The dynamics of some cubic vector fields with a center

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Abstract. *We study planar cubic vector fields*

$$V_H = yP_2 \frac{\partial}{\partial x} - xQ_2 \frac{\partial}{\partial y}$$

with a center, having Darboux first integral $H = P_1^2 Q_3$. *We give the bifurcation diagram of the phase portraits of the vector fields, in* 2-*dimensional parameter space.*

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1. Introduction

In [P1], H. Poincaré defined a center of a real vector field on the plane as isolated singularity surrounded by closed integral curves. He showed in [P2] that necessary and sufficient conditions for a polynomial vector field with a singular point with pure imaginary eigenvalues, to have a center at this point is the annihilation of an infinite number of polynomials in the coefficients of the vector field. The problem of explicitly finding a finite basis for these algebraic conditions, called the center problem, was solved in the case of quadratic vector fields by contributions of H. Dulac, W. Kapteyn and others.

It is well known that each quadratic vector field with a center also possesses an explicit first integral (constant of motion) defined and analytic at least in an open domain around the center. This is the reason why such systems are called integrable.

For polynomial vector fields of degree $n \geq 3$ the center problem is not solved. In [M] Malkin found necessary and sufficient conditions for a cubic vector field with no quadratic terms, to have a center. In [LS] Lunkevich and Sibirsky proved that the integrability of the systems satisfies these conditions.

In [S] Schlomiuk gives a background for work on the center problem for a general cubic vector field. Algebraic particular integrals are used in exploring conditions

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for the center; because the problem of calculating a finite basis for these conditions, using the computer, were leading to enormous expressions.

The center problem plays an important role in the Hilbert's 16th problem which asks for the maximum number of limit cycles of a polynomial vector field of degree n. One way to produce limit cycles is by perturbing a vector field which has a continuous family of closed orbits. In order to do it we have to describe the global geometry of the some class of vector fields and then to make the perturbation.

In recent years progress has been made concerning singularities which are centers. This progress was due partly to studies of specific classes of vector fields, partly to theoretical development. In [RS] a detailed study of the cubic vector fields symmetric with respect to a center is done. In [MM-JR] a new two-parameter family of cubic isochronous centers is found. In [RST] the centers in the reduced Kukles systems are studied, and in [T] the perturbation of isochronous centers of Kukles systems is studied. The authors of [RST] said that more work concerning specific classes of systems needs to be done. This article is a part of this effort; we want to describe the geometry of a specific class of vector fields with a center. Our next step in future will be the study of perturbed vector fields created from this class of vector fields.

When we place a center at the origin, a cubic vector field can be written in the form

$$\begin{aligned} \dot{x} &= -y + \sum_{i+j=2}^{3} a_{ij} x^i y^j \\ \dot{y} &= x + \sum_{i+j=2}^{3} b_{ij} x^i y^j, \qquad a_{ij}, b_{ij} \in \mathbb{R}. \end{aligned} \tag{1}$$

We study such cubic vector fields denoted by $V_H = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$, under two conditions:

a) the first Darboux integral of $V_H = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ is

$$H(x,y) = P_1(x,y)^2 Q_3(x,y),$$

where P_1, Q_3 are polynomials of degree 1, respectively 3;

b)

$$\begin{aligned} P(x,y) &= yP_2(x,y) \\ Q(x,y) &= -xQ_2(x,y) \end{aligned}$$

where P_2, Q_2 are polynomials of degree 2.

We need to explain why these conditions are taken.

The condition a) gives us a vector field which is not Hamiltonian. The Hamiltonian vector fields are studied quite often, see [PS] where quadratic Hamiltonian vector fields with a center are studied. We wanted to study cubic Hamiltonian vector fields under some conditions; and noticed that if we take a first integral $H_1 = P_1 Q_3$, there is only one 3-parameter vector field satisfying condition b) and it has the bifurcation diagram induced by the bifurcation diagram of the vector field with a first integral $H = P_1^2 Q_3$. More about this statement will be said at the end of the article, in *Remark 5*.

Moreover, Hamiltonian vector field X_H with Hamiltonian $H = P_1^2 Q_3$ satisfies $X_H = P_1 W_H$ where W_H is a vector field with a first integral $H = P_1^2 Q_3$. We obtain V_H from W_H satisfying condition b).

Condition b) is taken because of easier computation with less parameters. Without condition b) we would compute with 7 parameters of the system (13 to describe the two invariant curves, minus 6 to delete the action of the affine group).

Then, we give the bifurcation diagram of the phase portraits of the vector fields V_H under the conditions a) and b).

2. Basic notions

First we briefly recall the main notions of invariant algebraic curves. Given a polynomial system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{2}$$

of degree n, define the differential operator

$$DF = \dot{F} = F_x P + F_y Q. \tag{3}$$

Definition 1.

(1) An invariant algebraic curve *of the system (2) is a curve in the complex plane given by the equation $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y]$, such that there exists $K(x,y) \in \mathbb{C}[x,y]$ satisfying*

$$DF(x,y) = F(x,y)K(x,y) \tag{4}$$

where $K(x,y) \in \mathbb{C}_{n-1}[x,y]$.

(2) A Darboux factor *is a polynomial $F(x,y)$ such that $F(x,y) = 0$ is an invariant curve.*

(3) Any analytic function satisfying (4) for some $K(x,y) \in \mathbb{C}_{n-1}[x,y]$, is an analytic Darboux factor*. The polynomial $K(F) = K(x,y)$ is called the* cofactor *of the analytic Darboux factor.*

(4) A non-constant function $F(x,y)$ satisfying $DF(x,y) \equiv 0$ is a first integral.

Definition 2. *A* Darboux function *is a function $Z(x,y)$ of the form*

$$\prod_{j=0}^{k} F_j^{\alpha_j}, \qquad \alpha_j \in \mathbb{C}$$

with $F_j \in \mathbb{C}[z,\bar{z}] = \mathbb{C}[x,y]$, for each $j = 0,\dots,k$.

An important class of strata of centers of polynomial systems in the class of strata of Darboux integrable systems i.e. systems which have a Darboux function as a first integral. For more details see [MM-JR], [S] and [So].

3. Systems satisfying conditions a) and b)

In the paper of Sokulski [So], the Darboux integral $H = P_1^\alpha Q_3^\beta$ is given, where P_1 and Q_3 are the invariant curves of the vector field V_H. The vector field

$$V_H = \alpha Q_3(\frac{\partial P_1}{\partial y}\frac{\partial}{\partial x} - \frac{\partial P_1}{\partial x}\frac{\partial}{\partial y}) + \beta P_1(\frac{\partial Q_3}{\partial y}\frac{\partial}{\partial x} - \frac{\partial Q_3}{\partial x}\frac{\partial}{\partial y})$$

has a center at the origin if the linear part of V_H is $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.

Let $P_1(x,y) = ax + by + c$, $Q_3(x,y) = dx^3 + ex^2y + fxy^2 + gy^3 + hx^2 + jxy + ky^2 + lx + my + n$ and $ck \neq 0$. We take $\alpha = 2$, $\beta = 1$, and divide $H = P_1^2 Q_3$ by c^2k. We define new coefficients, they are the quotients obtained after that dividing; or just simply take $c = k = 1$. These new coefficients will be denoted by capital letters.

If $V_H = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ satisfies condition b), the first integrals in the form $P_1^2 Q_3$ are

$$\begin{aligned} H_A(x,y) &= (Ax+1)^2(Dx^3 - 2Axy^2 + (1 - \tfrac{3}{2}AL)x^2 + y^2 + Lx - \tfrac{L}{2A}), \\ & \quad A \neq 0, \quad A, D, L \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} H(x,y) &= (Ax + By + 1)^2((A - 3A^2L + \tfrac{3}{2}B^2L)x^3 + B(-2 + 6AL - \tfrac{3B^2L}{A})x^2y \\ & \quad +(-2A + 3B^2L)xy^2 - \tfrac{B}{2A}(-2A + 3B^2L)y^3 + (1 + \tfrac{3}{2}(\tfrac{B^2L}{A} - AL))x^2 \\ & \quad - 3BLxy + y^2 + Lx + \tfrac{BL}{A}y - \tfrac{L}{2A}), \quad A, B \neq 0, \quad A, B, L \in \mathbb{R}. \end{aligned}$$

If $h \neq 0$, then by taking $c = h = 1$ we get the integral

$$\begin{aligned} H_B(x,y) &= (By+1)^2(-2Bx^2y + Gy^3 + x^2 + (1 - \tfrac{3}{2}BL)y^2 + Ly - \tfrac{L}{2B}), \\ & \quad B \neq 0, \quad B, G, L \in \mathbb{R}. \end{aligned}$$

We can see that $H_B(x,y) = H_A(y,x)$ depending on three real parameters and it is not necessary to study both of them.

The system with the first integral H_A is

$$\begin{aligned} \dot{x} &= y(2 - 2Ax - 4A^2x^2) \\ \dot{y} &= -x(2 + (4A + 3D - 6A^2L)x + 5ADx^2 - 6A^2y^2), \quad A \neq 0, \end{aligned} \tag{5}$$

the system with the first integral H_B is

$$\begin{aligned} \dot{x} &= y(2 + (4B + 3G - 6B^2L)y + 5BGy^2 - 6B^2x^2) \\ \dot{y} &= -x(2 - 2By - 4B^2y^2), \quad B \neq 0, \end{aligned} \tag{6}$$

and the system with the first integral H is

$$\begin{aligned}
\dot{x} &= y(-4A-6B^2L+4A^2x+6AB^2Lx+8A^3x^2+12AB^2x^2-48A^2B^2Lx^2\\
&\quad +18B^4Lx^2-14ABy+9B^3Ly+10A^2Bxy-15AB^3Lxy\\
&\quad -10AB^2y^2+15B^4Ly^2)\\
\dot{y} &= -x(-4A-6B^2L-14A^2x+30A^3Lx-21AB^2Lx-10A^3x^2\\
&\quad +30A^4Lx^2-15A^2B^2Lx^2+4ABy+6B^3Ly+10A^2Bxy-30A^3BLxy\\
&\quad +15AB^3Lxy+12A^3y^2+8AB^2y^2-42A^2B^2Ly^2+12B^4Ly^2), \quad A,B\neq 0.
\end{aligned} \tag{7}$$

In this paper we study systems (5) and (7) with $L=0$. The complete study of systems (5) and (7) is quite long and we divided it into two natural parts, by means of the number of parameters of the systems. This part for $L=0$ is simpler than the part for $L\neq 0$ described in [Ž1].

4. Phase portraits of systems (5) for $L=0$ and their bifurcation diagram

We begin by studying the nature of singularities of the system

$$\begin{aligned}
\dot{x} &= yP_{2A}(x,y)=y(2-2Ax-4A^2x^2)\\
\dot{y} &= -xQ_{2A}(x,y)=-x(2+(4A+3D)x+5ADx^2-6A^2y^2).
\end{aligned} \tag{8}$$

Notice that we can simplify (8) by changing the coordinates

$$\begin{aligned}
u &= Ax\\
v &= Ay,
\end{aligned}$$

and by substituting $M=\frac{D}{A}$, then we get

$$\begin{aligned}
\dot{u} &= v\bar{P}_{2A}(u,v)=v(2-2u-4u^2)\\
\dot{v} &= -u\bar{Q}_{2A}(u,v)=-u(2+(3M+4)u+5Mu^2-6v^2).
\end{aligned} \tag{9}$$

We will study system (9).

Remark 1. *Notice that the function P_{2A} is symmetric under*

$$\begin{aligned}
(x,y,A)&\mapsto(-x,y,-A),\\
(x,y,A)&\mapsto(-x,-y,-A),
\end{aligned}$$

and Q_{2A} is symmetric under

$$\begin{aligned}
(x,y,A,D)&\mapsto(-x,y,-A,-D),\\
(x,y,A,D)&\mapsto(-x,-y,-A,-D).
\end{aligned}$$

There is another symmetry, between Q_{2A} and P_{2B}

$$(x,y,A,D)\mapsto(y,x,B,G),$$

and between P_{2A} and Q_{2B}; where P_{2B}, Q_{2B} are polynomials obtained from H_B in the same way as P_{2A}, Q_{2A}.

Remark 2. *From* Definition 1 *we can see that the cofactors of system (8) are: $K_{1A}(x,y) = 2Ay(1-2Ax)$ is the cofactor of $P_1(x,y) = 0$; and $K_{2A}(x,y) = -2K_{1A}(x,y)$ is the cofactor of $Q_3(x,y) = 0$.*

Proposition 1. *System (9) has the following singular points on the coordinate axes:*

(1) the origin which is a center;

(2) $S_{1A}(M) = (\frac{-3M-4+\sqrt{9M^2-16M+16}}{10M}, 0)$, for $M \neq 0$, which is a saddle;

(3) $S_{3A}(M) = (\frac{(-3M-4-\sqrt{9M^2-16M+16}}{10M}, 0)$, for $M \neq 0$, which is a saddle for $M \in (-\infty, -16/11] \cup (0,1]$, a center for $M \in (-16/11, 0) \cup (1, \infty)$.;

(4) $S = (-1/2, 0)$ if $M = D = 0$, which is a saddle.

Proof.

(2) For studying the singularity S_{1A}, we first translate the singularity into the origin; then we look for the eigenvalues, and see that the eigenvalues are real functions of M. We find that $\lambda_1(M) = -\lambda_2(M)$ and $\lambda_1^2(M) = \lambda_2^2(M)$ is a positive function of M.

(3) In the case when $M = 1$ and $M = -16/11$, we see that we have the nilpotent case i.e. the linear part is $u\frac{\partial}{\partial v}$. Then, we can simply see [D], or we can find the principal part defined through the Newton diagram (see [BM], [Ž]). The vector field $V_{H_A t}$, obtained by translation, is locally topologically equivalent to its principal part. After a quasi-homogeneous blowing-up of type of quasi-homogeneity $(2,1)$, we have a decomposition of the singularity into the elementary singularities, and, after blowing-down, we see that we have a saddle.

Proposition 2.

(i) System (9) has the following singular points:

(1) $T_{1A}(M) = (-1, \sqrt{M-1}/\sqrt{3})$, for $M \geq 1$, which is a saddle with eigenvalues $\lambda_1 = -4\sqrt{3}\sqrt{M-1}$ and $\lambda_2 = 2\sqrt{3}\sqrt{M-1}$;

(2) $T_{2A}(M) = (-1, -\sqrt{M-1}/\sqrt{3})$, for $M \geq 1$, which is a saddle with eigenvalues $\lambda_1 = 4\sqrt{3}\sqrt{M-1}$ and $\lambda_2 = -2\sqrt{3}\sqrt{M-1}$;

(3) $T_{3A}(M) = (1/2, \sqrt{16+11M}/(2\sqrt{6})$, for $M \geq -16/11$, which is a saddle with eigenvalues $\lambda_1 = \sqrt{\frac{3}{2}}\sqrt{16+11M}$ and $\lambda_2 = -\sqrt{\frac{3}{2}}\sqrt{16+11M}$;

(4) $T_{4A}(M) = (1/2, -\sqrt{16+11M}/(2\sqrt{6})$, for $M \geq -16/11$, which is a saddle with eigenvalues $\lambda_1 = -\sqrt{\frac{3}{2}}\sqrt{16+11M}$ and $\lambda_2 = \sqrt{\frac{3}{2}}\sqrt{16+11M}$;

(ii) System (9) has singular points at infinity:

(1) three pairs if $M > 0$*, then the singular points in* $P_2(\mathbb{R})$ *are:*

(a) $(0 : 1 : 0)$*, which is not elementary;*

(b) $(1 : \sqrt{M/2} : 0)$*, which is a node;*

(c) $(1 : -\sqrt{M/2} : 0)$*, which is a node;*

(2) two pairs if $M = 0$*, then the singular points are:*

(a) $(0 : 1 : 0)$*, which is not elementary;*

(b) $(1 : 0 : 0)$*, which is not elementary;*

(3) one pair if $M < 0$*, then the singular point is:*

(a) $(0 : 1 : 0)$*, which is not elementary.*

Proof.

(i) Both singular points

$$T_{1A}(1) = T_{2A}(1) = S_{3A}(1) \text{ and } T_{3A}(-16/11) = T_{4A}(-16/11) = S_{3A}(-16/11)$$

are saddles.

(ii) For the study of singular points at infinity we use variables Z, U with the change $Z = 1/u$, $U = v/u$ corresponding to the needed two charts. We have the vector field

$$(-2UZ^3+2UZ^2+4UZ)\frac{\partial}{\partial Z}+(-2U^2Z^2+2U^2Z+4U^2-2Z^2-(3M+4)Z-5M+6U^2)\frac{\partial}{\partial U}$$

with singularities $(Z = 0,\ U = \sqrt{M/2})$, $(Z = 0,\ U = -\sqrt{M/2})$ i.e. $(1 : \sqrt{M/2} : 0)$, $(1 : -\sqrt{M/2} : 0)$. These singular points are nodes for $M \neq 0$.

In another chart we see the singular point $(0 : 1 : 0)$ which is nonelementary; we check the topological type using the Newton diagram and the blowing-up. Analogously for the singularity $(1 : 0 : 0)$ in the case $M = 0$.

□

The system with the first integral $H_B(x, y) = (By + 1)^2(x^2 + y^2 + Gy^3 - 2Bx^2y$ can be checked in the same way as we have done for $H_A(x, y) = (Ax + 1)^2(x^2 + y^2 + Dx^3 - 2Axy^2$.

The phase portraits of system (8) are drawn on the Poincaré disc. Due to the symmetry of system (9), it is only necessary to draw the bifurcation diagram for systems (8) on the semi-plane with $A > 0$. The bifurcation diagram appears in *Figure 1.*

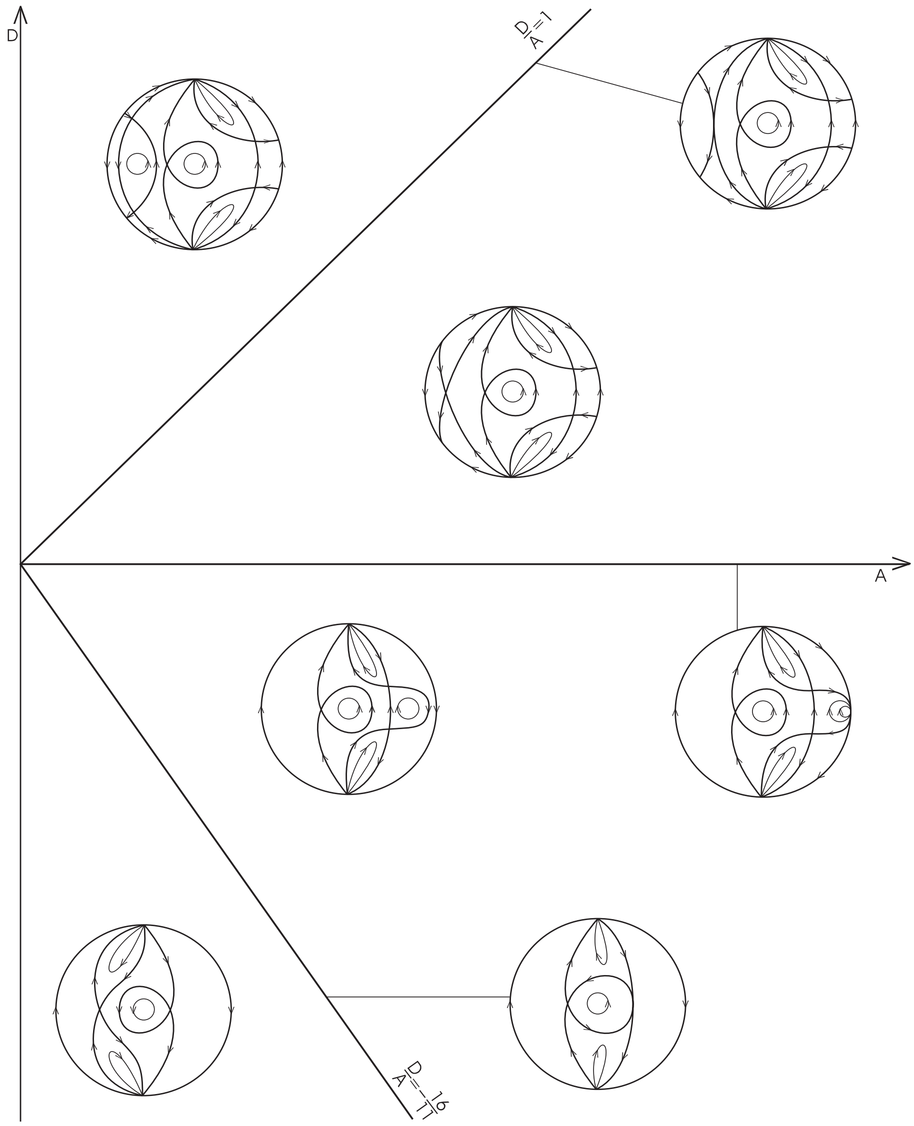

Figure 1.

5. Invariant curves of systems (8)

The zero level invariant curves of (8) are two algebraic curves, a straight line $P_{1A}(x,y) = Ax + 1 = 0$, and a cubic curve $Q_{3A}(x,y) = Dx^3 - 2Axy^2 + x^2 + y^2 = 0$, which are factors of $H_A(x,y) = (Ax+1)^2(Dx^3 - 2Axy^2 + x^2 + y^2) = 0$.

Changes of the phase portraits are related to changes of invariant curves. For a family of cubic curves depending on parameters, a point in the parameter space will be called a bifurcation point if any neighborhood of this point contains points with cubics of different affine types. We use the term bifurcation to indicate any qualitative, topological modification obtained for the curves we consider as a consequence of the variation of parameters.

If $A = 0$, then the straight line becomes the line at infinity; for $D = 0$ the number of the connected components of $Q_{3A} = 0$ is changed, for $D = 0$ is 2, for $D < 0$ is 1, and for $D > 0$ is 3 (it is obvious from $y^2 = \frac{x^2(Dx+1)}{2Ax-1}$); if $A = D = 0$, then the cubic curve becomes the reducible conic $x^2 + y^2 = 0$; if $D = -2A$, then the cubic curve becomes the reducible curve $(x^2 + y^2)(1 - 2Ax) = 0$.

Example 1. *In this example we describe the invariant curves of system (8) for $A = D$. The singularity $(-1/A, 0)$ lies on the curves $P_{1A} = 0$ and $Q_{3A} = 0$, these curves pass through this point. The origin lies on the curve $Q_{3A} = 0$, but the curve does not pass through. The saddles $(1/(2A), 3/(2\sqrt{2}A))$ and $(1/(2A), -3/(2\sqrt{2}A))$ lie on the curve $H_A = P_{1A}^2 Q_{3A} = C$, $C \neq 0$ and we have a saddle connection. The singularity $(-2/(5A), 0)$ lies on the curve $H_A = P_{1A}^2 Q_{3A} = C$, for some other $C \neq 0$, this point is a self-intersecting point of the curve.*

Phase portraits in Figure 1 *are determined by bifurcations of curves $P_{1A} = 0$ and $Q_{3A} = 0$, and also by bifurcations of the curve $H_A = C$, $C \neq 0$.*

6. Phase portraits of systems (7) for $L = 0$ and their bifurcation diagram

We have to find singularities of the system

$$\begin{aligned} \dot{x} &= yP_2(x, y) = y\left(2 + 7By - 2Ax + 5B^2y^2 - 5ABxy - 2(2A^2 + 3B^2)x^2\right) \\ \dot{y} &= -xQ_2(x, y) = -x\left(2 + 7Ax - 2By + 5A^2x^2 - 5ABxy - 2(3A^2 + 2B^2)y^2\right). \end{aligned} \tag{10}$$

Notice that we can simplify (10) in the similar way as (8). We take a change of the coordinates

$$\begin{aligned} u &= Ax \\ v &= By, \end{aligned}$$

and the substitution $F = \frac{A^2}{B^2}$, then we get

$$\begin{aligned} \dot{u} &= v\bar{P}_2(u, v) = v(2F + 7Fv - 2Fu + 5Fv^2 - 5Fuv - 2(2F + 3)u^2) \\ \dot{v} &= -u\bar{Q}_2(u, v) = -u(2 + 7u - 2v + 5u^2 - 5uv - 2(3F + 2)v^2). \end{aligned} \tag{11}$$

We will study system (11).

Remark 3. *Notice that the function P_2 is symmetric under*

$$\begin{aligned} (x, y, A, B) &\mapsto (x, -y, A, -B), \\ (x, y, A, B) &\mapsto (-x, y, -A, B), \end{aligned}$$

analogously for Q_2.

Remark 4. *The cofactor of* $P_1(x,y)=0$ *is* $K_1(x,y)=-2Bx+2Ay-5ABx^2+(4B^2-4A^2)xy+5ABy^2$ *and the cofactor of* $Q_3(x,y)=0$ *is* $K_2(x,y)=-2K_1(x,y)$.

Proposition 3. *System (11), for* $F>0$, *has the following singular points on the coordinate axes:*

(1) *the origin which is a center;*

(2) $S_1=(-1,0)$ *which is a saddle with eigenvalues* $\lambda_1=(3F+\sqrt{9F^2+72F})/2$ *and* $\lambda_2=(3F-\sqrt{9F^2+72F})/2$;

(3) $S_2=(0,-1)$ *which is a saddle with eigenvalues* $\lambda_1=(-3F+\sqrt{9F^2+72F^3})/2$ *and* $\lambda_2=(-3F-\sqrt{9F^2+72F^3})/2$;

(4) $S_3=(-2/5,0)$ *with eigenvalues* $\lambda_1=\frac{6}{5}\sqrt{F(9F-4)/5}$ *and* $\lambda_2=\frac{-6}{5}\sqrt{F(9F-4)/5}$; S_3 *is a saddle for* $F>4/9$, *a center for* $0<F<4/9$ *and a cusp for* $F=4/9$;

(5) $S_4=(0,-2/5)$ *with eigenvalues* $\lambda_1=\frac{6}{5}\sqrt{(9-4F)/5}$ *and* $\lambda_2=\frac{-6}{5}\sqrt{(9-4F)/5}$; S_4 *is a saddle for* $0<F<9/4$, *a center for* $F>9/4$ *and a cusp for* $F=9/4$.

Proposition 4.

(i) *If* $F=1$ *and* $A=B$, *then* $P_2(x,y)=Q_2(y,x)$ *and the singular points of (11), which are not on the coordinate axes, are:*

  (1) $T_+^+=((5+\sqrt{105})/20,(5+\sqrt{105})/20)$ *which is a saddle,*

  (2) $T_-^+=((5-\sqrt{105})/20,(5-\sqrt{105})/20)$ *which is a center.*

(ii) *If* $F=1$ *and* $A=-B$, *then* $P_2(x,y)=Q_2(-y,-x)$ *and the singular points of (11), which are not on the coordinate axes, are:*

  (1) $T_+^-=((5+\sqrt{105})/20,-(5+\sqrt{105})/20)$ *which is a saddle,*

  (2) $T_-^-=((5-\sqrt{105})/20,-(5-\sqrt{105})/20)$ *which is a center.*

(iii) *If* $F=1$ *then (11) has three pairs of singular points at infinity:*

  (1) $(1:-1:0)$ *which is not elementary,*

  (2) $(2:3+\sqrt{5}:0)$ *which is a node,*

  (3) $(2:3-\sqrt{5}:0)$ *which is a node.*

In the following theorem we prove that the system (10) has four real singular points, besides S_1, S_2, S_3, S_4 and the origin. They are not on the coordinate axes, except for some special $F\in\mathbb{R}$.

Theorem 1.

(i) *System (11) for* $F>0$, $F\neq 1$, *has the following singular points:*

  (1) $T_1(F)=(u_1(F),v_1(F))=(-F/(F-1),1/(F-1))$ *which is a saddle with eigenvalues* $\lambda_1(F)=-6F/(F-1)$ *and* $\lambda_2(F)=3F/(F-1)$;

(2) $T_2(F) = (u_2(F), v_2(F))$, *where* $u_2(F), v_2(F)$ *are the real algebraic functions;* $T_2(F)$ *is a center if* $F \in (0,1) \cup (1, (13+5\sqrt{145})/48)$, *and a saddle if* $F \in ((13+5\sqrt{145})/48, +\infty)$;

(3) $T_3(F) = (u_3(F), v_3(F))$, *where* $u_3(F), v_3(F)$ *are the real algebraic functions;* $T_3(F)$ *is a saddle if* $F \in (0,1) \cup (1, (13+5\sqrt{145})/48)$, *and a center if* $F \in ((13+5\sqrt{145})/48, +\infty)$*; if* $F = (13+5\sqrt{145})/48$, *then* $T_2((13+5\sqrt{145})/48) = T_3((13+5\sqrt{145})/48)$, *and this singular point is a center;*

(4) $T_4(F) = (u_4(F), v_4(F))$, *where* $u_4(F), v_4(F)$ *are the real algebraic functions;* $T_4(F)$ *is a saddle if* $F \in (0,4/9) \cup (9/4, +\infty)$, *and a center if* $F \in (4/9,1) \cup (1,9/4)$*; if* $F = 4/9$, *then* $T_4(4/9) = (-2/5, 0) = S_3$, *and this is a cusp; if* $F = 9/4$, *then* $T_4(9/4) = (0, -2/5) = S_4$, *and this is a cusp.*

(ii) System (11) for $F > 0$, $F \neq 1$, *has four pairs of singular points at infinity and they are all nodes.*

Proof.

(i) (1) We prove it by a straightforward calculation.

(2) Functions u_2, v_2 are irrational functions with explicit, but very long expressions found by Maple; the only discontinuity is $F = 1$. The graphs of the functions u_2, v_2 are given in *Figure 2*.

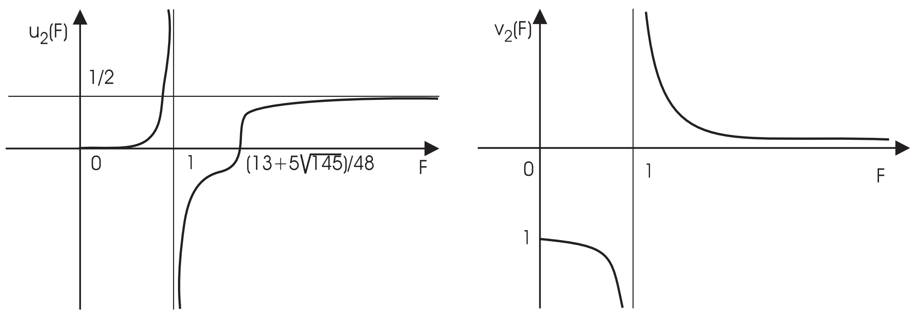

Figure 2.

The straightforward calculation shows that eigenvalues λ_1, λ_2 are pure imaginary for $F < (13+5\sqrt{145})/48$; and for $F > (13+5\sqrt{145})/48$ the eigenvalues are real and $\lambda_1 \lambda_2 < 0$.

(3) The function v_3 is found by Maple and it has the form

$$\begin{aligned} v_3(F) &= -\frac{1}{6}\frac{K(F)^{1/3}}{L(F)} - \frac{1}{6}M(F) + \frac{10}{3}\frac{F(4F+1)}{L(F)} \\ &\quad -i\frac{\sqrt{3}}{2}\left(\frac{1}{3}\frac{K(F)^{1/3}}{L(F)} - \frac{1}{3}M(F)\right) \end{aligned} \qquad (12)$$

where

$$\begin{aligned} J(F) &= -F(497664F^8 - 412992F^7 - 2227248F^6 + 1399011F^5 \\ &\quad +3698405F^4 - 1805109F^3 - 2815128F^2 + 1092528F + 1119744) \\ K(F) &= -41472F^7 - 249776F^6 - 30507F^5 + 539445F^4 + 148330F^3 \\ &\quad -337788F^2 - 13824F + 110592 + J(F)^{1/2}(288F^3 + 261F^2 - 261F - 288) \\ L(F) &= (F-1)(32F^2 + 61F + 61) \\ M(F) &= \tfrac{3456F^5+1180F^4-7855F^3-2225F^2+5640F+2304}{L(F)K(F)^{1/3}}. \end{aligned}$$

We want to prove that $v_3(F)$ is a real function at $\mathbb{R}^+$. Since $J(F) < 0$, for each $F > 0$, then $\sqrt{J(F)}$ is a pure imaginary number for each $F > 0$. We denote $\sqrt{J(F)} = d(F)i$, where $d(F)$ is a real function, then

$$K(F) = P_7(F) + d(F)i(F-1)(288F^2 + 549F + 288)$$

where

$$\begin{aligned} P_7(F) &= -41472F^7 - 249776F^6 - 30507F^5 + 539445F^4 + 148330F^3 \\ &\quad -337788F^2 - 13824F + 110592. \end{aligned}$$

Let

$$P_5(F) = 3456F^5 + 1180F^4 - 7855F^3 - 2225F^2 + 5640F + 2304$$

then

$$M(F) = \frac{P_5(F)}{L(F)K(F)^{1/3}};$$

let

$$\begin{aligned} f_1(F) &= \tfrac{10}{3}\tfrac{F(4F+1)}{L(F)} \\ f_2(F) &= -\tfrac{1}{6L(F)}(K(F)^{1/3} + \tfrac{P_5(F)}{K(F)^{1/3}}) - i\sqrt{3}(K(F)^{1/3} - \tfrac{P_5(F)}{K(F)^{1/3}}) \end{aligned}$$

then $v_3(F) = f_1(F) + f_2(F)$. Notice that $f_1(F) \in \mathbb{R}$ for each $F > 0$, and we need to show that $Im(f_2(F)) = 0$ for each $F > 0$. That is true if $\mid K(F)^{1/3} \mid^2 = P_5(F)$, but this is a straightforward calculation. The function

$$u_3(F) = -\frac{8v_3(F)^2F^2 + 9v_3(F)^2F + 8v_3(F)^2 - 9v_3(F)F + 4v_3(F) - 6F - 4}{15v_3(F)F + 10v_3(F) - 6F - 14}$$

is also real, because $v_3(F)$ is real. The graphs of the functions u_3, v_3 are in *Figure 3*.

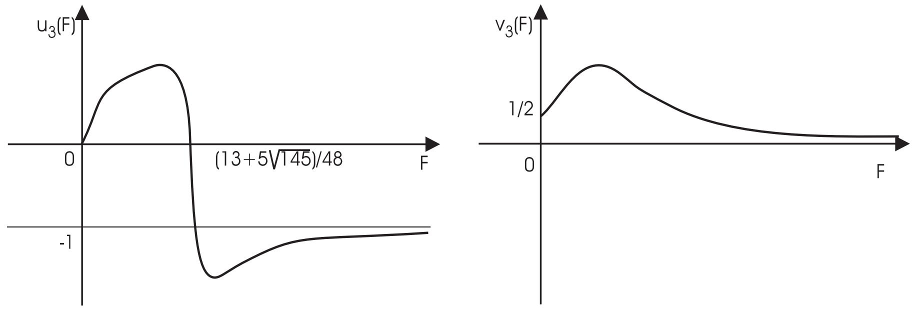

Figure 3.

The topological type is checked analogously to the part (2) of this proof. To study the singularities T_2 and T_3 for $F = (13+5\sqrt{145})/48$ we consider limit $F \to (13+5\sqrt{145})/48^{+}$ and $F \to (13+5\sqrt{145})/48^{-}$, and compute that $T_2((13+5\sqrt{145})/48) = T_3((13+5\sqrt{145})/48)$. We compute the normal form for the nilpotent case and prove that we have a center.

(4) Proof that functions u_4, v_4 are real is analogous to the part (3), because v_4 is the conjugate complex function of the function v_3 in the form (12) The graphs of the functions u_4, v_4 are in *Figure 4*.

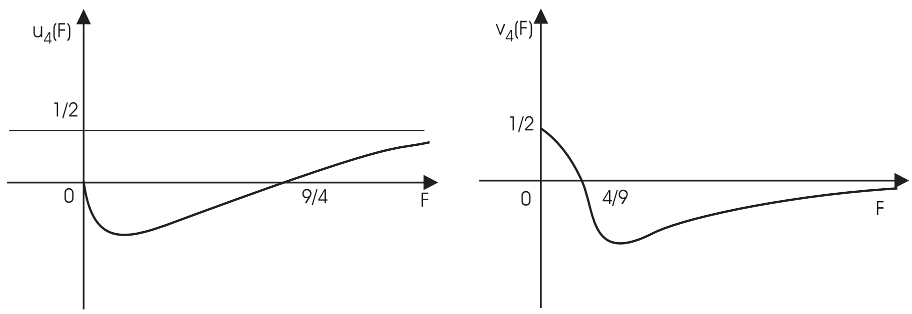

Figure 4.

For $F = 4/9$ and $F = 9/4$ we have a nilpotent case and a coincidence of singularities T_4 and S_3, i.e. T_4 and S_4, see *Proposition 3.*

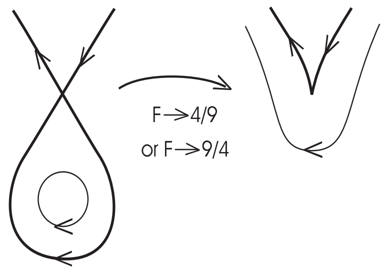

Figure 5.

(ii) Analogously to *Proposition 2* we make the change of coordinates and find four pairs of functions; then we prove, in the same way as in (i), that these functions are real for $F > 0$, $F \neq 1$.

□

This proposition shows "moving" of the points T_1, T_2, T_3, T_4 on the Poincaré disc, if the parameter goes to infinity, or it goes to zero, or the coordinate functions have discontinuity.

Proposition 5. *For* $T_1(F)$, $T_2(F)$, $T_3(F)$, $T_4(F)$ *defined in* Theorem 1*, the following statements hold:*

(1)

$$\begin{aligned} \lim_{F\to\infty} T_1(F) &= \lim_{F\to\infty} T_3(F) = (-1,0) = S_{3A}(1), \\ \lim_{F\to\infty} T_2(F) &= \lim_{F\to\infty} T_4(F) = (1/2,0) = S_{3A}(-16/11); \end{aligned}$$

(2)

$$\begin{aligned} T_1(0) &= T_2(0) = (0,-1) = S_{3B}(1), \\ T_3(0) &= T_4(0) = (0,1/2) = S_{3B}(-16/11), \end{aligned}$$

where the point S_{3B} *is defined analogously as the point* S_{3A} *in* Proposition 1*;*

(3)

$$\begin{aligned} T_3(1) &= ((5+\sqrt{105})/20, (5+\sqrt{105})/20) = T_+^+, \\ T_4(1) &= ((5-\sqrt{105})/20, (5-\sqrt{105})/20) = T_-^+, \end{aligned}$$

(4)

$$\begin{aligned} \lim_{F\to 1^+} T_1(F) &= \lim_{F\to 1^+} T_2(F) = (-\infty,+\infty), \\ \lim_{F\to 1^-} T_1(F) &= \lim_{F\to 1^-} T_2(F) = (-\infty,+\infty). \end{aligned}$$

The bifurcation diagram of the phase portraits of system (10), drawn by using all facts from Propositions and *Theorem 1*, appears in *Figure 6.* Due to the symmetry of system (11), it is only necessary to draw the bifurcation diagram for systems (10) for $A > 0$, $B > 0$.

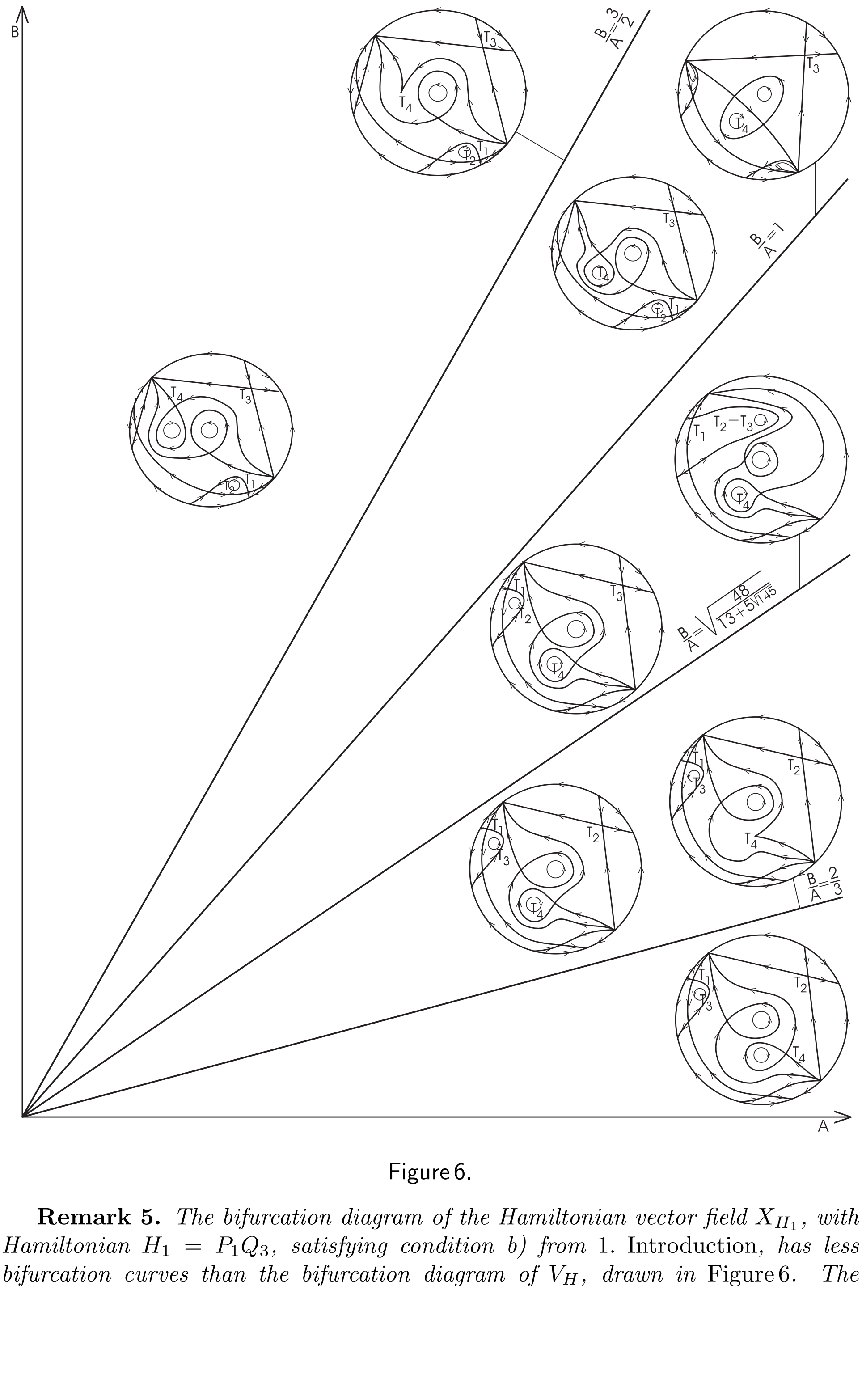

Figure 6.

Remark 5. *The bifurcation diagram of the Hamiltonian vector field* X_{H_1}*, with Hamiltonian* $H_1 = P_1Q_3$*, satisfying condition b) from* 1. Introduction*, has less bifurcation curves than the bifurcation diagram of* V_H*, drawn in* Figure 6. *The* *bifurcation curves of* X_{H_1} *are:* $\frac{B}{A}=\frac{1}{\sqrt{3}}$, $\frac{B}{A}=1$ *and* $\frac{B}{A}=\sqrt{3}$. *In the bifurcation diagram of* X_{H_1} *the "replacement" of* T_2 *and* T_3 *is done at bifurcation line* $\frac{B}{A}=1$, *and also at* $\frac{B}{A}=1$ *we have a coincidence of* T_3 *and* T_4.

We find interesting the study of the perturbed systems of (8) and (10). The next step will be the study of the existence of limit cycles of the system

$$\dot{x}=f(x)+\varepsilon g(x),$$

which is nonintegrable.

7. Invariant curves of systems (10)

The zero level invariant curves of (10) are two algebraic curves, a straight line $P_1(x,y)=Ax+By+1=0$, and a cubic curve $Q_3(x,y)=Ax^3-2Bx^2y-2Axy^2+By^3+x^2+y^2=0$, which are factors of $H(x,y)=(Ax+By+1)^2(Ax^3-2Bx^2y-2Axy^2+By^3+x^2+y^2)=0$.

If $A=B=0$ then the straight line becomes the line at infinity, and the cubic curve becomes the reducible conic $x^2+y^2=0$.

The discriminant of the cubic equation $Ax^3+(-2By+1)x^2-2Ay^2x+By^3+y^2=0$ is a function of y. We find the number of the connected components of $Q_3=0$ using the discriminant, in the same way as in [RS]. The cubic curve has 3 connected components for each $A,B\geq 0$.

Example 2. *In this example we describe the invariant curves of the system (6.1.) for* $A=B$. *The singularity* $(-1,0)$ *lies on the curves* $P_1=0$ *and* $Q_3=0$, *these curves pass through this point. The origin lies on the curve* $Q_3=0$, *but the curve does not pass through. The saddles* $(-2/5,0)$ *and* $(0,-2/5)$ *lie on the curve* $H=P_1^2Q_3=C$, $C\neq 0$ *and we have a saddle connection. The singularity* $((5+\sqrt{105})/20,(5+\sqrt{105})/20))$ *lies on the curve* $H=P_1^2Q_3=C$, *for some other* $C\neq 0$, *this point is a self-intersecting point of the curve. The singularity* $((5-\sqrt{105})/20,(5-\sqrt{105})/20))$ *lies on the curve* $H=C$, $C\neq 0$ *but the curve does not pass through.*

The phase portraits in *Figure 6* are determined by the bifurcations of the curve $H=C$, $C\neq 0$.

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