Least squares estimation of regression coefficients of singular random fields observed on a sphere*

V. V. ANH†, N. N. LEONENKO‡ AND M. M. SHARAPOV§

Abstract. *We present some results on the rate of convergence to the normal law of the least square estimates of the regression coefficient of random fields with long range dependence observed on a sphere.*

Key words: *long-range dependence, random fields with singular spectrum, regression on a sphere*

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1. Introduction

This paper investigates regression models for homogeneous and isotropic random fields observed on a sphere of an increasing radius. This problem was considered by Yadrenko (1983), Ivanov and Leonenko (1989), and others. We consider regression models with singular random noise (or noise field with long-range dependence (LRD)).

A continuous-parameter homogeneous isotropic random field is said to be singular or with LRD if its covariances decrease to zero at infinity but their integral diverges. An alternative definition is available via Tauberian-Abelian theorems, which requires the spectral density to be unbounded at the origin (see Leonenko (1999)). Such processes and fields arise in hydrology, meteorology, turbulence theory, etc. See, e.g., Beran (1994) and the references therein.

Statistical problems for discrete-parameter singular random fields were studied by many authors. Beran (1994) and Leonenko(1999) contain an extensive bibliography of the subject. In particular, Samarov and Taqqu (1988), Yajima (1988, 1991), Künsch, Beran and Hampel (1993), Dahlhaus (1995), Robinson and Hidalgo (1997)

*Partially supported by the Australian Research Council grant A69804041

†Centre in Statistical Science and Industrial Mathematics, Queensland University of Technology, GPO Box 2434, Brisbane, Q 4001, Australia, e-mail: `v.anh@fsc.qut.edu.au`

‡School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF2 4HY, UK, and Department of Mathematics, Kyiv University (National), Volodimirska str., 64, 252601 Kyiv, Ukraine, e-mail: `leonenkon@Cardiff.ac.uk`

§Department of Cybernetics, Kyiv University (National), Volodimirska str., 64, 252601 Kyiv, Ukraine

considered regression models with LRD errors in discrete time. These results indicate that the efficiency of least squares estimates (LSE) for the coefficients of linear regression with long memory errors may still be very good in comparison with the efficiency of LSE for coefficients of linear regression with random noise which has zeroes in the spectrum (see Adenstedt (1974), Holevo (1976), Rasulov (1976)). The works of Yajima (1988, 1991) contain some central limit theorems (under conditions on cumulants of all orders) for LSE of regression coefficients of long-memory time series. Yajima (1988, 1991) also obtained a formula for the asymptotic covariance matrix of LSE and generalized LSE, and conditions under which they have equal asymptotic efficiency. Dahlhaus (1995) proved the asymptotic normality of generalized LSE. Künsch, Beran and Hampel (1993) discussed the effect of long-memory errors on standard independence-based inference rules in the context of certain experimental designs. Robinson and Hidalgo (1997) proved the central limit theorem for the time series regression estimates which include generalized LSE, in the presence of long-memory processes in both errors and stochastic regressors. Koul (1992), Koul and Mukherjee (1993, 1994) considered the asymptotic properties of various robust estimates of regression coefficients. A theory of semiparametric / nonparametric regression with discontinuities (eg. change points) and LRD errors was developed in Anh *et al.* (1999) and Gao and Anh (1999). Leonenko and Sharapov (1999) presented Gaussian and non-Gaussian limit distributions of the LSE of regression coefficients of long-memory time series.

Statistical problems for continuous-parameter random processes and fields with LRD were studied by Ivanov and Leonenko (1989), Chambers (1996), Comte (1996), Leonenko and Šilac-Benšić (1996), Leonenko and Benšić (1998), among others. In particular, Leonenko and Šilac-Benšić (1996) and Leonenko and Benšić (1998) presented the Gaussian and non-Gaussian limit distributions of LSE of regression coefficients of long memory random fields with the continuous parameter. These results were obtained using the methods of Dobrushin and Major (1979) and Taqqu (1979).

Statistical problems for singular random fields observed on a sphere are also of interest. In this paper we present some results on the rate of convergence to the normal law of the LSE of regression coefficients of random fields with LRD observed on a sphere. We use the method proposed by Leonenko (1988) (see also Ivanov and Leonenko (1989), p.64-70).

The problem of estimatiing the LRD exponent (or Hurst's parameter) will be considered elsewhere. Some simple cases have been considered in Beran (1994), Robinson (1995), Comte (1996), Hall, Koul and Turlach (1997), among others.

2. On the efficiency of LSE

Let $\mathbb{R}^n$, $n \geq 2$, be an n-dimensional Euclidean space, $s(r) = \{x \in \mathbb{R} : |x| = r\}$ an a sphere of radius $r > 0$, (ρ, u), $\rho \geq 0$, $u \in s(1)$, the spherical coordinates of point $x \in \mathbb{R}^n$, $\sigma(x)$ the Lebesgue measure on sphere (see, Yadrenko (1983), Leonenko (1999)).

Let (Ω, F, P) be a complete probability space and

$$\eta(\omega, x) = \eta(x) : \Omega \times \mathbb{R}^n \to \mathbb{R}$$

a measurable mean-square continuous random field.
A. Suppose the random field

$$\xi(x) = ag(x) + \eta(x),$$

or in spherical coordinates

$$\tilde{\xi}(\rho, u) = a\tilde{g}(\rho, u) + \tilde{\eta}(\rho, u),$$

is observed on a sphere $s(r) \subset \mathbb{R}^n$, where $g(x) = \tilde{g}(\rho, u)$ is a known function, a is an unknown parameter and $\eta(x) = \tilde{\eta}(\rho, u)$ is a homogeneous isotropic mean-square continuous random field with $E\tilde{\eta}(\rho, u) = 0$ and covariance function

$$B(|x - y|) = E\eta(x)\eta(y)$$

or in spherical coordinates

$$\tilde{B}(\rho) = E\tilde{\eta}(\rho_1, u_1)\tilde{\eta}(\rho_2, u_2),$$

where

$$\rho = |x - y| = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\gamma}, \quad \cos\gamma = \frac{< x, y >}{|x| \cdot |y|},$$

$x = (\rho_1, u_1)$, $\rho \geq 0$, $u_1 \in s(1)$, $y = (\rho_2, u_2)$, $\rho_2 > 0$, $u_2 \in s(1)$.

It is known (see Yadrenko (1983)) that the covariance function $\tilde{B}(\rho)$ has the spectral representation

$$B(|x|) = \int_{\mathbb{R}^n} e^{i<\lambda,x>} F(d\lambda)$$

or in spherical coordinates

$$\tilde{B}(\rho) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right) \int_0^\infty \frac{J_{(n-2)/2}(\lambda\rho)}{(\lambda\rho)^{(n-2)/2}} dG(\lambda), \tag{2.1}$$

where $J_v(z)$ is the Bessel function of the first kind of order $v > -1/2$ and $F(\cdot)$ is a spectral measure of the field $\eta(x)$, or $G(\lambda)$ is a bounded nondecreasing function (the spectral function) such that $G(\lambda) = \int_{|u|<\lambda} F(du)$.

Later the following function will be important:

$$b_m(r) = 2^{n-1}\Gamma\left(\frac{n}{2}\right)\pi^{n/2} \int_0^\infty J^2_{m+\frac{n-2}{2}}(\lambda r)(\lambda r)^{2-n} dG(\lambda), \quad m = 0, 1, 2, ...$$

For example, if for some $\gamma \in (-1, n - 2)$

$$dG(\lambda) = |s(1)|\lambda^\gamma h(\lambda)d\lambda,$$

where $h(\lambda)$ is continuous in a neighborhood of zero, $h(0) \neq 0$, $h(\lambda)$ is bounded on $[0, \infty)$, then (see Ivanov and Leonenko (1989), p.136)

$$b_m(r) = (2\pi)^n h(0)c_1(n, m, \gamma)|s(1)|r^{-1-\gamma}(1 + o(1))$$

as $r \to \infty$, where $|s(1)|$ is the area of surface of the unit sphere $s(1)$, and

$$c_1(n,m,\gamma) = \frac{\Gamma\left(n-2-\gamma\right)\Gamma\left(\frac{2m+\gamma+1}{2}\right)}{2^{n-2-\gamma}\Gamma^2\left(\frac{n-1-\gamma}{2}\right)\Gamma\left(\frac{2m+2n-\gamma-3}{2}\right)}.$$

Let $S_m^l(u)$, $u \in s(1)$ be a real orthogonal spherical harmonics of degree $m \in \{0,1,2,...\}$, $l = 1,2,...,h(n,m)$ (see for example Müller (1966), Leonenko (1999)), where

$$h(n,m) = (2m+n-2)\frac{(m+n-3)!}{(n-2)!m!}$$

is a number of such harmonics.

B. Suppose $g(x) \in L_2(s(r),\sigma(dx))$, and

$$\int\limits_{s(r)} g^2(x)\sigma(dx) < \infty$$

for every fixed $r > 0$.

We introduce Fourier coefficients

$$g_m^l(r) = \int\limits_{s(1)} \tilde{g}(r,u)S_m^l(u)\sigma(du).$$

We consider the LSE for the unknown parameter a, i.e., the value of a which minimizes the functional

$$Q(a) = \int\limits_{s(r)} [\xi(x) - ag(x)]^2\sigma(dx).$$

The LSE has the following simple form

$$\hat{a}_r = \int\limits_{s(r)} \xi(x)g(x)\sigma(dx) \Big/ \int\limits_{s(r)} g^2(x)\sigma(dx). \tag{2.2}$$

In view of the spectral decomposition of the covariance function (2.1) and the additional theorem for the Bessel function (see, for example, Yadrenko (1983)):

$$2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)J_{(n-2)/2}\left(\sqrt{r_1^2+r_2^2-2r_1r_2\cos\gamma}\right)\left(r_1^2+r_2^2-2r_1r_2\cos\gamma\right)^{(2-n)/n}$$

$$= c_1^2(n)\sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} S_m^l(u_1)S_m^l(u_2)J_{m+\frac{n-2}{2}}(r_1)J_{m+\frac{n-2}{2}}(r_2)(r_1r_2)^{(2-n)/2},$$

where

$$c_1^2(n) = 2^{n-1}\Gamma\left(\frac{n}{2}\right)\pi^{n/2},$$

we obtain (see Ivanov and Leonenko (1989), p.135-136)

$$var\ \hat{a}_r = \frac{\int\limits_{s(r)}\int\limits_{s(r)} g(x)g(y)B(|x-y|)\sigma(dx)\sigma(dy)}{\left(\int\limits_{s(r)} g^2(x)\sigma(dx)\right)^2}$$

$$= c_1^2(n)\sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} A_{m,l}(r)b_m(r) / \sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} \left[g_m^l(r)\right]^2,$$

where

$$A_{m,l}(r) = \frac{\left[g_m^l(r)\right]^2}{\sum\limits_{m=0}^{\infty}\sum\limits_{l=1}^{h(n,m)} \left[g_m^l(r)\right]^2}.$$

Clearly,

$$A_{m,l}(r) \geq 0,\ \sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} A_{m,l}(r) = 1$$

for every $r > 0$. Thus for every r a discrete probability distribution $\{A_{m,l}(r)\}$, $(m,l) \in T = \{0,1,...\} \times \{1,2,...,h(n,m)\}$, is defined.

A theorem of Yadrenko (1983), p. 171, shows that the best linear unbiased estimate (BLUE) a_r^* of parameter a based on the observation $\tilde{\xi}(\rho,u)$ is of the form

$$a_r^* = \int\limits_{s(1)} l(r,u)\tilde{\xi}(r,u)\sigma(du) / \sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} \frac{\left[g_m^l(r)\right]^2}{b_m(r)},$$

where the optimal weight function

$$l(r,u) = \sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} g_m^l(u)S_m^l(u)/b_m(r).$$

Under assumptions A, B and if

$$\sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} \left[g_m^l(r)\right]^2/b_m(r) < \infty,$$

the variance of the estimate a_r^* has the form

$$var\ a_r^* = c_1^2(n) / \left[\sum_{m=0}^{\infty}\sum_{l=1}^{h(n,m)} \frac{\left[g_m^l(r)\right]^2}{b_m(r)}\right].$$

For every r a discrete probability distribution $\{A_{m,l}(r)\}$, $(m,l) \in T = \{0,1,...\} \times \{1,2,...,h(n,m)\}$ is defined.

Then the efficiency of LSE

$$eff(\hat{a}_r, a_r^*) = var\ a_r^*/var\ \hat{a}_r$$

$$= \left[\left(\sum_{m=0}^{\infty} \sum_{l=1}^{h(n,m)} A_{m,l}(r) b_m(r) \right) \left(\sum_{m=0}^{\infty} \sum_{l=1}^{h(n,m)} \frac{A_{m,l}(r)}{b_m(r)} \right) \right]^{-1}.$$

Suppose that the discrete probability distribution $\{A_{m,l}(r)\}$, $(m, l) \in T$, is concentrated only at point $(m_0, l_0) \in T$ for all $r > 0$ (that is, $A_{m_0,l_0} = 1$ and the rest $A_{m,l}(r) = 0$). Then $eff(\hat{a}_r, a_r^*) = 1$ for every covariance function $B(\cdot)$ and every $r > 0$.

In particular, if $\tilde{g}(\rho, u) = \tilde{p}(\rho)$ is a radial function, then $g_0^l(r) = \tilde{p}(r)/\sqrt{|s(1)|}$ and the remainder $g_m^l(r)$ vanishes. Hence $A_{0,1}(r) = 1$, then the remainder $A_{m,l}(r) = 0$ and the discrete probability distribution is concentrated on one point $(m_0, l_0) = (0, 1)$.

Or if $\tilde{g}(r, u) = S_{m_0}^{l_0}(u)$, $u \in s(1)$, then by the orthogonal property of special harmonics $A_{m_0}^{l_0} = 1$, the remainder $A_m^l(r) = 0$. Then the discrete probability distribution is concentrated on one point $(m_0, l_0) \in T$. These results (see also Leonenko (1999), pp. 335-340) are in contrast with the corresponding results for random processes observed on sets like $[-r, r]$ or $[0, r]$ (see Grenander and Rosenblatt (1984), Ibragimov and Rozanov (1978), among others).

3. Main results

In what follows we need some extra assumptions.

C. Let $\eta(x) = G(\varepsilon(x))$, where $\varepsilon(x)$, $x \in \mathbb{R}^n$, $n \geq 2$, is a homogeneous isotropic Gaussian mean square continuous random field with $E\varepsilon(x) = 0$ and covariance function

$$R(|x|) = (1 + |x|^2)^{-\alpha/2}, \quad 0 < \alpha < n,$$

where $G(u)$ is a non-random function such that $EG(\varepsilon(0)) = 0$, $EG^2(\varepsilon(0)) < \infty$.

Note that under assumption C

$$\int_{\mathbb{R}^n} R(x) dx = \infty$$

and there exists a spectral density $f_\alpha(|u|)$, $u \in \mathbb{R}^n$ of the form (see Leonenko (1999), p.67)

$$f_\alpha(|u|) = \left[\pi^{n/2} 2^{(\alpha-n)/2} \Gamma\left(\frac{\alpha}{2}\right) \right]^{-1} K_{(n-\alpha)/2}(|u|) |u|^{(\alpha-n)/2}$$

$$= \left[\Gamma\left(\frac{n-\alpha}{2}\right) / 2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right) \right] |u|^{\alpha-n} (1 - \theta(|u|)), \quad 0 < \alpha < n,$$

where $\theta(|u|) \to 0$ as $|u| \to 0$, and

$$K_\nu(z) = \frac{1}{2} \int_0^\infty s^{\nu-1} e^{-\frac{1}{2} z \left(s + \frac{1}{s}\right)} ds$$

is the Bessel function of the third kind of order ν (or McDonald's function). Note that $K_\nu(z) \sim \Gamma(\nu)2^{\nu-1}z^{-\nu}$, $z \to 0$, $\nu > 0$ and for large values of z we have

$$K_\nu(z) = \left(\frac{\pi}{2}\right)^{1/2} z^{-1/2}e^{-z}\left(1 + \frac{4\nu^2-1}{8z} + \dots\right).$$

Let $H_m(u) = (-1)^m \exp\{u^2/2\}d^m/du^m \exp\{-u^2/2\}$, $u \in \mathbb{R}$, $m = 0,1,2,\dots$, be the Chebyshev-Hermite polynomials with the leading coefficient equal to 1. As is well known, they form a complete orthogonal system in the Hilbert space $L_2(\mathbb{R}, \phi(u)du)$, where

$$\phi(u) = \exp\{-u^2\}/\sqrt{2\pi}, u \in \mathbb{R}$$

and

$$EH_m(\xi_1)H_q(\xi_2) = \delta_m^q \rho^m m!, \quad m \geq 0,\ q \geq 0, \tag{3.1}$$

where (ξ_1, ξ_2) is a Gaussian vector with $E\xi_1 = E\xi_2 = 0$, $E\xi_1^2 = E\xi_2^2 = 1$, $E\xi_1\xi_2 = \rho$, δ_m^q is the usual Kronecker symbol.

Under assumption C the function $G(u)$, $u \in \mathbb{R}$, has the following representation in the Hilbert space $L_2(\mathbb{R}, \phi(u)du)$:

$$G(u) = \sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(u), \quad C_k = \int_{-\infty}^{\infty} G(u)H_k(u)\phi(u)du. \tag{3.2}$$

Note that

$$C_0 = EG(\varepsilon(0)) = 0.$$

Under assumption C the LSE has the form

$$\hat{a}_r = a + \int_{s(r)} g(x)G(\varepsilon(x))\sigma(dx) \,/ \int_{s(r)} g^2(x)\sigma(dx). \tag{3.3}$$

It is obvious that $E\hat{a}_r = a$. From (3.1) and (3.3) we obtain the following expression for the variance of the LSE $\hat{a}_r$:

$$\begin{aligned} var\ \hat{a}_r = \left[\int_{s(r)} g^2(x)\sigma(x)\right]^{-2} \int_{s(r)}\int_{s(r)} EG(\varepsilon(x))G(\varepsilon(y)) \times \\ \times g(x)g(y)\sigma(dx)\sigma(dy) = \\ = \sum_{k=1}^{\infty} \left(\frac{C_k^2}{k!}\right) \psi_k^2(r) \,/\, W^4(r), \end{aligned} \tag{3.4}$$

where C_k are defined in (3.2) and

$$\psi_k^2(r) = \frac{C_k^2}{k!} \int_{s(r)}\int_{s(r)} g(x)g(y)R^k(|x-y|)\sigma(dx)\sigma(dy),$$

$$W^2(r) = \int\limits_{s(r)} g^2(x)\sigma(dx) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(n,m)} \left[g_m^l(r)\right]^2.$$

D. Consider the non-negative regression function

$$g(x) = g_{rad}(|x|)\psi\left(\frac{x}{|x|}\right),$$

where $\psi(u)$, $u \in s(1)$, is a continuous function defined on the unit sphere $s(1)$ and $g_{rad}(|x|) = \tilde{g}_{rad}(\rho)$, $\rho = |x|$, is a radial function such that $\tilde{g}_{rad}(|x|) \neq 0$ for $|x| \neq 0$.

Under assumptions C, D

$$W^2(r) = \tilde{g}_{rad}^2(r) r^{n-1} \int\limits_{s(1)} \psi^2(u)\sigma(du)$$

and

$$\begin{aligned}\psi_k^2(r) &= \left(\frac{C_k^2}{k!}\right) \tilde{g}_{rad}^2(r) r^{2(n-1)} \int\limits_{s(1)} \int\limits_{s(1)} \psi(u)\psi(v) R^k(r|u-v|)\sigma(du)\sigma(dv) \\ &= \left(\frac{C_k^2}{k!}\right) \tilde{g}_{rad}^2(r) r^{2(n-1)} \int\limits_{s(1)} \int\limits_{s(1)} \psi(u)\psi(v) \\ &\quad \times \frac{\sigma(du)\sigma(dv)}{(1+r^2|u-v|^2)^{k\alpha/2}}, \quad 0 < \alpha < \frac{n-1}{k}. \end{aligned} \tag{3.5}$$

Let $k = 1$ or $k = 2$ and $C_k \neq 0$. Then under assumptions C and D, using (3.5) and Lebesgue's dominated convergence theorem we obtain

$$\psi_k^2(r) = \left(\frac{C_k^2}{k!}\right) \tilde{g}_{rad}^2(r) r^{-k\alpha} r^{2(n-1)} l_k(\alpha, n)(1 + o(1)) \quad \text{as } r \to \infty,$$

where

$$l_k(\alpha, n) = \int\limits_{s(1)} \int\limits_{s(1)} \psi(u)\psi(v) \frac{\sigma(du)\sigma(dv)}{|u-v|^{k\alpha}}, \quad 0 < \alpha < \frac{n-1}{k}. \tag{3.6}$$

The exact form of constants $l_k(\alpha, n)$ can be found in Ivanov and Leonenko (1989, p.60-61) for the case $\psi(u) \equiv 1$ by using a geometrical probability method.

Thus under conditions C and D with $C_1 \neq 0$, $0 < \alpha < n - 1$,

$$var\ \hat{a}_r = \left[C_1^2 l_1(\alpha) \Big/ \left(\int\limits_{s(1)} \psi^2(u)\sigma(du)\right)^2\right] r^{-\alpha}(1 + o(1)) \quad \text{as } r \to \infty.$$

Let us now consider the random variable

$$\Xi_r = (\hat{a}_r - a) \Big/ \left[\operatorname{sgn}\{C_1\} r^{-\alpha} \int\limits_{s(1)} \int\limits_{s(1)} \frac{\psi(u)\psi(v)}{(r^{-2} + |u-v|^2)^{\alpha/2}} \sigma(du)\sigma(dv)\right].$$

Let X and Y be arbitrary random variables, and consider the uniform (or Kolmogorov's) distance between distributions of random variables X and Y via the formula

$$\mathcal{K}(X,Y) = \sup_{u\in\mathbb{R}} \left| P(X \le u) - P(Y \le u) \right|.$$

Let $\mathcal{N}$ be the standard normal random variable with density function $\phi(u)$.

The main result of this paper describes the rate of convergence to the normal law of the random variables Ξ_r as $r \to \infty$. This result is presented in the following theorem.

Theorem 1. *Suppose that assumptions A, B, C, D hold for* $0 < \alpha < (n-1)/2$, *and*

$$C_1 = \int_{-\infty}^{\infty} uG(u)\phi(u)du \neq 0.$$

Then the following quantity exists:

$$\lim_{r\to\infty} \sup r^{\alpha/3} \mathcal{K}(\Xi_r, \mathcal{N})$$

and is bounded by

$$\frac{3}{2}\left[\frac{l_2(\alpha,n)}{\pi l_1(\alpha,n)}\right]^{1/3} [c(G)]^{1/3},$$

where the constants $l_i(\alpha,n), i = 1,2$ *are defined by (3.6) and*

$$c(G) = C_1^{-2}\left[\int_{-\infty}^{\infty} G^2(u)\phi(u)du - C_1^2\right].$$

4. Proof of the main result

Before proving *Theorem 1*, we mention a result of Petrov (1971).

Lemma 1. *Let X and Y be two arbitrary random variables and* $\mathcal{K}(X,\mathcal{N}) \le K$. *Then for every* $\varepsilon > 0$

$$\mathcal{K}(X+Y,\mathcal{N}) \le K + P\{|Y| > \varepsilon\} + \frac{\varepsilon}{\sqrt{2\pi}}.$$

Proof. See Petrov (1971), p.28. □

Proof of Theorem 1. Expansion (3.2) implies the following expansion in the Hilbert space $L_2(\Omega)$:

$$G(\varepsilon(x)) = \sum_{k=1}^{\infty} \frac{C_k}{k!} H_k(\varepsilon(x)),$$

where C_k's are defined by (3.2).

We now consider random variables

$$\eta_k(r) = \int\limits_{s(r)} g(x) H_k(\varepsilon(x)) \sigma(dx), \quad k = 1, 2, \dots$$

In order to apply *Lemma 1*, we represent Ξ_r as

$$\Xi_r = (X_r + Y_r)/\psi_1(r), \tag{4.1}$$

where

$$X_r = C_1 \int\limits_{s(r)} g(x)\varepsilon(x)\sigma(dx), \qquad Y_r = \sum_{k \geq 2} \frac{C_k}{k!} \eta_k(r).$$

Note that X_r is a Gaussian random variable with $EX_r = 0$ and $varX_r = \psi_1^2(r)$. So we have

$$\mathcal{K}(X_r/\psi_1(r), \mathcal{N}) = 0. \tag{4.2}$$

From assumption C we obtain that

$$\frac{k!}{C_k^2} \psi_k^2(r) \leq \frac{m!}{C_m^2} \psi_m^2(r), \quad 1 \leq m \leq k,$$

and thus, for $0 < \alpha < (n-1)/2$,

$$varY_r = \sum_{k \geq 2} \psi_k^2(r) \leq \int\limits_{s(r)} \int\limits_{s(r)} g(x) g(y) B^2(x-y) \sigma(dx)\sigma(dy) \left[\sum_{k \geq 2} \frac{C_k^2}{k!} \right]$$

$$= r^{2(n-\alpha-1)} \tilde{L}^2(r) \tilde{g}_{rad}^2(r) l_{2,r}(\alpha, n) \sum_{k \geq 2} \frac{C_k^2}{k!}, \tag{4.3}$$

where by assumption E and Lebesgue's dominated convergence theorem

$$l_{2,r}(\alpha, n) = \int\limits_{s(1)} \int\limits_{s(1)} \psi(u)\psi(v) \left(r^{-2} + |u-v|^2 \right)^{-\alpha} \sigma(du)\sigma(dv) \to l_2(\alpha, n) \tag{4.4}$$

as $r \to \infty$ for $0 < \alpha < (n-1)/2$, where $l_2(\alpha, n)$ is defined by (3.6).

Similarly, for $0 < \alpha < n-1$,

$$l_{1,r}(\alpha, n) = \int\limits_{s(1)} \int\limits_{s(1)} \psi(u)\psi(v) \left(r^{-2} + |u-v|^2 \right)^{-\alpha/2} \sigma(du)\sigma(dv) \to l_1(\alpha, n) \tag{4.5}$$

as $r \to \infty$, where $l_1(\alpha, n)$ is defined by (3.6).

From (4.3) and Chebyshev's inequality we obtain that for every $\varepsilon > 0$

$$P\left(Y_r/\psi_1(r) > \varepsilon\right) \leq \varepsilon^{-2} var\left(Y_r/\psi_1(r)\right)$$

$$\leq \varepsilon^{-2}\psi_1^{-2}(r) l_{2,r}(\alpha,n) r^{2(n-\alpha-1)} \tilde{L}^2(r) \tilde{g}_{rad}^2(r) \sum_{k\geq 2} C_k^2/k!$$

$$= \varepsilon^{-2} \frac{1}{r^\alpha} \frac{l_{2,r}(\alpha,n)}{l_{1,r}(\alpha,n)} \left[C_1^{-2} \sum_{k\geq 2} C_k^2/k! \right], \quad 0<\alpha<(n-1)/2. \tag{4.6}$$

Using *Lemma 1* with $X = X_r/\psi_1(r)$ and $Y = Y_r/\psi_1(r)$ we obtain from (4.2) and (4.6) that for any $\varepsilon > 0$

$$K(\Xi_r, \mathcal{N}) \leq \frac{\varepsilon}{\sqrt{2\pi}} + \frac{1}{\varepsilon^2} \frac{1}{r^\alpha} c(G) \frac{l_{2,r}(\alpha,n)}{l_{1,r}(\alpha,n)}, \tag{4.7}$$

where

$$c(G) = C_1^{-2} \sum_{k\geq 2} \frac{C_k^2}{k!} = C_1^{-2} \left[\int_{-\infty}^{\infty} G^2(u)\phi(u)du - C_1^2 \right] < \infty,$$

because by Parseval's equality

$$EG^2(\varepsilon(0)) = \sum_{k\geq 0} \frac{C_k^2}{k!} = \int_{-\infty}^{\infty} G^2(u)\phi(u)du < \infty.$$

In order to minimize the right-hand side of inequality (4.7), set

$$\varepsilon = (1/r^\alpha)^{1/3} (8\pi)^{1/6} \left[c(G) l_{2,r}(\alpha,n)/l_{1,r}(\alpha,n) \right]^{1/3}.$$

Thus we derive the following inequality:

$$K(\Xi_r, \mathcal{N}) \leq (1/r^\alpha)^{1/3} \frac{3}{2} \left(\frac{c(G) l_{2,r}(\alpha,n)}{\pi l_{1,r}(\alpha,n)} \right)^{1/3}. \tag{4.8}$$

From (4.8), (4.4) and (4.5) we obtain the statement of the theorem. □

5. Extensions and generalizations

In view of the results of Illicheva and Leonenko (1995) the asymptotic normality of the normalized LSE takes place for all $\alpha \in (0, n-1)$ (see assumption C and (3.5)) if $C_1 \neq 0$, whereas *Theorem 1* gives the convergence rate to zero of the Kolmogorov distance between the normalized LSE and the normal law only for $\alpha \in (0, (n-1)/2)$. Nevertheless, our method is applicable also to the larger interval $\alpha \in (0, n-1)$ but at the price of a slower convergence rate. For simplicity, we consider the case of radial regression function $g(x) = \tilde{g}_{rad}(|x|)$, $x \in \mathbb{R}^n$.

E. Let $\varepsilon(x)$, $x \in \mathbb{R}^n$, be a real-valued mean-square continuous isotropic Gaussian field with $E\varepsilon(x) = 0$, $E\varepsilon^2(x) = 1$ and covariance function $B(x) = \tilde{B}(|x|) = cov(\varepsilon(0), \varepsilon(x)) \downarrow 0$ as $|x| \to \infty$, and $\eta(x) = G(\varepsilon(x))$, where $EG(\varepsilon(x)) = 0$, $EG^2(\varepsilon(x)) < \infty$, $x \in \mathbb{R}^n$.

F. Suppose that the regression function $g(x) = \tilde{g}_{rad}(|x|)$, $x \in \mathbb{R}^n$, is such that $\tilde{g}(|x|) > 0$ if $|x| \neq 0$, and $\tilde{g}_{rad}(|x|) \leq \tilde{g}_{rad}(|y|)$ for $|x| \leq |y|$.

In contrast to condition C we assume the following condition.

G. There exists $\delta \in (0,1)$ such that

$$\gamma(r,n) = (n-1)r^{(1-n)(1+\delta)} \int\limits_{s(r)} \int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy)$$

$$= (n-1)r^{(n-1)(1-\delta)}\tilde{B}(r) \int\limits_{s(1)} \int\limits_{s(1)} \frac{\tilde{B}(r|u-v|)}{\tilde{B}(r)}\sigma(du)\sigma(dv) \to \infty$$

as $r \to \infty$.

Note that if assumption C holds, then $\gamma(r,n) \to \infty$ as $r \to \infty$. Thus, the random field $\varepsilon(x)$, $x \in \mathbb{R}^n$, satisfying assumption G is a random field with LRD.

Let U_1 and U_2 be two independent random vectors selected in accordance with the uniform law of the sphere $s(r) \subset \mathbb{R}^n$. Then (see, for example, Ivanov and Leonenko (1989), p.29) the density function $f_r(u)$ of the Euclidean distance $|U_1 - U_2|$ between U_1 and U_2 is

$$f_r(u) = \pi^{-1/2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} r^{1-n} u^{n-2} \left[1 - \left(\frac{u}{2r}\right)^2\right]^{\frac{n-3}{2}}, \quad 0 < u < 2r$$

where $\Gamma(\cdot)$ is the gamma function.

Using randomization we obtain for every function $f(|x-y|)$, $x, y \in \mathbb{R}^n$:

$$\int\limits_{s(r)} \int\limits_{s(r)} f(|x-y|)\sigma(dx)\sigma(dy) = |s(r)|^2 Ef(|U_1 - U_2|)$$

$$= r^{2(n-1)}|s(1)|^2 \int\limits_0^{2r} f(u) f_r(u) du$$

$$= q_1(n) r^{n-1} \int\limits_0^{2r} z^{n-2} f(z) \left[1 - \left(\frac{z}{2r}\right)^2\right]^{\frac{n-3}{2}} dz, \tag{5.1}$$

where $q_1(n) = 4\pi^{n-1/2}\Gamma^{-1}\left(\frac{n}{2}\right)\Gamma^{-1}\left(\frac{n-1}{2}\right) = 2^n \pi^{n-1}/(n-2)!$ and $|s(r)| = 2\pi^{n/2}\Gamma^{-1}\left(\frac{n}{2}\right)$ is the area of the unit sphere $s(1)$.

Under assumptions E, F and G as in the proof of *Theorem 1* we represent

$$\Xi_r = (X_r + Y_r)/\psi_1(r),$$

where

$$K(X_r/\psi_1(r), \mathcal{N}) = 0, \quad Y_r = \sum_{k \geq 2} \frac{C_k}{k!} \eta_k(r),$$

$$\eta_k(r) = \tilde{g}_{rad}(r) \int\limits_{s(r)} H_k(\varepsilon(x))\sigma(dx), \quad k = 1, 2, ...$$

$$\psi_1^2(r) = C_1^2 \tilde{g}_{rad}^2(r) \int\limits_{s(r)} \int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy).$$

Let $\chi(A)$ be the indicator function. From (5.1) with $f(|x-y|) = \chi(|x-y| < r^\delta)$ we obtain

$$var\left(\frac{Y_r}{\psi_1(r)}\right) \le \psi_1^{-2}(r)\left(\sum_{k\ge 2}\frac{C_k^2}{k!}\right)\tilde{g}_{rad}^2(r)\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}^2(|x-y|)\sigma(dx)\sigma(dy)$$

$$= \frac{1}{\psi_1^2(r)}\left(\sum_{k\ge 2}\frac{C_k^2}{k!}\right)\tilde{g}_{rad}^2(r)\left[\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}^2(|x-y|)\chi(|x-y| < r^\delta)\sigma(dx)\sigma(dy)\right.$$

$$\left.+ \int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}^2(|x-y|)\chi(|x-y| \ge r^\delta)\sigma(dx)\sigma(dy)\right]$$

$$\le \left[C_1^{-2}\sum_{k\ge 2}\frac{C_k^2}{k!}\right]\left[\frac{\int\limits_{s(r)}\int\limits_{s(r)} \chi(|x-y| < r^\delta)\sigma(dx)\sigma(dy)}{\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy)}\right.$$

$$\left. + sup\{\tilde{B}(z), z \ge r^\delta\}\frac{\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}(|x-y|)\chi(|x-y| \ge r^\delta)\sigma(dx)\sigma(dy)}{\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy)}\right]$$

$$\le c(G)\left[\frac{\int\limits_{s(r)}\int\limits_{s(r)} \chi(|x-y| < r^\delta)\sigma(dx)\sigma(dy)}{\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy)} + \tilde{B}(r^\delta)\right]$$

$$\le c(G)\left[\frac{q_1(n)(1+o(1))}{(n-1)r^{(1-n)(\delta+1)}\int\limits_{s(r)}\int\limits_{s(r)} \tilde{B}(|x-y|)\sigma(dx)\sigma(dy)} + \tilde{B}(r^\delta)\right]$$

$$= c(G)\left[\frac{q_1(n)(1+o(1))}{\gamma(r,n)} + \tilde{B}(r^\delta)\right] \tag{5.2}$$

as $r \to \infty$.

Using *Lemma 1* with $X = X_r/\psi_1(r)$ and $Y = Y_r/\psi_1(r)$ we obtain from (5.2) that for any $\varepsilon > 0$

$$K(\Xi_r, \mathcal{N}) \le \frac{\varepsilon}{\sqrt{2\pi}} + \frac{1}{\varepsilon^2}c(G)\left[\frac{q_1(n)(1+o(1))}{\gamma(r,n)} + \tilde{B}(r^\delta)\right]. \tag{5.3}$$

In order to minimize the right-hand side of (5.3), set

$$\varepsilon = \left[2\sqrt{2\pi}c(G)\right]^{1/3}\left[\frac{q_1(n)(1+o(1))}{\gamma(r,n)}+\tilde{B}(r^\delta)\right]^{1/3}.$$

Thus, from (5.3) we obtain

Theorem 2. *Under assumptions F, G, J with $C_1 \neq 0$ there exists*

$$\lim_{r\to\infty}\sup\left[\frac{q_1(n)}{\gamma(r,n)}+\tilde{B}(r^\delta)\right]^{-1/3} K(\Xi_r,\mathcal{N}) \leq \frac{3}{2}\left(\frac{c(G)}{\pi}\right)^{1/3},$$

where $q_1(n)$ is defined as in (5.1).

6. Conclusion remarks

Suppose we observe $\xi(x)$, $x \in s(r)\chi(x \in A)$, $A \in s(r)$. Then a regression model based on $\xi(x)$ can be reduced to a regression model on spheres by choosing $g(x) = \overline{g}(x)\chi(x \in A)$, $\eta(x) = G(\varepsilon(x))$, where $G(u) = \chi(u \in A)$. Thus, a regression model on spheres is suitable to investigate the statistical inference in *signal plus noise* problems involving observations on a sphere such as the earth, which is a natural example of a sphere with a large radius ($r \to \infty$).

The assumption $C_1 \neq 0$ is satisfied for a large class of sets $A \in s(r)$ with some symmetry properties. If $C_1 = ... = C_{m-1} = 0$, but $C_m \neq 0$ for some integer $m \geq 1$, then limiting distributions of LSE of regression coefficients of the random field observed on a sphere can be found in Ill'icheva and Leonenko (1995). They have a non-Gaussian structure for $m \geq 2$. An extension of *Theorems 1* and *2* of this paper to the case $m \geq 2$ is in progress. Some simple aspects of this case have been dealt with in Leonenko and Anh (2001).

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