Unique symmetric (66,26,10) design admitting an automorphism of order 55

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Abstract. *We have proved that the first known symmetric (66,26,10) design, constructed by Tran van Trung, is up to isomorphism the only symmetric (66,26,10) design admitting an automorphism of order 55. A full automorphism group of that design is isomorphic to* $Frob_{55} \times D_{10}$.

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1. Introduction and preliminaries

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$,

2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,

3. every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Elements of the set $\mathcal{P}$ are called points and elements of the set $\mathcal{B}$ are called blocks.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of $G-$orbits by t, point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. We shall denote points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$, (i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r-1}\}$). Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k \,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir} \gamma_{jr} = \lambda \Omega_j + \delta_{ij} \cdot (k - \lambda) \,. \tag{2}$$

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Definition 1. *The* $(t \times t)$*-matrix* (γ_{ir}) *with entries satisfying properties* (1) *and* (2) *is called the orbit structure for parameters* (v, k, λ) *and orbit distribution* $(\omega_1, \dots, \omega_t)$*,* $(\Omega_1, \dots, \Omega_t)$*.*

The first step of the construction of designs is to find all orbit structures (γ_{ir}) for some parameters and orbit distribution. The next step, called indexing, is to determine for each number γ_{ir} exactly which points from the point orbit $\mathcal{P}_r$ are incident with the representative of the block orbit $\mathcal{B}_i$. Because of a large number of possibilities, it is often necessary to involve a computer in both construction steps.

Definition 2. *The set of indices of points of the orbit* $\mathcal{P}_r$ *indicating which points of* $\mathcal{P}_r$ *are incident with the representative of the block orbit* $\mathcal{B}_i$ *is called the index set for the position* (i, r) *of the orbit structure.*

The first symmetric $(66, 26, 10)$ design is constructed by Tran van Trung (see [8]). A full automorphism group of that design is isomorphic to $Frob_{55} \times D_{10}$. I. Matulić-Bedenić, K. Horvatić-Baldasar and E. Kramer (see [6]) have constructed 18 mutually nonisomorphic symmetric $(66, 26, 10)$ designs with Frobenius automorphism group of order 39. Later on, M.-O. Pavčević and E. Spence (see [7]) have proved that there are up to isomorphism 558 symmetric $(66, 26, 10)$ designs admitting a dihedral automorphism group of order 10 acting with orbit distribution $(1, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$ on the sets of points and blocks, and 22 symmetric $(66, 26, 10)$ designs having an elementary abelian automorphism group of order 25. Finally, it has been proved (see [2]) that there are up to isomorphism three symmetric $(66, 26, 10)$ designs with automorphism group isomorphic to $Frob_{55}$. Among constructed symmetric $(66, 26, 10)$ designs there are 590 mutually nonisomorphic ones.

Our aim is to construct symmetric $(66, 26, 10)$ designs admitting an abelian automorphism group of order 55, which will complete the classification of symmetric $(66, 26, 10)$ designs with an automorphism group of order 55.

2. Z_{55} acting on a symmetric (66,26,10) design

From now on by G we shall denote a cyclic group of order 55 presented as follows:

$$G \cong \langle \rho, \sigma \mid \rho^{11} = 1,\, \sigma^5 = 1,\, \rho^\sigma = 1 \rangle.$$

Let α be an automorphism of a symmetric design. By $F(\alpha)$ we shall denote the number of points fixed by α. In that case, the number of blocks fixed by α is also $F(\alpha)$.

Lemma 1. *Let* ρ *be an automorphism of a symmetric* $(66, 26, 10)$ *design. If* $|\rho| = 11$*, then* $F(\rho) = 0$*.*

Proof. It is known that $F(\rho) < k + \sqrt{k - \lambda}$ and $F(\rho) \equiv v (mod\ |\rho|)$. Therefore, $F(\rho) \in \{0, 11, 22\}$. One cannot construct fixed blocks for $F(\rho) \in \{11, 22\}$. □

Lemma 2. *Let* G *be a cyclic automorphism group of order* 55 *of a symmetric* $(66, 26, 10)$ *design* $\mathcal{D}$*.* G *acts semistandardly on* $\mathcal{D}$ *with orbit distribution* $(11, 55)$*.*

Proof. Group $\langle \rho \rangle$ acts on $\mathcal{D}$ with orbit distribution $(11, 11, 11, 11, 11, 11)$. Since $\langle \rho \rangle \triangleleft G$, σ maps $\langle \rho \rangle$-orbits on $\langle \rho \rangle$-orbits. Therefore, only possibilities for orbit distributions for the group G are $(11, 11, 11, 11, 11, 11)$ and $(11, 55)$. In the case of orbit distribution $(11, 11, 11, 11, 11, 11)$, automorphism σ would act on $\mathcal{D}$ with 66 fixed points (blocks), which is not possible. For orbit distribution $(11, 55)$ one can construct only one orbit structure, namely:

OS	11	55
11	1	25
55	5	21

□

3. Construction of the design

It would be difficult to proceed with indexing of orbit structure OS. For example, there are $\binom{55}{25}$ possibilities for index sets for the position $(1, 2)$ in the OS. Therefore, we shall use a principal series $\langle 1\rangle \triangleleft \langle\rho\rangle \triangleleft G$ of the automorphism group G. Our aim is to find all orbit structures for the group $\langle\rho\rangle$ corresponding to the structure OS. We shall construct designs from those orbit structures for $\langle\rho\rangle$, having in mind the action of the permutation σ on $\langle\rho\rangle$-orbits.

Theorem 1. *Up to isomorphism there is only one symmetric* $(66, 26, 10)$ *design with an automorphism of order* 55*. Full automorphism group of that design is isomorphic to* $Frob_{55} \times D_{10}$ *and its* 2*-rank is* 31*.*

Proof. The only orbit structure for the group $\langle\rho\rangle$ corresponding to the orbit structure OS is

OS1	11	11	11	11	11	11
11	1	5	5	5	5	5
11	5	5	5	5	5	1
11	5	5	5	5	1	5
11	5	5	5	1	5	5
11	5	5	1	5	5	5
11	5	1	5	5	5	5

We shall proceed with indexing of the orbit structure OS1, knowing that σ acts on the set of six $\langle\rho\rangle$-orbits of blocks as permutation $(1)(2, 3, 4, 5, 6)$, and on the set of point orbits as $(1)(2, 6, 5, 4, 3)$. It is sufficient to determine index sets for the first and second row of orbit structure OS1. Also, in the first row we have to determine index sets only for positions $(1, 1)$ and $(1, 2)$. We shall assume that automorphisms ρ and σ act on the set of points as follows:

$$\rho = (I_0, I_1, \dots, I_{10}),\ I = 1, 2, \dots, 6,$$

$$\sigma = (1_i)(2_i, 6_i, 5_i, 4_i, 3_i),\ i = 0, 1, \dots, 10.$$

Because of that action, index sets for positions $(1, 2)$ and $(2, 1)$ have to be basic blocks for symmetric $(11, 5, 2)$ designs. As representatives of the first and second block orbit we shall choose the first blocks in those orbits, with respect to the lexicographical order. Index sets which could occur in designs are:

$$0 = \{0\},\ \dots\ ,\ 10 = \{10\},\ 11 = \{0, 1, 2, 3, 4\},\ \dots\ ,\ 472 = \{6, 7, 8, 9, 10\}.$$

To eliminate isomorphic structures during indexing, we have used automorphisms of the orbit structure OS1 which commute with permutation representation of σ on the set of $\langle\rho\rangle$-orbits, and permutations $\alpha_l, \beta_{1,k}, \beta_{2,k} \in N_S(G)$, $2 \le l \le 10$, $1 \le k \le 10$, which act on the set of points as follows:

$r_i\alpha_l = r_j, j \equiv (i \cdot l)(mod\ 11),\ for\ r = 1, \dots, 6,\ i = 0, 1, \dots, 10,$

$1_i\beta_{1,k} = 1_j, j \equiv (i + k)(mod\ 11),\ for\ i = 0, 1, \dots, 10,$

$r_i\beta_{1,k} = r_i,\ for\ r = 2, \dots, 6,\ i = 0, 1, \dots, 10,$

$1_i\beta_{2,k} = 1_i,\ for\ i = 0, 1, \dots, 10,$

$r_i\beta_{2,k} = r_j, j \equiv (i + k)(mod\ 11),\ for\ r = 2, \dots, 6,\ i = 0, 1, \dots, 10.$

We have constructed up to isomorphism only one design, which is presented in terms of index sets as follows:

0	280	280	280	280	280
20	20	450	450	20	5
20	450	450	20	5	20
20	450	20	5	20	450
20	20	5	20	450	450
20	5	20	450	450	20

Group Z_{55} is a subgroup of the group $Frob_{55} \times D_{10}$. Therefore, it is clear that the constructed design is isomorphic to the one constructed by Tran van Trung with full automorphism group isomorphic to $Frob_{55} \times D_{10}$. By means of the computer program developed by V. Tonchev we have computed that 2-rank of the design is 31. The constructed design is self-dual. □

In [2] we have proved the following theorem:

Theorem 2. *Up to isomorphism there are three symmetric* $(66, 26, 10)$ *designs with automorphism group* $Frob_{55}$*. Let us denote them by* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *and* $\mathcal{D}_3$*. Full automorphism groups of designs* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are isomorphic to* $Frob_{55}$*, and full automorphism group of the design* $\mathcal{D}_3$ *is isomorphic to* $Frob_{55} \times D_{10}$*. Designs* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are dually isomorphic.* $2-$*ranks of these three designs are* 31*.*

Remark 1. *Design* $\mathcal{D}_3$ *is isomorphic to the design constructed by Tran van Trung.*

Thereby we have proved the following statement:

Theorem 3. *Up to isomorphism there are three symmetric* $(66, 26, 10)$ *designs with an automorphism group of order 55, namely the designs from* Theorem 2.

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