $a_0 = a$. Suppose that $a_i \in A$ is defined for each $i \in N, i < k$. We define a_k as follows. Consider the mapping $fq_{a_{k-1}} : \lim X \to Y_{a_{k-1}}$, where $q_{a_{k-1}} : \lim Y \to Y_{a_{k-1}}$ is a natural projection. By *Theorem 1* and *Lemma 7* there exists $a_k \in A$, $a_k \geq a_{k-1}$, and a mapping $f_{a_{k-1}b} : X_b \to Y_{a_{k-1}}$ such that every diagram commutes for each $b \geq a_k$. Hence, a_k is defined for every $k \in \mathbb{N}$. We obtain an increasing sequence $E = \{a_0, a_1, ..., a_k, ...\}$. There exists $b = \sup a_k \in A$ since A is complete. By the definition of a_k there exists a mapping $f_{a_k b} : X_b \to Y_{a_k}$ for every $k \in \mathbb{N}$. The collection $\{f_{a_k b} : k \in N\}$ induces the mapping $f_b : X_b \to \lim\{Y_{a_k}, q_{a_k a_l}, E\}$. From the continuity of $\mathbf{X}$ it follows that Y_b is homeomorphic to $\lim\{Y_{a_k},\ q_{a_k a_l},\ E\}$. This means that $f_b : X_b \to Y_b$. It is clear that $b \geq a$. Hence, the subset $B(f)$ of A is cofinal in A and the mappings $f_b : X_b \to Y_b$, $b \in B(f)$, such that each diagram (5) commutes, induce the mapping f.

If f is a homeomorphism h, then there exists the set $B(h)$ for the mapping f and the set $B(h^{-1})$ for f^{-1}. Let $B(h) = B(h) \cap B(h^{-1})$. From the commutative diagram

$$\begin{array}{ccc} X_b & \overset{p_b}{\longleftarrow} & \lim \mathbf{X} \\ g_b \uparrow\downarrow f_b & & h^{-1} \uparrow\downarrow h \\ Y_b & \overset{q_b}{\longleftarrow} & \lim \mathbf{Y} \end{array} \tag{7}$$

it follows that $g_b f_b$ and $f_b g_b$ are the identity. Hence, f_b is a homeomorphism. Let us observe that by *Lemma 2* $\lim \mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is homeomorphic to $\lim \mathbf{X} = \{X_a,\ p_{ab},\ B(f)\}$ □

In the remaining parts of this section we discuss the necessary and sufficient conditions for surjectivity of the bonding mappings of the inverse σ-system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit (by *Lemma 5*) is $C(\lim \mathbf{X})$. We adopt the notion of hyper-onto representation ([10, p. 183, Definition (1.186)], [4, p. 439]) as follows.

A continuum X *is said to have a hyper-onto representation* provided that there exists an inverse σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that: (i) X is homeomorphic to $\lim \mathbf{X}$, (ii) each X_a is a metric space and (iii) each mapping $C(p_{ab}) : C(X_b) \to C(X_a)$ is a surjection.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ satisfying (i) through (iii) is called a *hyper-onto representation for* X.

A continuous mapping $f : X \to Y$ is said to be *confluent* [12, p. 284, Definition 13.12] if for each subcontinuum Q of Y and each component K of $f^{-1}(Q)$ we have $f(K) = Q$.

A continuous mapping $f : X \to Y$ is said to be *weakly confluent* [10, p. 22] if for each subcontinuum Q of Y there exists a component K of $f^{-1}(Q)$ such that $f(K) = Q$. Every monotone surjection is weakly confluent.

It is clear that, as in [10, p. 186, Theorem (190)], the following theorem holds.

Theorem 8. *A continuum X has a hyper-onto representation if and only if there exists an inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ satisfying (i) and (ii) and such that every bonding mapping $p_{ab} : X_b \to X_a$ is weakly confluent.*

In the sequel we investigate the hyper-onto representation of some classes of continua.

2.3. Hyper-onto representation of locally connected and rim-metrizable continua

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $Bd(U))$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a space X is rim-metrizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

Lemma 8. *[15, Theorem 1.2]. Let X be a nondegenerate rim-metrizable continuum and let Y be a continuous image of X under a light mapping $f : X \to Y$. Then $w(X) = w(Y)$.*

Lemma 9. *[15, Theorem 3.2]. Let X be a rim-metrizable continuum and let $f : X \to Y$ be a monotone mapping onto Y. Then Y is rim-metrizable.*

Let us prove the following theorem.

Theorem 9. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

1) *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and* $\lim X$ *is homeomorphic to* $\lim M(\mathbf{X})$,

2) *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed,*

3) *If $\mathbf{X}$ is σ-complete, then $M(\mathbf{X})$ is σ-complete,*

4) *If every X_a is a metric space and limX is locally connected (a rim-metrizable continuum), then every M_a is metrizable.*

Proof. 1) The proof of 1) is broken into several steps.

a) Let $X = \{X_a, p_{ab}, A\}$ be an inverse system with limit X and the projections $p_a : X \to X_a, a \in A$. For every mapping $p_a : X \to X_a$ there exists a monotone-light factorization $p_a = \ell_a m_a$, where $m_a : X \to M_a$ is monotone and $\ell_a : M_a \to X_a$ is light [1, p. 451, Theorem 6.2.22]. We have a collection of spaces M_a, $a \in A$.

b) For every bonding mapping $p_{ab} : X_b \to X_a, b \geq a$, we define $m_{ab} : M_b \to M_a$ as follows. Let x be a point of M_b, $x_b = \ell_b(x)$ and $x_a = p_{ab}(x_b)$. Then x is a component in $p_b^{-1}(x_b)$. This means that there exists a unique component y of $p_a^{-1}(x_a)$ containing x since $p_b^{-1}(x_b) \subset p_a^{-1}(x_a)$. Set $m_{ab}(x) = y \in M_a$. The mapping $m_{ab} : M_b \to M_a$ is defined. From the definition of $m_{ab} : M_b \to M_a$ it follows

$$p_a = \ell_a m_a, \tag{8}$$

$$p_{ab}\ell_b = \ell_a m_{ab}, \tag{9}$$

and

$$m_{ab}m_b = m_a. \tag{10}$$

c) *Transitivity.* Let as prove that $m_{ac} = m_{ab}m_{bc}$. Let x be any point of M_c. Set $x_c = \ell_c(x)$. This means that there exists a component C of $p_c^{-1}(x_c)$ such that $m_c(C) = x$. Let $x_b = p_{bc}(\ell_c(x))$. It is clear that C is contained in some component D of $p_b^{-1}(x_b)$. Let $x_a = p_{ab}(x_b)$. It follows that D is contained in some component E of $p_a^{-1}(x_a)$. Hence,

$$m_{bc}(x) = m_b(D). \tag{11}$$

This means that $m_{ab}m_{bc}(x) = m_{ab}m_b(D) = m_a(D) = m_a(E) = m_c(C)$ since $m_{ab}m_b = m_a$ and $D \subset C$. On the other hand $m_{ac}(x) = m_a(C)$. Hence, for every $x \in M_c$ we have

$$m_{ac}(x) = m_{ab}m_{bc}(x). \tag{12}$$

The proof of the transitivity is completed.

d) We infer that $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ is an inverse system. Let us prove that $\lim \mathbf{X}$ and $\lim M(\mathbf{X})$ are homeomorphic. Let x be any point of $\lim M(\mathbf{X})$. From (10) it follows that the collection $\{m_a(x) : a \in A\}$ is a point of $\lim M(\mathbf{X})$. This means that the collection $\{m_a : a \in A\}$ induces a continuous mapping $m : \lim \mathbf{X} \to \lim M(\mathbf{X})$ which assigns to the point x the point $m(x) = \{m_a(x) : a \in A\} \in \lim M(X)$. If x and y are distinct points of $\lim \mathbf{X}$**,** then there exists an $a \in A$ such that $p_a(x) \neq p_a(y)$. It is clear that $m_a(x) \neq m_a(y)$. This means that the mapping m is 1-1. Similarly, one can prove that m is a surjection. Hence m is a homeomorphism.

2) Obvious.

3) It suffices to prove the continuity of $M(\mathbf{X})$. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be continuous. Let $a_1 \leq a_2 \leq ... \leq a_\alpha...., \alpha < \tau$, be a transfinite sequence in A. We have a transfinite well-ordered inverse system $\{X_{a_\alpha}, p_{a_\alpha a_\beta}, \alpha < \tau\}$ whose limit space is $X_{a_\tau} \in \mathbf{X}$. We have also a well-ordered inverse system $\{M_{a_\alpha}, m_{a_\alpha a_\beta}, \alpha < \tau\}$. We must prove that the inverse system $\{M_{a_\alpha}, m_{a_\alpha a_\beta},\ \alpha < \tau\}$ has the limit homeomorphic to M_{a_τ} and that the homeomorphism is induced by the mappings $m_{a_\alpha a_\tau}$. Let Y be the limit of $\{M_{a_\alpha}, m_{a_\alpha a_\beta},\ \alpha < \tau\}$ and let $n_{a_\alpha} : Y \to M_{a_\alpha}$ be the natural projection, $\alpha < \tau$. For each point $x \in M_{a_\tau}$ the collection $\{m_{a_\alpha a_\tau}(x) : \alpha < \tau\}$ is a thread in $\{M_{a_\alpha}, m_{a_\alpha a_\beta},\ \alpha < \tau\}$. Define $H(x) = (m_{a_\alpha a_\tau}(x) : \alpha < \tau) \in Y$. We have a continuous mapping $H : M_{a_\tau} \to Y$ induced by mappings $m_{a_\alpha a_\tau}$ such that $Hm_{a_\alpha a_\tau} = n_{a_\alpha}$, $\alpha < \tau$. Let us prove that H is a homeomorphism. It suffices to prove that H is onto and 1-1. If $y \in Y$, then $y_{a_\alpha} = n_{a_\alpha}(y)$ and $m_{a_\alpha a_\beta}(y_{a_\beta}) = y_{a_\alpha}$. Every $m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha})$ is non-empty and $m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) \supset m^{-1}_{a_\beta a_\tau}(y_{a_\beta})$, $\alpha < \beta < \tau$, since $m_{a_\alpha a\tau} = m_{a_\alpha a_\beta}m_{a_\beta a_\tau}$. We infer that $\bigcap\{m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) : \alpha < \tau\}$ is non-empty subset of M_{a_τ}. For each point $x \in \bigcap\{m^{-1}_{a_\alpha a_\tau}(y_{a_\alpha}) : \alpha < \tau\}$ we have $H(x) = y$. Thus, H is onto. Finally, let us prove that H is 1-1. Let x, y be a pair of distinct point of M_{a_τ}. We consider two cases. First, let $\ell_{a_\tau}(x) \neq \ell_{a_\tau}(y)$. This means that there exists an $\alpha < \tau$ such that $p_{a_\alpha a_\tau}(\ell_{a_\tau}(x)) \neq p_{a_\alpha a_\tau}(\ell_{a_\tau}(y))$ since X_{a_τ} is the limit of the system $\{X_{a_\alpha}, p_{a_\alpha a_\beta}, \alpha < \tau\}$. From (9) it follows that $\ell_{a_\alpha} m_{a_\alpha a_\tau}(x) = p_{a_\alpha a_\tau}(\ell_{a_\tau}(x))$ and $\ell_{a_\alpha} m_{a_\alpha a_\tau}(y) = p_{a_\alpha a_\tau}(\ell_{a_\tau}(y))$. Thus, $\ell_{a_\alpha} m_{a_\alpha a_\tau}(x) \neq \ell_{a_\alpha} m_{a_\alpha a_\tau}(y)$. It is clear that $m_{a_\alpha a_\tau}(x) \neq m_{a_\alpha a_\tau}(y)$. Because of the definition of H it follows that $H(x) \neq H(y)$. Consider the case $\ell_{a_\tau}(x) = \ell_{a_\tau}(y)$. Set $z = \ell_{a_\tau}(x) = \ell_{a_\tau}(y)$. From $x \neq y$ it follows that there exist two different components C, D of $p^{-1}_{a_\tau}(z)$ such that $m_{a_\tau}(C) = x$ and $m_{a_\tau}(D) = y$. For every $\alpha < \tau$ we have the point $z_{a_\alpha} = p_{a_\alpha a_\tau}(z)$ such that $\bigcap\{p^{-1}_{a_\alpha a_\tau}(z_{a_\alpha}) : \alpha < \tau\} = z$ since X_{a_τ} is the limit of the system $\{X_{a_\alpha},\ p_{a_\alpha a_\beta}, \alpha < \tau\}$. It follows that $\bigcap\{p^{-1}_{a_\tau} p^{-1}_{a_\alpha a_\tau}(z_{a_\alpha}) : \alpha < \tau\} = p^{-1}_{a_\tau}(z)$ or $\bigcap\{p^{-1}_{a_\alpha}(z_{a_\alpha}) : \alpha < \tau\} = p^{-1}_{a_\tau}(z)$. We infer that every component of $p^{-1}_{a_\tau}(z)$ is contained in some component of $p^{-1}_{a_\alpha}(z_{a_\alpha})$. If we suppose that for every $\alpha < \tau$ there exists a component K_{a_α} of $p^{-1}_{a_\alpha}(z_{a_\alpha})$ which contains both C and D, then we have the continuum $\bigcap\{K_{a_\alpha} : \alpha < \tau\}$ [1, Corollary 6.1.19] containing C and D. This is impossible since C and D are components. Hence, there exists an $\alpha < \tau$ such that C and D are in different components of $p^{-1}_{a_\alpha}(z_{a_\alpha})$. We infer that $m_{a_\alpha}(C) \neq m_{a_\alpha}(D)$. From (10) it follows that $m_{a_\alpha a_\tau} m_{a_\tau}(C) = m_{a_\alpha}(C)$ and $m_{a_\alpha a_\tau} m_{a_\tau}(D) = m_{a_\alpha}(D)$. This means that $m_{a_\alpha a_\tau} m_{a_\tau}(C) \neq m_{a_\alpha a_\tau} m_{a_\tau}(D)$ or $m_{a_\alpha a_\tau}(x) \neq m_{a_\alpha a_\tau}(y)$ since $m_{a_\tau}(C) = x$ and $m_{a_\tau}(D) = y$. From the definition of q_b it follows that $H(x) \neq H(y)$. The continuity is proved.

4) If X is rim-metrizable, then apply *Lemmas 8* and *9*. If X is locally connected, then apply [8, Theorem 1]. □

Theorem 10. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-directed inverse system of compact spaces and surjective bonding mappings* p_{ab}*. If* $\lim \mathbf{X}$ *is a locally connected space (rim-metrizable continuum), then there exists an* $a \in A$ *such that the projection* p_b *is monotone, for every* $b \geq a$*.*

Proof. Let $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ be the inverse system of compact metric space M_a and monotone bonding mappings m_{ab} (*Theorem 9*) whose limit is homeomorphic to $\lim \mathbf{X}$. From Theorem *7* and *Lemma 7* it follows that there exists an $a \in A$ such that for every $b \geq a$ there exists a homeomorphism $h_b : X_b \to M_b$ such that $h_b p_b = m_b$, where $m_b : \lim M(X) \to M_b$ is a projection. Clearly, m_b is monotone. Hence, p_b is monotone since $h_b p_b = m_b$ and $h_b : X_b \to M_b$ is a homeomorphism. □

Theorem 11. *If* X *is a locally connected or rim-metrizable continuum, then* X *has a hyper-onto representation.*

Proof. By *Theorems 9* and *10* there exists an inverse σ-system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ such that the bonding mappings p_{ab} are monotone. From *Theorem 8* it follows that $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ is a hyper-onto representation of X. □

2.4. Hyper-onto representation of chainable continua

A *chain* $\{U_1, ..., U_n\}$ is a finite collection of sets U_i such that $U_i \bigcap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A continuum X is said to be *chainable* or *arc-like* if each open covering of X can be refined by an open covering $u = \{U_1, ..., U_n\}$ such that $\{U_1, ..., U_n\}$ is a chain.

Theorem 12. *[7, Theorem* 2^**]. Every chainable continuum* X *is homeomorphic with the inverse limit of an inverse system* $\{Q_a, q_{ab}\}$ *of metric chainable continua* Q_a*.*

Remark 1. *One can assume that* q_{ab} *are onto mappings since a closed connected subset* C *of a chainable continuum is chainable [8, Lemma 12].*

Theorem 13. *[12, p. 262, Theorem 12.46]. If $f : X \to Y$ is a mapping of the metric continuum X onto an arc-like continuum Y, then f is weakly confluent.*

Representation in *Theorem 12* is not an hyper-onto since $\{Q_a, q_{ab}\}$ is not an σ-system. Now we shall prove that every chainable continuum has the hyper-onto representation.

Theorem 14. *If X is a chainable continuum, then X has the hyper-onto representation $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ such that each Q_Δ is a metric chainable continuum and each $p_{\Delta\Gamma}$ is a weakly confluent surjection.*

Proof. Let $\mathbf{Q} = \{Q_a, q_{ab}, A\}$ be an inverse system as in *Theorem 12*. Using *Theorem 5*, for $\tau = \aleph_0$, we obtain the inverse system $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ which is a σ-directed and σ-complete inverse system such that $\lim Q$ and $\lim Q_\sigma$ are homeomorphic. Every Q_Δ is chainable since we may assume that $\mathbf{Q}^\Delta = \{Q_b, q_{bb'}, \Delta\}$ is an inverse sequence since Δ is countable and $Q_\Delta = \lim \mathbf{Q}^\Delta$. Let $u = \{U_1, ..., U_n\}$ be an open covering of Q_Δ. There exists a $b \in \Delta$ and an open covering $u_b = \{U_1^b, ..., U_m^b\}$ of Q_b such that $\{q_b^{-1}(U_1^b), ..., q_b^{-1}(U_m^b)\}$ refines the covering $u = \{U_1, ..., U_n\}$. There is a chain $\{V_1^b, ..., V_p^b\}$ which refines u_b since Q_b is chainable. It is clear that $\{q_b^{-1}(V_1^b), ..., q_b^{-1}(V_p^b)\}$ is a chain which refines the covering u. Hence, Q_Δ is chainable. Further, one can assume that every $p_\Delta : \lim Q_\sigma \to Q_\Delta$ is onto since a closed connected subset C of an chainable continuum is chainable [8, Lemma 12]. From *Theorem 13* it follows that every bonding mapping $p_{\Delta\Gamma}$ is weakly confluent. Finally, we infer that every chainable continuum has the hyper-onto representation. □

2.5. Hyper-onto representation of hereditarily indecomposable continua

A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua [4, p. 61]. A continuum that is not decomposable is said to be *indecomposable*. A continuum is said to be *hereditarily indecomposable* [4, p. 61] provided that all of its nondegenerate subcontinua are indecomposable.

Now we obtain the hyper-onto representation for rim-metrizable hereditarily indecomposable continua.

Theorem 15. *If X is a rim-metrizable non-metric hereditarily indecomposable continuum, then X has an hyper-onto representation $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric hereditarily indecomposable continuum and each p_{ab} is a monotone surjection.*

Proof. Using *Theorem 9* we obtain an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a metric continuum and every p_{ab} is a surjective monotone mapping. It remains to prove that each X_a is hereditarily indecomposable. This easy follows from the fact that the projection $p_a, a \in A$, are monotone surjections. □

3. Hyperspaces which are products

In [11, Question 2.0], Nadler asked the following question: If C(X) is a finite-dimensional Cartesian product then must X be an arc or a circle? For a metric continuum X Illanes [3, Theorem A.] answered by the following theorem.

Theorem 16. *[3, Theorem A.]. If X is a metric continuum, then $C(X)$ is a finite-dimensional Cartesian product if and only if X is an arc or a circle.*

Let $\mathcal{K}$ be a class of non-metric finite-dimensional continua which are the limit of σ-directed inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric finite-dimensional continua $X_a, \dim X_a \leq \dim X$, and weakly confluent bonding mappings p_{ab}. By virtue of *Theorems 6* and *11* the class $\mathcal{K}$ contains all locally connected non-metric continua and all non-metric rim-metrizable continua. Moreover, the class $\mathcal{K}$ contains all non-metric chainable continua since every continuous mapping of a continuum onto a chainable continuum is weakly confluent (*Theorem 13*) and every chainable non-metric continuum has a hyper-onto representation (*Theorem 14*).

We start with the following theorem.

Theorem 17. *Let a continuum X be in class $\mathcal{K}$. If $C(X)$ is homeomorphic to $Y \times Z$, then X is a generalized arc or a generalized circle.*

Proof. From $X \in \mathcal{K}$ it follows that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua such that X is homeomorphic to $\lim \mathbf{X}$ and $\dim X_a \leq \dim X$. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ with surjective bonding mappings $C(p_{ab})$ [10, Theorem (0.49.1)]. If $C(X) \approx Y \times Z$, then there exist the inverse systems $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ and $\mathbf{Z} = \{Z_a, r_{ab}, A\}$ such that Y_a and Z_a are subspace of $C(X_a)$ and q_{ab}, r_{ab} are the restriction of $C(p_{ab})$ on Y_a and Z_a. It is clear that q_{ab}, r_{ab} are surjections. Then $Y \times Z$ is homeomorphic to $\lim\{Y_a \times Z_a, q_{ab} \times r_{ab}, A\}$ [1, p. 143]. Let us observe that $q_{ab} \times r_{ab}$ is a surjection. It follows that $C(X)$ is homeomorphic to $\lim\{Y_a \times Z_a, q_{ab} \times r_{ab}, A\}$ and to $\lim\{C(X_a), C(p_{ab}), A\}$. By *Theorem 7* it follows that there exists a cofinal subset B of A such that $Y_b \times Z_b$ is homeomorphic to $C(X_b)$ for each $b \in B$. From *Theorem 16* it follows that each X_b is an arc or a circle. By [2, Theorem 3] we infer that X is locally connected. Using *Theorem 10* we may assume that p_{bc} is monotone for every $b, c \in B$. If there exists a subset D of B which is cofinal in A and for each $d \in D$ X_d is an arc, then $X = \lim\{X_c, p_{cd}, C\}$ is an arc [2, Theorem 5]. If there is no a subset D of C which is cofinal in A such that for each $d \in D$ X_d is an arc, then there exists a subset E of C cofinal in A such that X_e is a generalized circle, $e \in E$. From *Lemma 4* it follows that $X = \lim\{X_c, p_{cd}, E\}$ is a generalized circle. □

Problem 1. *Is it true that $C(X)$ is homeomorphic to $Y \times Z$ for every generalized arc (for every generalized circle)?*

From *Theorems 17* and *14* there follows the following result.

Corollary 1. *Let X be a chainable non-metric continuum. If $C(X)$ is homeomorphic to $Y \times Z$, then X is a generalized arc.*

Theorem 18. *If X is a non-metric rim-metrizable (or locally connected) finite-dimensional continuum and $C(X)$ is homeomorphic to $Y \times Z$, then X is a generalized arc or a generalized circle.*

Proof. By virtue of *Theorem 6* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua such that X is homeomorphic to $\lim \mathbf{X}$ and $\dim X_a \leq \dim X$. From *Theorem 10* it follows that we may assume that p_{ab} are monotone surjections. Now, apply *Theorem 17*. □

Problem 2. *Is the converse of* Theorem 18 *true?*

Problem 3. *Is it true that every non-metric finite-dimensional continuum X is a generalized arc or a generalized circle if $C(X)$ is homeomorphic to $Y \times Z$?*

If $C(X)$ is an infinite-dimensional product and X is a locally connected metric continuum, then we have the following result.

Theorem 19. *[11, Theorem 3.15]. Let X be a Peano continuum. If $C(X) \approx Y \times Z$, then one of the following must hold:*

(3.15.1) *X is a circle,*

(3.15.2) *X contains no free arc,*

(3.15.3) *The closure of any component of $\cup F(X)$ is a free arc (in X) which is disjoint from any free arc (in X) not contained in it.*

Let us recall that for any continuum M, $F(M)$ is defined by $F(M) = \{A \in C(M) : A \subset J$ for some free arc in M and A is nondegenerate$\}$ [11, p. 60]. It follows that $\cup F(M) = \{p \in M : p \in J$ for some free arc J in $M\}$ [11, p. 60].

In a non-metric case we have the following theorem.

Theorem 20. *Let X be a locally connected non-metric continuum. If $C(X) \approx Y \times Z$, then one of the following must hold:*

(a) *X is a generalized circle,*

(b) *X contains no free arc,*

(c) *The closure of any component of $\cup F(X)$ is a free arc (in X) which is disjoint from any free arc (in X) not contained in it.*

Proof. From *Theorem 11* it follows that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric locally connected continua such that X is homeomorphic to $\lim \mathbf{X}$. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ with surjective bonding mappings $C(p_{ab})$ [10, Theorem (0.49.1)]. If $C(X) \approx Y \times Z$, then there exist the inverse systems $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ and $\mathbf{Z} = \{Z_a, s_{ab}, A\}$ such that Y_a and Z_a are subspace of $C(X_a)$ and q_{ab}, s_{ab} the restriction of $C(p_{ab})$ onto Y_a and Z_a. Then $Y \times Z$ is homeomorphic to $\lim\{Y_a \times Z_a, q_a \times r_a, A\}$ [1, p. 143]. It follows that $C(X)$ is homeomorphic to $\lim\{Y_a \times Z_a, q_a \times r_a, A\}$ and to $\lim\{C(X_a), C(p_{ab}), A\}$. By *Theorem 7* it follows that there exists a cofinal subset B of A such that $Y_b \times Z_b$ is homeomorphic to $C(X_b)$ for each b∈B. From *Theorem 19* it follows that each X_b is either an arc or a circle or (3.15.3) is satisfied. Consider the following sets : $C = \{b \in B : X_b$ is an arc$\}$, $D = \{b \in B : X_b$ contains no free arc$\}$ and $E = \{b \in B : X_b$ satisfies (c)$\}$. It is clear that if C is cofinal in B, then D and E are not cofinal in B since a monotone image of an arc is an arc. From *Lemma 3* it follows that X is a generalized arc. Similarly, if D is cofinal in B, then C is not cofinal in B. Let us prove that in this case X contains no free arc. Suppose that X contains a free arc L with the end-points x and y. This means that $U = L \setminus \{0, 1\}$ is an open set in X. There exists a $d \in D$ and an open set $U_d \subset X_d$ such that $p_d^{-1}(U_d) \subset U \subset L$. Let us observe that $L_d = p_d(L)$ is an arc since p_d is monotone. We infer that $U_d \subset L_d$. This means U_d is an interval (x_d, y_d) of the arc L_d. It follows that (x_d, y_d) is open in X. This is impossible since X_d contains no free arc. It remains to consider the case when E is cofinal in B. Let K be any component of $\cup F(X)$ and let $z \in K$. There exists a free arc J with end points x and y such that

$p \in J \setminus \{x, y\}$. Now, $U = J \setminus \{0,1\}$ is open in X. As in case when D is cofinal in B we infer that there exists an $e_0 \in E$ such that, for every $e \geq e_0$, $p_e(z)$ is in some free arc in X. This means that $p_e(K)$ is contained in some component K_e of $\cup F(X_e)$. From the monotonicity of $p_{e_1e_2}, e_0 \leq e_1 \leq e_2$, it follows that $p^{-1}_{e_1e_2}(K_{e_1}) = K_{e_2}$. Similarly, $p^{-1}_{e_1e_2}(Cl(K_{e_1})) = Cl(K_{e_2})$. Now, we have the inverse system $\{Cl(K_e), p_{ef} |Cl(K_f)$, $e_0 \leq e_1 \leq e_2\}$ whose limit is $Cl(K)$. Let us prove that $Cl(K)$ is an arc. This follows from *Lemma 3* since every $Cl(K_e)$ is an arc. Similarly, one can prove that $Cl(K)$ is a free arc. It remains to prove that $Cl(K)$ is disjoint from any free arc (in X) not contained in it. Suppose that there exists a free arc J in X such that $Cl(K) \cap J \neq \emptyset$ and $J \setminus Cl(K) \neq \emptyset$. This means that $p_e(Cl(K)) \cap p_e(J) \neq \emptyset$ for every $e \geq e_0$. Moreover, if $z \in p_e(Cl(K)) \cap p_e(J) \neq \emptyset$ then there exists a $e_0 \in E$ such that, for every $e \geq e_0$, $p_e(z)$ is in some free arc J_e in X_e. By (c) it follows that $p_e(J)$ is contained in $p_e(Cl(K))$ for every $e \geq e_0$. Thus, J is contained in $\lim\{Cl(K_e), p_{ef} |Cl(K_f)$, $e_0 \leq e_1 \leq e_2\} = Cl(K)$, a contradiction. □

Now we consider the non-metric continua for which $C(X) \approx X \times I$, where $I = [0,1]$.

Theorem 21. *[10, p. 342, Theorem (10.3)]. If X is a finite-dimensional metric continuum such that $C(X)$ is homeomorphic to $X \times I$, then X is an arc.*

For non-metric finite-dimensional continua we shall prove the following theorem.

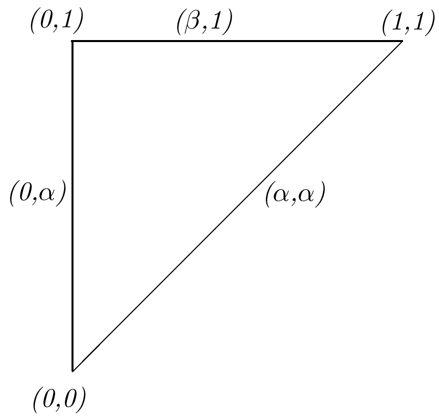
Theorem 22. *If X is a finite-dimensional non-metric rim-metrizable (or locally connected) continuum such that $C(X)$ is homeomorphic to $X \times I$, then X is a generalized arc.*

Proof. By virtue of *Theorem 6* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua such that X is homeomorphic to $\lim \mathbf{X}$ and $\dim X_a \leq \dim X$. By virtue of *Theorem 10* we may assume that p_{ab} are monotone surjections. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. If $C(X) \approx X \times I$, then there exist the inverse systems $\mathbf{X} \times I = \{X_a \times I, p_{ab} \times id, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ with homeomorphic limits $C(X)$. From *Theorem 7* it follows that there exists a cofinal subset B of A such that $X_b \times I$ is homeomorphic to $C(X_b)$ for each $b \in B$. From *Theorem 21* it follows that each X_b is a metric arc. From [2, Theorem 5] it follows that X is a generalized arc. □

Remark 2. *Let us observe that the proof above is valid for chainable non-metric continua. This means that if X is a finite-dimensional non-metric chainable continuum such that $C(X)$ is homeomorphic to $X \times I$, then X is a generalized arc.*

Problem 4. *Is the arc X in* Theorem 22 *a metric arc? Moreover, is an arc L a metric arc if $C(L)$ is homeomorphic to $L \times I$?*

Remark 3. *Let us prove that there exists an $\alpha \in L$ such that $[0,\alpha]$ is a metric arc and a $\beta \in L$ such that $[\beta, 1]$ is metrizable, where 0 and 1 are end-points of L. The following Figure shows $C(X)$.*



The Figure is obtained as follows. Every member of $C(X)$ is a subarc $[\alpha,\ \beta]$ of L, where $\alpha \leq \beta$. Let $T = \{(\alpha,\ \beta) : \alpha \leq \beta\}$. We define $H : C(L) \to T$ by $H([\alpha,\ \beta]) = (\alpha,\ \beta) \in T$. It is easy to see that H is a homeomorphism. Let p_1 and p_2 be projections of the triangle T such that $p_1(\alpha,\ \beta) = \alpha$ and $p_2(\alpha,\ \beta) = \beta$. Let $h : X \times I \to C(X)$ be a homeomorphism. There exists a point $x = (\alpha,\ t) \in X \times I$ such that $h(x) = (0, 0)$. Consider the metric arc $I = \{\alpha\} \times [0, 1]$ which contains the point x. Now, $h(I)$ contains the point $(0, 0)$. The projection $p_1 h(I)$ is a non-degenerate arc on the vertical side of the triangle. Since the vertical side of the triangle is homeomorphic to L, we obtain that there exists an $\alpha \in L$ such that $[0, \alpha]$ is a metric arc. Similarly, considering the point $(1, 1)$ we see that there exists a $\beta \in L$ such that $[\beta, 1]$ is metrizable.

Remark 4. *The long segment V [1, p. 297] is a non-metric arc. From the above* Remark *it follows that $C(V)$ is not homeomorphic to $V \times I$ since for each $\alpha \in V$ the segment $[\alpha, \omega_1]$ is non-metrizable.*

Problem 5. *Let V be the long segment. Is $C(V)$ homeomorphic to $V \times V$? If it is, what is a homeomorphism?*

Theorem 23. *Let L be a generalized arc. If X is a finite-dimensional non-metric rim-metrizable (or locally connected) continuum such that $C(X)$ is homeomorphic to $X \times L$ and $w(X) \geq w(L)$, then X is a generalized arc.*

Proof. a) Suppose that $w(X) = w(L)$. By virtue of *Theorem 6* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua such that X is homeomorphic to $\lim \mathbf{X}$ and $dim X_a \leq \ dim X$. By virtue of *Theorem 11* we may assume that p_{ab} are monotone surjections. Similarly, there exists an inverse σ-system $\mathbf{L} = \{I_a, q_{ab}, A\}$ of the metric arcs $I_a = [0, 1]$ such that L is homeomorphic to $\lim \mathbf{L}$. By virtue of *Theorem 11* we may assume that q_{ab} are monotone surjections. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. If $C(X) \approx X \times L$, then there exist the inverse system $\mathbf{X} \times L = \{X_a \times I_a,\ p_{ab} \times q_{ab},\ A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ with homeomorphic limits $C(X)$. From *Theorem 7* it follows that there exists a cofinal subset B of A such that $X_b \times I_a$ is homeomorphic to $C(X_b)$ for each $b \in B$. By *Theorem 21* it follows that each X_b is a metric arc. Finally, from [2, Theorem 5] it follows that X is a generalized arc.

b) $w(X) > w(L)$. Set $\tau = w(L)$. From *Theorem 5* it follows that there exists a τ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of continua X_a with $w(X_a) = \tau$ and monotone bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. If $C(X) \approx X \times L$, then there exist the inverse system $\mathbf{X} \times L = \{X_a \times L, p_{ab} \times id, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ with homeomorphic limit $C(X)$. From *Theorem 7* it follows that there exists a cofinal subset B of A such that $X_b \times L$ is homeomorphic to $C(X_b)$ for each $b \in B$. By a) of this proof it follows that each X_b is a generalized arc. Finally, from [2, Theorem 5] it follows that X is a generalized arc. □

4. Hyperspaces which are cones

The cone over X [10, p. 19] is the decomposition space of the upper semi-continuous decomposition $(X \times [0,1])/_{(X \times \{1\})}$ of $X \times [0,1]$ obtained by "shrinking $X \times \{1\}$ to a point". The cone over X will be denoted by $Cone(X)$, its *base* $X \times \{0\}$ by $B(X)$, and its *vertex* $X \times \{1\} \in Cone(X)$ by v.

A space X has the *cone = hyperspace property* [10, p. 303] if there exists a *Rogers homeomorphism* $H : C(X) \to Cone(X)$, i.e., a homeomorphism such that $H(X(1)) = B(X)$, where $X(1) = \{\{x\} : x \in X\}$.

Theorem 24. *[10, p. 308, Theorem (8.6)]. Let X be a metric continuum such that dim(X) $< \infty$ and such that X has the cone = hyperspace property. Then X is an arc, a circle, or an indecomposable continuum such that each nondegenerate proper subcontinuum of X is an arc.*

Now we shall prove that this is true for non-metric continua.

Theorem 25. *[10, p. 308, Theorem (8.6)]. Let X be a non-metric continuum such that dim(X) $< \infty$ and such that X has the cone = hyperspace property. Then X is an arc, a generalized circle, or an indecomposable continuum such that each nondegenerate proper subcontinuum of X is a generalized arc.*

Proof. By virtue of *Theorem 6* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to $\lim \mathbf{X}$and $dim X_a \leq dim X$. Now, $C(X)$ is a limit of inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$. Moreover, we have the inverse system $\mathbf{X} \times I = \{X_a \times I, p_{ab} \times id, A\}$ whose limit is $X \times I$. Let $B(X_a)$ be a base and ν_a a vertex of $Cone(X_a)$. Let $Cone(p_{ab})$ be a mapping such that $Cone(p_{ab})(\nu_b) = \nu_a$. It follows that $Cone(X)$ is the inverse limit of the system $Cone(\mathbf{X}) = \{Cone(X_a), Cone(p_{ab}), A\}$. If X has the *cone = hyperspace*, let $H : C(X) \to Cone(X)$ be a Rogers homeomorphism, i.e., a homeomorphism such that $H(BX)) = X(1)$. Now, from *Theorem 7* it follows that there exists an $a \in A$ such that for every $b \geq a$ there exists a homeomorphism $H_b : C(X_b) \to Cone(X_b)$ such that $H_b C(p_b) = Cone(p_b)H$. It follows that H_b is a Rogers homeomorphism. By virtue of *Theorem 24* X_b is an arc, a circle, or an indecomposable continuum such that each nondegenerate proper subcontinuum of X_b is a generalized arc. Let $B = \{b \in A : b \geq a\}$. We have the following cases:

1) There exists a subset C of B cofinal in A such that each $X_c, c \in C$, is an arc,

2) There exists a subset C of B cofinal in A such that each $X_c, c \in C$, is a circle,

3) There exists a subset C of B cofinal in A such that each $X_c, c \in C$, is an indecomposable continuum such that each nondegenerate proper subcontinuum of

X_c is an arc.

If 1), then X is a generalized arc. Let us prove that X is metrizable. Let $x = (0,0)$ and $y = (1,1)$ be the points as in figure of *Remark 3*. There exists a metrizable arc L in $Cone(X)$ with the end-points $H^{-1}(x)$ and $H^{-1}(y)$. Then $H(L)$ is a metrizable arc containing the points x and y. It is clear that $p_1(H(L)) = X$. Thus, X is metrizable. If 2), then X is a generalized circle (*Lemma 4*). Consider the case 3). We have the inverse system $\{\mathrm{X}_c, \mathrm{p}_{cd}, \mathrm{C}\}$ with the limit X. Let us prove that X is indecomposable. Suppose that X is decomposable, $X = X_1 \cup X_2$, where X_1, X_2 are subcontinua of X and $X_1 \neq X_2$. There exists a $c \in C$ such that $p_c(X_1) \neq p_c(X_2)$. Moreover, $X_c = p_c(X_1) \cup p_c(X_2)$. This is impossible since X_c is indecomposable. Hence, X is indecomposable. It remains to prove that each nondegenerate proper subcontinuum of X is a generalized arc. Let K be a nondegenerate proper subcontinuum of X. There exists a subset D of C which is cofinal in C and every $p_d(K)$ is a nondegenerate subcontinuum of X_d. This means that $p_d(K)$ is an arc. We have the inverse system $\{p_d(K), p_{de}|p_e(K), D\}$ whose limit is K. From *Lemma 3* it follows that X is a generalized arc. □

Problem 6. *Is it true that X in the case* 2) *of the above proof is metrizable?*

Problem 7. *Is every nondegenerate proper subcontinuum of X in the case* 3) *of the proof above is a metrizable arc?*

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