Minisum location in space with restriction to curves and surfaces

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Abstract. *The minisum location problem is well-known and has extensively been studied in the case of the unknown location being somewhere in the space. Also the accompanying Weiszfeld iteration method [1, 2, 3] is well understood nowadays, even for noneuclidean distances. We introduce as a side condition for the unknown optimal location that it lies on some given curve or surface in space. For the straight line, the plane, the sphere, and the circle the corresponding Weiszfeld-like iteration methods are developed and numerical examples are given.*

Key words: *minisum location problem, side conditions, Weiszfeld iteration*

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1. The problem for general curves and surfaces

Let points be given

$$\boldsymbol{x_i} = (x_i, y_i, z_i) \quad (i = 1, \dots, n \geq 3), \tag{1}$$

not all of them being collinear. Looking for an optimal location point (x, y, z) in space means to determine $\boldsymbol{x} = (x, y, z)$ such that the sum of Euclidean distances $d_i(x, y, z)$ from the unknown location to point i is minimized. Thus, with

$$e_i(x, y, z) = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2, \tag{2}$$

$$d_i(x, y, z) = \sqrt{e_i(x, y, z)}, \tag{3}$$

the optimization problem

$$F(x, y, z) = \sum_{i=1}^{n} d_i(x, y, z) \longrightarrow \min \tag{4}$$

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has to be solved. Because F is strictly convex [3], the optimal location uniquely exists; it lies in the convex hull of the given points. Also, the conditions

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{pF}{\partial z} = 0 \tag{5}$$

are necessary and sufficient for the global minimum.

As (similar for y and z)

$$\frac{1}{2}\,\frac{\partial e_i(x,y,z)}{\partial x} \;=\; (x - x_i), \tag{6}$$

$$\frac{1}{2}\,\frac{\partial d_i(x,y,z)}{\partial x} \;=\; \frac{\frac{\partial e_i(x,y,z)}{\partial x}}{d_i(x,y,z} \;=\; \frac{x - x_i}{d_i(x,y,z)} \tag{7}$$

conditions (5) can be written as (similar for y and z)

$$x = \frac{\sum\limits_{i=1}^{n} \frac{x_i}{d_i(x,y,z)}}{\sum\limits_{i=1}^{n} \frac{1}{d_i(x,y,z)}}\;. \tag{8}$$

It is well-known [1, 2] that the Weiszfeld fixpoint iteration based on (8), i.e.

$$x^{(k+1)} = \frac{\sum\limits_{i=1}^{n} \frac{xi}{d_i(x^{(k)},y^{(k)},z^{(k)})}}{\sum\limits_{i=1}^{n} \frac{1}{d_i(x^{(k)},y^{(k)},z^{(k)})}}, \quad k = 0,1,2\ldots \tag{9}$$

with similar formulae for $y^{(k+1)}$ and $z^{(k+1)}$, linearly converges to the optimum (x,y,z) up to very rare cases ($d_i(x^{(k)},y^{(k)},z^{(k)}) = 0$ for some k).

As starting values $(x^{(0)},y^{(0)},z^{(0)})$ one takes some point in the convex hull of the given points, preferably the mean $(\overline{x},\overline{y},\overline{z})$ of the given points, i.e. $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and so on. Now the mean is the solution of

$$G(x,y,z) = \sum_{i=1}^{n} e_i(x,y,z) \longrightarrow \min\;. \tag{10}$$

This is the reason why we will later consider (10) and (4) subsequently and therefore the solution of (10) normally gives a good approximation for the solution of (4) at least in the above case.

Now we add as a side condition to the minimization problems (10) and (4) that an optimal location point must lie on some given parametric curve or surface, i.e. either on

$$x = f(t),\ y = g(t),\ z = h(t),\ t \in [t_1, t_2] \tag{11}$$

or on

$$x = f(u,v),\ y = g(u,v),\ z = h(uv),\ u \in [u_1,u_2],\ v \in [v_1,v_2]\ . \tag{12}$$

Inserting (11) into (10) and (4), we get

$$G(t) = \sum_{i=1}^{n} e_i\big(f(t), g(t), h(t)\big) \longrightarrow \min\ , \tag{13}$$

$$F(t) = \sum_{i=1}^{n} d_i\big(f(t), g(t), h(t)\big) \longrightarrow \min \tag{14}$$

and inserting (12) into (10) and (4), we get

$$G(u,v) = \sum_{i=1}^{n} e_i\big(f(u,v), g(u,v), h(u,v)\big) \longrightarrow \min\ , \tag{15}$$

$$F(u,v) = \sum_{i=1}^{n} d_i\big(f(u,v), g(u,v), h(u,v)\big) \longrightarrow \min\ . \tag{16}$$

Instead of three variables (x,y,z) we now have just one variable t in (13) and (14) and just two variables (u,v) in (15) and (16). We will now write down the necessary conditions for an optimum for (14) and (16); because of (6) and (7) those for (13) and (15) are received by replacing $d_i(t)$ or $d_i(u,v)$ by 1, respectively.

For (14) this condition is

$$\sum_{i=1}^{n} \frac{f'(t)\big(f(t)-x_i\big)+g'(t)\big(g(t)-y_i\big)+h'(t)\big(h(t)-z_i\big)}{d_i(t)} = 0\ , \tag{17}$$

where we wrote $d_i(t)$ instead of $d_i\big(f(t), g(t), h(t)\big)$, and for (16) those conditions are

$$\sum_{i=1}^{n} \frac{\frac{\partial f}{\partial u}(u,v)\big(f(u,v)-x_i\big)+\frac{\partial g}{\partial u}(u,v)\big(g(u,v)-y_i\big)+\frac{\partial h}{\partial u}\big(h(u,v)-z_i\big)}{d_i(u,v)} = 0 \tag{18}$$

and

$$\sum_{i=1}^{n} \frac{\frac{\partial f}{\partial v}(u,v)\big(f(u,v)-x_i\big)+\frac{\partial g}{\partial v}(u,v)\big(g(u,v)-y_i\big)+\frac{\partial h}{\partial v}\big(h(u,v)-z_i\big)}{d_i(u,v)} = 0 \tag{19}$$

where we wrote $d_i(u,v)$ instead of $d_i\big(f(u,v), g(u,v), h(u,v)\big)$. As neither (17) nor (18) plus (19) can generally be solved, we will discuss some special cases in the following sections. For (13) and (15) equations (17) or (18) plus (19) with $d_i = 1$ can be solved directly for the straight line, the plane, the sphere and some rotated circle in space; modified Weiszfeld iteration methods based on (17) or (18) plus (19) will be developed for (14) and (16) in the case of those special curves and surfaces; convergence proofs are omitted because they ought to be made the same way as for (9).

2. The straight line in space

A general straight line in space is given by

$$x = f(t) = a + pt,\ y = g(t) = b + qt,\ z = h(t) = c + rt\ , \tag{20}$$

where (a, b, c) and (p, q, r) are prescribed by $p^2 + q^2 + r^2 \neq 0$. Condition (17) transformed on fixpoint form like (8) turns out to be

$$t = \frac{p \sum_{i=1}^{n} \frac{x_i - a}{d_i(t)} + q \sum_{i=1}^{n} \frac{y_i - b}{d_i(t)} + r \sum_{i=1}^{n} \frac{z_i - c}{d_i(t)}}{(p^2 + q^2 + r^2) \sum_{i=1}^{n} \frac{1}{d_i(t)}} \tag{21}$$

and the Weiszfeld iteration consists of writing $t = t^{(k)}$ on the right-hand side and $t = t^{(k+1)}$ on the left-hand side of (21). The explicit solution of (13) is obtained via (21) by putting $d_i(t) = 1$ and can and will be used as a starting value for the Weiszfeld iteration.

Here and in the following sections we will use the following artificial values for the given data points (1):

$$\begin{array}{c|ccccccc} x_i & 2 & 3 & 4 & 2 & 5 & 6 & 4 \\ y_1 & 8 & 7 & 3 & 2 & 4 & 2 & 3 \\ z_i & 2 & 4 & 5 & 6 & 3 & 5 & 8 \end{array} \tag{22}$$

Also k always denotes the iteration number as in (9). For $k = 0$ the results for t or (u, v) are for the objective function G, i.e. for (10) or (15) which are used as starting values for the objective function F, i.e. for (4) or (16), respectively. The iterates for the optimal location on the curve or surface are (x, y, z).

In the case of a straight line (20) and with given $(a, b, c) = (0, 1, 2)$ and $(p, q, r) = (3, 2, 1)$ we got for the data (22)

k	t	G or F	x	y	z
0	1.4388	86.1327	4.316	3.878	3.439
1	1.4790	22.5298	4.437	3.958	3.479
2	1.4846	22.5290	4.454	3.969	3.485
3	1.4853	22.5289	4.456	3.971	3.485
4	1.4854	22.5289	4.456	3.971	3.485
5	1.4854	22.5289	4.456	3.971	3.485

The small number of iterations is typical of a large number of calculated examples, also in the following sections.

3. The plane

A general plane in space is given by

$$\begin{aligned} x &= f(u,v) = a + p_1 u + p_2 v\,, \\ y &= g(u,v) = b + q_1 u + q_2 v\,, \\ z &= h(u,v) = c + r_1 u + r_2 v\,, \\ & p_1^2 + q_1^2 + r_1^2 > 0,\ p_2^2 + q_2^2 + r_2^2 > 0\,. \end{aligned} \tag{23}$$

Equations (18) and (19) may be written as

$$(p_1^2 + q_1^2 + r_1^2)u + (p_1 p_2 + q_1 q_2 + r_1 r_2)v = \frac{\sum\limits_{i=1}^{n} \frac{p_1(x_i-a)+q_1(y_i-b)+r_1(z_i-c)}{d_i(u,v)}}{\sum\limits_{i=1}^{n} \frac{1}{d_i(u,v)}}\,, \tag{24}$$

$$(p_1 p_2 + q_1 q_2 + r_1 r_2)u + (p_2^2 + q_2^2 + r_2^2)v = \frac{\sum\limits_{i=1}^{n} \frac{p_2(x_i-a)+q_2(y_i-a)+r_2(z_i-c)}{d_i(u,v)}}{\sum\limits_{i=1}^{n} \frac{1}{d_i(u,v)}}\,. \tag{25}$$

Matrix A in the linear left-hand part of (24) and (25), i.e.

$$A = \begin{pmatrix} p_1^2 + q_1^2 + r_1^2 & p_1 p_2 + q_1 q_2 + r_1 r_2 \\ p_1 p_2 + q_2 q_2 + r_1 r_2 & p_2^2 + q_2^2 + r_2^2 \end{pmatrix} \tag{26}$$

is positive definite because $p_1^2 + q_1^2 + r_1^2 > 0$ (else (23) would be a straight line) and because

$$\det A = (p_1 q_2 - p_2 q_2)^2 + (p_1 r_2 - p_2 r_1)^2 + (r_1 q_2 - r_2 q_1)^2 > 0\,. \tag{27}$$

Thus, problem (15), where $d_i(u,v)$ within (24) and (25) has to be replaced by 1, has a unique solution (u,v) that can easily be calculated because explicitly

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} p_2^2 + q_2^2 + r_2^2 & -(p_1 p_2 + q_1 q_2 + r_1 r_2) \\ -(p_1 p_2 + q_1 q_2 + r_1 r_2) & p_1^2 + q_1^2 + r_1^2 \end{pmatrix}\,. \tag{28}$$

The optimum (x,y,z) is given by inserting (u,v) into (23). In the case of (16) we define a Weiszfeld-like iteration by

$$\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = A^{-1} \begin{pmatrix} w_1(u^{(k)}, v^{(k)}) \\ w_2(u^{(k)}, v^{(k)}) \end{pmatrix}\,, \quad k = 0,1,2\ldots \tag{29}$$

where w_1 and w_2 are the right-hand sides in (24) and (25). The above results (u,v) for (15) can and will be used as starting values $(u^{(0)}, v^{(0)})$ for (29).

Using again the data (22) and $(a, b, c) = (2, 5, 0)$, $(p_1, q_1, r_1) = (2, 5, 3)$, and $(p_2, q_2, r_2) = (-3, -2, 4)$, we obtain by this way

k	u	v	G or F	x	y	z
0	0.4116	0.5888	151.1314	1.057	5.880	3.590
1	0.4551	0.5166	30.6833	1.360	6.242	3.432
2	0.4710	0.4913	30.6413	1.468	6.373	3.378
3	0.4771	0.4820	30.6354	1.508	6.421	3.359
4	0.4794	0.4785	30.6345	1.523	6.440	3.352
5	0.4802	0.4772	30.6344	1.529	6.447	3.350
6	0.4805	0.4767	30.6344	1.531	6.449	3.349
7	0.4807	0.4766	30.6344	1.532	6.450	3.348
8	0.4807	0.4765	30.6344	1.532	6.451	3.348

4. The sphere

A sphere is given in its parametric form by

$$\begin{aligned} x &= f(u,v) = a + r\cos u \sin v\,, \\ y &= g(u,v) = b + r\sin u \sin v\,, \\ z &= h(u,v) = c + r\cos v\,, \end{aligned} \tag{30}$$

where (a, b, c) is its center and $r \neq 0$ its radius. Here we will at first consider the minimization of G (15). In this case within equations (18) and (19) $d_i(u, v)$ has to be replaced by 1 and will give in our special case (30)

$$r\sin v\left(-\sin u\sum_{i=1}^{n}(a-x_i)+\cos u\sum_{i=1}^{n}(b-y_i)\right)=0\,, \tag{31}$$

$$r\cos v\left(\cos u\sum_{i=1}^{n}(a-x_i)+\sin u\sum_{i=1}^{n}(b-y_i)\right)-r\sin v\sum_{i=1}^{n}(c-z_i)=0\,. \tag{32}$$

The case of $\sin v = 0$, i.e. $v = 0$ or $v = \pi$, in (31) is not relevant because $G(u, 0)$ and $G(u, \pi)$ (later also $F(u, 0)$ and $F(u, \pi)$) are constant. Thus, (31) gives

$$tg\,u = \frac{\sum_{i=1}^{n}(b-y_i)}{\sum_{i=1}^{n}(a-x_i)} \tag{33}$$

with two solutions u and $u + \pi$. With those values (32) gives

$$tg\,v = \frac{\cos u\sum_{i=1}^{n}(a-x_i)+\sin u\sum_{i=1}^{n}(b-y_i)}{\sum_{i=1}^{n}(c-z_i)} \tag{34}$$

with corresponding solutions v and $-v$. But as

$$G(u,v) = G(u+\pi,-v),\ G(u+\pi,v) = G(u,-v) \tag{35}$$

(also valid for F), we need to consider out of four possible combinations just two: (u,v) and $(u,-v)$. The pair (u,v) will be the global minimum unless $G(u,-v) < G(u,v)$ when it will be $(u,-v)$.

For the function F (16) in all the equations (31), (32), (33), and (34) each term within all sums has to be divided by $d_i(u,v)$. Thus, (33) and (34) will read

$$tg\,u = \frac{\sum_{i=1}^{n} \frac{(b-y_i)}{d_i(u,v)}}{\sum_{i=1}^{n} \frac{(a-x_i)}{d_i(u,v)}}\ , \tag{36}$$

$$tg\,v = \frac{\cos u \sum_{i=1}^{n} \frac{(a-x_i)}{d_i(u,v)} + \sin u \sum_{i=1}^{n} \frac{(b-y_i)}{d_i(u,v)}}{\sum_{i=1}^{n} \frac{(c-z_i)}{d_i(u,v)}} \tag{37}$$

These two formulas indicate again a canonical Weiszfeld-like fixpoint method by putting $u = u^{(k+1)}$, $v = v^{(k+1)}$ on the left-hand sides and putting $u = u^{(k)}$, $v = v^{(k)}$ on the right-hand sides of (36) and (37). But now some modification is necessary. We have to choose at each iteration $(u^{(k+1)}, v^{(k+1)})$ unless $F(u^{(k+1)}, -v^{(k+1)}) < F(u^{(k+1)}, v^{(k+1)})$ when $(u^{(k+1)}, -v^{(k+1)})$ has to be chosen.

For the data (22), $(a,b,c) = (2,0,1)$ and radius $r = 3$ we obtained

k	u	v	G or F	x	y	z
0	1.1785	0.8790	127.4678	2.883	2.135	2.914
1	1.1108	0.8441	28.5061	2.995	2.009	2.993
2	1.1041	0.8386	28.5051	3.004	1.992	3.006
3	1.1034	0.8377	28.5051	3.004	1.990	3.007
4	1.1033	0.8376	28.5051	3.004	1.990	3.008

5. The circle in space

A circle in space with center (a,b,c) and so far parallel to the $x-y$ plane is given by

$$\begin{aligned} x &= f(t) = a + r\cos t\,, \\ y &= g(t) = b + r\sin t\,, \\ x &= h(t) = c,\ 0 \le t < 2\pi,\ r \ne 0\,. \end{aligned} \tag{38}$$

The general form of a circle in space is then described by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A(\beta)^T B(\gamma)^T \begin{pmatrix} a + r\cos t \\ b + r\sin t \\ c \end{pmatrix}\ , \tag{39}$$

where $B(\gamma)^T$ rotates (38) in the $y-z$ plane and where $A(\beta)^T$ rotates the result in the $x-z$ plane. For given angles β and γ these rotation matrices are

$$A(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & \sigma \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} , \tag{40}$$

$$B(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} . \tag{41}$$

If we rotate our given data points $\boldsymbol{x_i} = (x_i, y_i, z_i)$ by $B(\gamma)$ and then by $A(\beta)$, i.e., if we put

$$\widehat{\boldsymbol{x}}_{\boldsymbol{i}} = B(\gamma)\,\boldsymbol{x_i},\ \widetilde{\boldsymbol{x}}_{\boldsymbol{i}} = A(\beta)\,\widehat{\boldsymbol{x}}_{\boldsymbol{i}} \quad (i = 1, \dots, n) , \tag{42}$$

with $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}} = (\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i)$, then

$$d_i(t) = \sqrt{(a + r\cos t - \widetilde{x}_i)^2 + (b + r\sin t - \widetilde{y}_i)^2 + (c - \widetilde{z}_i)^2} \tag{43}$$

and (13) and (14) using (43) are defined. Hence, besides (1), $a, b, c, r \neq 0, \beta, \gamma$ are given and t is to be determined such that either (13) or (14) is minimized. For (13) we get

$$tg\, t = \frac{\sum_{i=1}^{n} (b - \widetilde{y}_i)}{\sum_{i=1}^{n} (a - \widetilde{x}_i)} \tag{44}$$

and for (14) we obtain

$$tg\, t = \frac{\sum_{i=1}^{n} \frac{(b-\widetilde{y}_i)}{d_i(t)}}{\sum_{i=1}^{n} \frac{(a-\widetilde{x}_i)}{d_i(t)}} \tag{45}$$

when $d_i(t)$ is defined by (43).

For (44) either t or $t+\pi$ is the desired solution, i.e. that one for which G is smaller. Introducing the Weiszfeld iteration for (45) like for (36) at each iteration, it has to be decided whether t or $t+\pi$ makes F smaller. The final t in both cases has to be inserted into (39) to get the value for the optimal (x, y, z).

Again for the data (22), $(a, b, c) = (0, 1, 2)$, $r = 3$, $\beta = 2$, $\gamma = 1$ we get

k	t	G or F	x	y	z
0	2.0027	298.0180	−3.310	0.329	−2.896
1	2.0252	45.0826	−3.262	0.314	−2.941
2	2.0269	45.0826	−3.259	0.313	−2.945
3	2.0270	45.0826	−3.258	0.312	−2.945
4	2.0270	45.0826	−3.258	0.312	−2.945

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