A note on a Whitney map for continua

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Abstract. *Let X be a non-metric continuum, and $C(X)$ the hyperspace of subcontinua of X. It is known that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X. On the other hand, there exist non-metrizable continua which admit and the ones which do not admit a Whitney map for $C(X)$. In this paper we investigate the properties of non-metrizable continua which admit a Whitney map and the ones which do not admit a Whitney map for $C(X)$. It is shown that there is no Whitney map on the hyperspace $C(X)$ if X is a non-metrizable locally connected or rim-metrizable continuum.*

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1. Introduction

Let X be a space. We denote by 2^X the set of all nonempty closed subsets of X, by *C(X)* the set of all nonempty closed connected subsets of X and by *X(n)*, n a positive integer, the set of all nonempty subsets consisting of at most n points [5]. We consider C(X) and X(n) as a subset of 2^X.

Let X and Y be compact spaces and let f : X $\to$ Y be a continuous map. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [9, 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by C(f).

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim X \to X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|X_b(n), A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$, $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a}:2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$ since $\{p_a(F):a \in A\}$ is a thread of the system $2^{\mathbf{X}}$. The mapping M is continuous and 1-1. It is also an onto mapping since for each thread $\{F_a:a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a):a \in A\}$ is

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nonempty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}$, $a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F by M(F) we have $P_a = 2^{p_a}$.

Lemma 1. *[5, Lemma 2.]. Let $X = \lim \boldsymbol{X}$. Then $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\boldsymbol{X})$ and $X(n) = \lim \boldsymbol{X}(n)$.*

If $F_a \in 2^{X_a}$, then $P_a^{-1}(F_a) = (2^{p_a})^{-1}(F_a) = \{F : F$ is a closed subset of $\lim \mathbf{X}$ and $p_a(F) = F_a\} \in 2^{\lim \mathbf{X}}$. Similarly, for the natural projection Q_a of the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ we have $Q_a = C(p_a)$. Moreover, if $C_a \in C(X_a)$, then $Q_a^{-1}(C_a) = (C(p_a))^{-1}(C_a) = \{C : C$ is a subcontinuum of $\lim \mathbf{X}$ and $p_a(C) = C_a\} \in C(\lim \mathbf{X})$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

In the sequel we shall use the following theorem.

Theorem 1. *[6, Lemma 2.2]. Let $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f: X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b: X_b \to Y$ such that $f = g_b p_b$.*

If the bonding mappings are not surjective, then we consider the inverse system $\{p_a(X), p_{ab}|p_b(X), A\}$ which has surjective bonding mappings. Moreover, $p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}$. Applying *Theorem 1* we obtain the following theorem.

Theorem 2. *Let $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with limit X. Let Y be a metric compact space. For each surjective mapping $f: X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping g_b: $p_b(X) \to Y$ such that $f = g_b p_b$.*

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, n = 0, 1, ..., by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y)\colon x,y \in \Delta_n\}$, where m(x,y) is a member of A such that $x,y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n\colon n \in \mathbb{N}\}$. It is clear that $card(\Delta) = card(\Delta_0)$. Moreover, Δ is directed by $\leq$ [12, Lemma 9.2]. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A,\ card(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Then A_σ is σ - directed by inclusion [12, Lemma 9.3]. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma}\colon X_\Gamma \to X_\Delta$ denote the map induced by the projections $p_\delta^\Gamma\colon X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system X^Γ. Now, we have the following theorem.

Theorem 3. *[12, Theorem 9.4] If $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then $\boldsymbol{X}_\sigma = \{X_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ is a σ - directed inverse system and $\lim \boldsymbol{X}$ and $\lim \boldsymbol{X}_\sigma$ are canonically homeomorphic.*

Theorem 4. *Let X be a compact space. There exists a σ - directed inverse system $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to $\lim \boldsymbol{X}$.*

Proof. Apply [8, pp. 152 , 164] and *Theorem 3.* □

Theorem 5. *If X is a locally connected compact space, then there exists a σ-directed inverse system $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected compact space, each p_{ab} is a monotone surjection and X is homeomorphic* *to lim**X**. Conversely, the inverse limit of such system is always a locally connected compact space.*

Proof. Apply *Theorem 4* and [8, p. 163, Theorem 2.]. □

2. Whitney map for C(X)

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [10, p. 24, (0.50)] we will understand any mapping g : $\Lambda \to [0,+\infty)$ satisfying

a) if A,B$\in \Lambda$ such that A$\subset$B, A$\neq$B, then g(A)$<$ g(B) and

b) g({x}) = 0 for each x$\in$X.

It is known that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X [1]. On the other hand, there exist non-metrizable continua which admit and the ones which do not admit a Whitney map for C(X) [1].

A continuous mapping f : X $\to$ Y is *light (zero-dimensional)* if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [3, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subset of cardinality larger than one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

The key theorem of this section is the following theorem.

Theorem 6. *Let* $\boldsymbol{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-directed inverse system of compact spaces* X_a *and surjective bonding mappings* p_{ab}*. Let X be the limit of* $\boldsymbol{X}$*. If there exists a Whitney map g for C(X) then there exists an* $a \in A$ *such that* $p_b : X \to X_b$ *is a light mapping for every* $b \geq a$*.*

Proof. From *Lemma 1* it follows that $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ is an inverse system whose limit is homeomorphic to C(X). Because of *Theorem 2*, for a Whitney map g : $C(X) \to [0,+\infty)$ there exists an a $\in$ A such that for every b $\geq$ a there exists a mapping $g_b : Q_b(C(X)) \to [0,+\infty)$ such that $g = g_b Q_b$, where Q_b is the natural projection $Q_b : \lim C(\mathbf{X}) \to C(X_b)$. Let b $\geq$ a be fixed. Now we will prove that the natural projection $p_b : \lim \mathbf{X} \to X_b$ is a light mapping. Suppose that there exists a point $x_b \in X_b$ such that $p_{ab}^{-1}(x_b)$ contains a non-degenerate component C. Let x be a point of C. Then {x}$\subset$ C and {x}$\neq$ C. This means that g({x}) $<$ g(C), i.e., 0 $\neq$ g(C). On the other hand, we have $Q_b(\{x\}) = Q_b(C)$. This means that $g_b Q_b(\{x\}) = g_b Q_b(C)$. From $g = g_b Q_b$ it follows that g({x}) = g(C). This contradicts 0 $\neq$ g(C). We infer that p_b is a light mapping. □

Theorem 7. *Let* $\boldsymbol{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-directed inverse system of compact spaces* X_a *and monotone surjections* p_{ab}*. Let X be the limit of* $\boldsymbol{X}$*. If there exists a Whitney map g for C(X) then there exists an* $a \in A$ *such that* $p_b : X \to X_b$ *is a homeomorphism for every* $b \geq a$*.*

Proof. From *Lemma 1* it follows that $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ is an inverse system whose limit is homeomorphic to C(X). Moreover, every $C(p_{ab})$ is a surjection. By *Theorem 1* for a Whitney map g : $C(X) \to [0,+\infty)$ there exists an a $\in$ A such that for every b $\geq$ a there exists a mapping $g_b : C(X_b) \to [0,+\infty)$ such that $g = g_b Q_b$, where Q_b is the natural projection $Q_b : \lim C(\mathbf{X}) \to C(X_b)$. Let b $\geq$ a be fixed.

Now we shall prove that the natural projection p_b: lim$\mathbf{X} \to X_b$ is a homeomorphism. It suffices to prove that p_b is 1-1. From *Theorem 6* it follows that there exists an $a_1 \in A$ such that for each $b \geq a_1$ p_b is light. Now, p_b for $b \geq a, a_1$ is light and monotone. This means that p_b is 1-1. Hence, p_b is a homeomorphism. □

Corollary 1. *If X is a limit of a σ-directed inverse system* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *of compact metric spaces* X_a *and monotone surjections* p_{ab}*, then there exists a Whitney map for C(X) if and only if X is metrizable.*

Proof. If X is metrizable, then there exists a Whitney g map for 2^X [10, pp. 25-27]. The restriction g|C(X) is a Whitney map for C(X). Conversely, if there exists a Whitney map for C(X), then there exists an $a \in A$ such that for every $b \geq a$ a mapping p_b is a homeomorphism. Hence, X is metrizable. □

The following Theorem generalizes Observation 3 from [1] which states that for any non-metrizable dendron (i.e., a Hausdorff continuum such that any two of its distinct points are separated by a third one) X there is no Whitney map for $C(X)$ since there is a canonical embedding of X(2) in C(X) (which maps any pair {x, y} with $x \neq y$ to the unique arc xy).

Theorem 8. *Let X be a locally connected compact space. Then there exists a Whitney map for C(X) if and only if X is metrizable.*

Proof. If X is metrizable, then there exists a Whitney g map for 2^X [10, pp. 25-27]. The restriction g|C(X) is a Whitney map for C(X). Conversely, let X be a locally connected compact space for which there exists a Whitney map $g : C(X) \to [0,+\infty)$. By virtue of *Theorem 5* there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a locally connected metric space, every p_{ab} is a monotone surjection and X is homeomorphic to lim$\mathbf{X}$. Apply *Corollary 1.* □

Corollary 2. *If X is a non-metric locally connected compact space, then there is no Whitney map for* $C(X)$.

Let τ be an infinite cardinal. A space X is said to be *rim-τ* if it has a basis $\mathcal{B}$ such that the weight $w(Bd(U)) \leq \tau$ for each $U \in \mathcal{B}$. In the sequel we shall use the following theorem.

Theorem 9. *[15, Theorem 1.4].Let* $f : X \to Y$ *be a light mapping of a non-degenerate continuum X onto a space Y. If X admits a basis of open sets whose boundaries have weight* $\leq w(Y)$*, then* $w(X) = w(Y)$.

Theorem 10. *Let X be a rim-τ continuum with* $w(X) > \tau$*. Then there is no Whitney map for C(X).*

Proof. There exists an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to lim$\mathbf{Y}$. From *Theorem 2.7* of [6] (see also the proof of *Theorem 3*) it follows that there exists a τ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A_\tau\}$ such that each X_a is homeomorphic to the limit of an inverse subsystem of $\mathbf{Y}$ of cardinality τ and X is homeomorphic to lim$\mathbf{X}$. We infer that $w(X_a) = \tau$. If we suppose that there exists a Whitney map for C(X), then the projection p_b must be light for every $b \geq a$ for some $a \in A$ (*Theorem 6*). From *Theorem 9* it follows that $w(X) = w(X_b) = \tau$. This is impossible since $w(X) > \tau$. □

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that Bd(U)) is metrizable for each $U \in \mathcal{B}$. Equivalently, a space X is rim-metrizable if and only if for each pair F,G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

Lemma 2. *[15, Theorem 1.2]. Let X be a nondegenerate rim-metrizable continuum and let Y be a continuous image of X under a light mapping $f: X \rightarrow Y$. Then w(X) = w(Y).*

Lemma 3. *[15, Theorem 3.2]. Let X be a rim-metrizable continuum and let $f: X \rightarrow Y$ be a monotone mapping onto Y. Then Y is rim-metrizable.*

It is clear that rim-metrizable continua are rim-τ for $\tau = \aleph_0$. Hence, we have the following theorem.

Theorem 11. *Let X be a rim-metrizable continuum. Then there exists a Whitney map for C(X) if and only if X is metrizable.*

Proof. *Theorem 11* follows from *Theorem 10.* We shall give an independent proof. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim\mathbf{X}$. From *Theorem 6* it follows that there exists an $a \in A$ such that for each $b \geq a$ the projection p_b is light. By *Lemma 2* we infer that $w(X) = w(X_b)$. This means that X is metrizable since $w(X_b) = \aleph_0$. □

If X is a continuous image of an ordered compact space, then X is rim-metrizable [7, Theorem 5.]. Hence we have the following corollary.

Corollary 3. *If a continuum X is a continuous image of an ordered compact space, then there exists a Whitney map for C(X) if and only if X is metrizable.*

In 1973 Heath, Lutzer and Zenor [4] introduced the concept of monotone normality which is a strengthening of normality.

A space X is *monotonically normal* [4] if points are closed and, for each $x \in X$ and an open set U with $x \in U$, there is an open H(x, U) with $x \in H(x, U) \subseteq U$ such that:

(1) (normality) $H(x, U) \bigcap H(y, V) = \emptyset$ unless $x \in V$ or $y \in U$, and

(2) (monotonicity) if $x \in U \subseteq V$, then $H(x, U) \subseteq H(x, V)$.

Every metrizable space is monotonically normal and every linearly ordered space is monotonically normal [4]. An arbitrary subspace of monotonically normal space is monotonically normal and a closed image of a monotonically normal space is a monotonically normal space [4]. It follows that every continuous image of an ordered compactum is monotonically normal. Moreover, we have the following excellent recent result of M.E. Rudin [13].

Theorem 12. *A space is compact and monotonically normal if and only if it is the continuous image of some compact, linearly ordered space.*

Thus, we have the following corollary.

Corollary 4. *Let X be a monotonically normal continuum. Then there exists a Whitney map for C(X) if and only if X is metrizable.*

Theorem 13. *Let X be a continuum which admits a Whitney map for C(X). Then each arc L in X is metrizable.*

Proof. It is clear that there exists a Whitney map for C(L). From *Theorem 11* it follows that L is metrizable. □

A *dendroid* is a hereditarily unicoherent continuum which is arcwise connected. If X is a dendroid and $x, y \in X$, then there exists a unique arc [x, y] in X with endpoints x and y.

Corollary 5. *If X is a dendroid which admits a Whitney map for C(X), then each arc in X is a metric arc.*

The dendroids in which every arc is a metric arc play an interesting role as the following theorem shows.

Theorem 14. *Let X be a dendroid. There exists an inverse system* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *such that each* X_a *is a dendroid with metrizable arcs, every* p_{ab} *is monotone and X is homeomorphic to* $lim\boldsymbol{X}$.

Proof. There exists an inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim\mathbf{Y}$. Let q_a be the natural projection of X onto Y_a. Applying the monotone-light factorization [16] to q_a, we get compact spaces X_a, monotone surjection $m_a : X \rightarrow X_a$ and light surjection $l_a : X_a \rightarrow Y_a$ such that $q_a = l_a \circ m_a$. By [8, Lemma 8] there exist monotone surjections $p_{ab} : X_b \rightarrow X_a$ such that $p_{ab} \circ m_b = m_a$, $a \leq b$. It follows that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system such that X is homeomorphic to $\lim\mathbf{X}$. Let us prove that X_a is a dendron. The space X_a is hereditarily unicoherent since m_a is monotone. Moreover, X_a is arcwise connected. Namely, if x_a, y_a are distinct points of X_a, then there exists a pair x, y of points of X such that $x_a = m_a(x)$ and $y_a = m_a(y)$. Let L be the arc with endpoints x and y. Now, $m_a(L)$ is a continuous image of an arc and, consequently, arcwise connected [14]. Hence, X_a is a dendroid. Since every map l_a is light, we infer that each arc in X_a is metrizable (*Theorem 2*). □

Theorem 15. *Let X be a rim-metrizable dendroid. There exists an inverse system* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *such that each* X_a *is a metric dendroid, every* p_{ab} *is monotone and X is homeomorphic to* $lim\boldsymbol{X}$.

Theorem 16. *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of dendroids and monotone surjective bonding mappings* p_{ab}. *The* $X = lim\boldsymbol{X}$ *is a dendroid.*

Proof. It is well known that X is hereditarily unicoherent [11, Theorem 3]. Let us prove that X is arcwise connected. Let x, y be a pair of distinct points in X. There exists an $a \in A$ such that $p_b(x) \neq p_b(y)$ for every $b \geq a$. There exists a unique arc L_b which contains $p_b(x)$ and $p_b(y)$. Let us prove that $p_{bc}(L_c) = L_b$. Now, $p_{bc}(L_b)$ is a continuous image of an arc and, consequently, arcwise connected [14]. It follows that there exists an arc M_b with endpoints $p_b(x)$ and $p_b(y)$. It follows that $M_b = L_b$ since X_b is hereditarily unicoherent. Moreover, $p_{bc}^{-1}(M_b) = p_{bc}^{-1}(L_b)$ is a continuum containing L_c since X_c is hereditarily unicoherent. This means that $p_{bc}(L_c) \subset L_b$. Finally, $p_{bc}(L_c) = L_b$ since L_b is the arc and $p_{bc}(L_c)$ contains $p_b(x)$ and $p_b(y)$. □

Now we consider the existence of transfinite sequences of subcontinua in a continuum X.

Theorem 17. *If a continuum X contains a transfinite increasing (decreasing) sequence of subcontinua, then X admits no Whitney map for C(X).*

Proof. Suppose that X is a continuum which contains the transfinite increasing sequence $C_0 \subset C_1 \subset ... \subset C_\xi \subset ...$, $\xi < \omega_1$ (decreasing sequence $C_0 \supset C_1 \supset ... \supset C_\xi \supset ...$, $\xi < \omega_1$) of subcontinua of X. Then $\omega(C_0) < \omega(C_1) < ... < \omega(C_\xi) < ...$, $\xi < \omega_1$ is an increasing transfinite sequence of real numbers ($\omega(C_0) > \omega(C_1) > ... > \omega(C_\xi) > ...$, $\xi < \omega_1$ is a decreasing transfinite sequence of real numbers). This is impossible since $w(\mathbb{R}) = \aleph_0$. □

Using *Theorem 17* we obtain the following theorem.

Theorem 18. *Let X be a continuum such that there exists a point x of X with the property that for each* $y \neq x$ *there exists a locally connected compact subspace* *(or rim-metrizable subcontinuum) C(x, y) containing x and y. If the density d(X) $> \aleph_0$, then there is no Whitney map for C(X).*

Proof. Suppose that there exists a Whitney map for C(X). Consider the subset E = X$\setminus\{$x$\}$. For each e$\in$E we consider a subcontinuum C(e, x). It is clear that X = $\bigcup\{$C(e, x) : e $\in$ E$\}$. If there exists a Whitney map for C(X), then there exists a Whitney map for hyperspace C(C(e,x)). From *Theorems 8* and *11* it follows that every subcontinuum C(e, x) is separable since it is metrizable. Hence if E is countable, then X is separable. This means that if d(X) $> \aleph_0$, then E is uncountable. Now, we define a separable subcontinuum $C_\alpha \subseteq$X for every countable ordinal $\alpha < \omega_1$ such that $C_\alpha \subseteq C_\beta$ if $\alpha < \beta$. Let e_1 be any point of E. Set C_1 = C(e_1, x). There exists a point $e_2 \in$E$\setminus C_1$ since C_1 is separable and d(X)$> \aleph_0$. Set $C_2 = C_1 \bigcup$ C(e_2, x). Suppose that C_α is defined for every $\alpha < \beta$ and define C_β. If β is a non-limit ordinal, then there exists a point $e_\beta \in$E$\setminus C_{\beta-1}$ since $C_{\beta-1}$ is separable and d(X)$> \aleph_0$. Set $C_\beta = C_{\beta-1} \bigcup$ C(e_β, x). If β is a limit ordinal, then we set C_β = Cl($\bigcup\{C_\alpha : \alpha < \beta\}$). It is clear that C_β is separable. We have strictly increased a transfinite sequence $C_1 \subset C_2 \subset ... \subset C_\alpha \subset ..., \alpha < \omega_1$. From *Theorem 17* it follows that there is no Whitney map for C(X), which is a contradiction. □

Corollary 6. *Let X be a continuum. If there exists a point x of X such that for each $y \neq x$ there exists a locally connected compact subspace (or rim-metrizable subcontinuum) C(x, y) containing x and y and if there is a Whitney map for C(X), the density d(X) $= \aleph_0$.*

Theorem 18 implies the following corollary.

Corollary 7. *Let X be an arcwise connected continuum. If the density d(X) $> \aleph_0$, then there is no Whitney map for C(X).*

An arc L in a space X is said to be a *free arc in X* provided that L without its endpoints is open in X.

Corollary 8. *Let X be an arcwise connected continuum. If X contains uncountable many free arcs, then there is no Whitney map for C(X).*

Proof. It is clear that d(X) $> \aleph_0$. Apply *Theorem 18.* □

Remark 1. *The cone over X [10, p. 19] is the decomposition space of the upper semi-continuous decomposition (X×[0, 1])$/_{(X\times\{1\})}$ of X×[0, 1] obtained by "shrinking X×$\{$1$\}$ to a point". The cone over X will be denoted by Cone(X), its base X×$\{$0$\}$ by B(X), and its vertex X×$\{$1$\}$ $\in$ Cone(X) by v. Let Ω_1 be the set of all ordinals $\alpha \leq \omega_1$, where ω_1 is the first uncountable ordinal. The space X = Cone(Ω_1) is a dendroid which contains uncountable many free arcs. Hence, there does not exist a Whitney map for C(X). Let us note that X is not locally connected. Moreover, X is not rim-metrizable. Thus, there exists a non locally connected and non rim-metrizable continuum X which admits no Whitney map for C(X).*

A point e of a dendroid X is said to be an *endpoint* of X if there exists no arc [a, b] in X such that x $\in [a, b]$ $\{a, b\}$. The set of all endpoints of a dendroid X is denoted by E(X).

Corollary 9. *[2, p. 317].For every point x of a dendroid X, X = $\bigcup\{[ex]$: e $\in$ E(X)$\}$.*

Corollary 10. *Let X be a dendroid. If the density d(X) $> \aleph_0$, then there is no Whitney map for C(X).*

Corollary 11. *If X is a dendroid such that there is a Whitney map for C(X), then d(X)* $= \aleph_0$.

3. Whitney map for 2^X

Now we shall give an alternate proof of *Theorem 1* from [1] using the inverse system method.

Theorem 19. *The following conditions are equivalent for a Hausdorff compact space X:*

(1.1) X is metrizable;

(1.2) there exists a Whitney map for 2^X*;*

(1.3) there exists a Whitney map for $X(2)$.

Proof. The implication from (1.1) to (1.2) is well known. The one from (1.2) to (1.3) is obvious. It remains to show that (1.3) implies (1.1). So, assume (1.3). Let X be a Hausdorff compact space for which there exists a Whitney map g : $X(2) \to [0,+\infty)$. By virtue of *Theorem 4* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that every X_a is a metric space, every p_{ab} is a surjection and X is homeomorphic to lim**X**. From *Lemma 1* it follows that $\mathbf{X}(2) = \{X_a(2), 2^{p_{ab}}|X_b(2), A\}$ is an inverse system whose limit is homeomorphic to X(2). Moreover, every $X_a(2)$ is a metric space and every $2^{p_{ab}}|X_b(2)$ is a surjection. By virtue of *Theorem 1*, for a Whitney map g : $X(2) \to [0,+\infty)$ there exists an a $\in$ A such that for every b $\geq$ a there exists a mapping $g_a : X_b(2) \to [0,+\infty)$ such that $g = g_b P_b$, where P_b is the natural projection $P_b : \lim\mathbf{X}(2) \to X_b(2)$. Now we shall prove that every natural projection $p_b : \lim\mathbf{X} \to X_b$ is a homeomorphism. It suffices to prove that p_b is 1-1. Suppose that there exists a point $x_b \in X_b$ such that $p_{ab}^{-1}(x_b)$ contains two different points x and y. Then $\{x\} \subset p_{ab}^{-1}(x_b)$ and $\{x\} \neq \{p_{ab}^{-1}(x_b)\}$. This means that $g(\{x\}) < g(\{p_{ab}^{-1}(x_b)\})$, i.e., $0 < g(\{p_{ab}^{-1}(x_b)\})$. On the other hand, $g(\{p_{ab}^{-1}(x_b)\}) = g_b P_b \{p_{ab}^{-1}(x_b)\} = g_b(\{x_b\})$ since $P_b\{p_{ab}^{-1}(x_b)\} = \{x_b\}$. This means that $g(\{p_{ab}^{-1}(x_b)\}) = 0$. This is impossible since $0 < g(\{p_{ab}^{-1}(x_b)\})$. We infer that p_b is a homeomorphism. Hence, X = lim**X** is a metric space since every X_b is a metric space. □

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