Mann iteration for generalized pseudocontractive maps in Hilbert spaces

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Abstract. *If X is a real Hilbert space, B is a nonempty, bounded, convex and closed subset, $T : B \to B$ is a generalized pseudocontraction; then the iteration*

$$\begin{aligned} x_1 &\in B, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ (\alpha_n)_n &\subset (0,1),\ \sum_{n=1}^{\infty} \alpha_n = \infty, \\ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| &< \infty,\ \lim_{n\to\infty} \alpha_n = 0, \end{aligned} \tag{1}$$

strongly converges to the fixed point of T.

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1. Preliminaries

In this note we study the convergence of the Mann iteration process (1) for generalized pseudocontractions. According to [8] the generalized pseudocontractions are more general than the pseudocontractions introduced by Browder.

Definition 1. *[8]. Let X be a Hilbert space, let B be a nonempty subset. A map $T : B \to B$ is said to be a generalized pseudocontraction if for $x, y \in B$ there exists $r > 0$ such that*

$$\langle Tx - Ty, x - y \rangle \leq r \|x - y\|^2 . \tag{2}$$

Clearly, (2) is equivalent to

$$\langle (I-T)x - (I-T)y, x - y \rangle \geq (1-r) \|x - y\|^2 .$$

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The map T is a *strong pseudocontraction* if there exists $k \in (0,1)$ such that for all $x, y \in B$,

$$\langle (I-T)x-(I-T)y, x-y\rangle \geq k\left\Vert x-y\right\Vert^2,$$

see, for example [6]. Remark that both generalized pseudocontractivity and strong pseudocontractivity generalize the pseudocontractivity, but in a different manner. Iteration (1),where T is a strong pseudocontraction in Banach spaces, was studied in [1], [2], [3], [4], [6], [9].

The following lemma can be found in [9] as *Lemma 4*. Also, it can be found in [4] as Lemma 1.2, with another proof. A more general case is in Lemma 2 from [5]. The proof from [5] is similar to the proof of Lemma 4 from [9].

Lemma 1. *[9], [4]. Let $(\rho_n)_n$ be a nonnegative real sequence satisfying*

$$\rho_{n+1} \leq (1-\lambda_n)\rho_n + \sigma_n,$$

where $\lambda_n \in (0,1), \forall n \in N, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \rho_n = 0$.

The normalized duality mapping J is the identity, when X is a Hilbert space, see [4]. Thus Lemma 1.1 from [4] becomes:

Lemma 2. *[4]. If X is a Hilbert space, then*

$$\left\Vert x+y\right\Vert^2 \leq \left\Vert x\right\Vert^2 + 2\left\langle y,(x+y)\right\rangle,$$

for all $x, y \in X$.

The following result is a corollary of Lemma 1 from [7]:

Lemma 3. *[7]. If X is a real Hilbert space, B is a nonempty, bounded, convex and closed subset, and $T : B \to B$ is a generalized pseudocontraction, then the sequence given by* (1) *satisfies*

$$\lim_{n\to\infty} \left\Vert x_{n+1} - x_n\right\Vert = 0.$$

In [7], the map T is nonexpansive. If we consider the proof of Lemma 1 from [7], we see that the result is true, when our assumptions are fulfilled.

2. Main result

We are now able to give the following result:

Theorem 1. *If X is a real Hilbert space, B is a nonempty, bounded, convex and closed subset, and $T : B \to B$ is a generalized pseudocontraction, then the iteration* (1) :

$$\begin{aligned} x_1 &\in B, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ (\alpha_n)_n &\subset (0,1),\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty, \\ \lim_{n\to\infty} \alpha_n &= 0\ . \end{aligned}$$

strongly converges to the fixed point of T.

Proof. Theorem 2.1 from [8] gives us the existence and the uniqueness of the fixed point of T. Let us denote this fixed point by q. Using *Lemma 3* and (2), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(x_n-q)+\alpha_n(Tx_n-q)\|^2 \\
&\leq (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle Tx_n-q,x_{n+1}-q\rangle \\
&= (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle Tx_n-q,x_n-q\rangle+ \\
&\quad +2\alpha_n\langle Tx_n-q,x_{n+1}-x_n\rangle \\
&\leq (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n r\|x_n-q\|^2 \\
&\quad +2\alpha_n\langle Tx_n-q,x_{n+1}-x_n\rangle \\
&\leq [1-\alpha_n(2(1-r)-\alpha_n)]\|x_n-q\|^2 \\
&\quad +2\alpha_n\langle Tx_n-q,x_{n+1}-x_n\rangle .
\end{aligned}$$

Let us denote

$$\begin{aligned}
A_n &: = \langle Tx_n-q,x_{n+1}-x_n\rangle , \\
\lambda_n &: = \alpha_n(2(1-r)-\alpha_n), \\
\rho_n &: = \|x_n-q\|^2, \\
\sigma_n &: = 2\alpha_n A_n.
\end{aligned}$$

Thus, we have

$$\rho_{n+1}\leq(1-\lambda_n)\rho_n+\sigma_n.$$

We observe that

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sigma_n}{\lambda_n} &= \lim_{n\to\infty}\frac{2\alpha_n\langle Tx_n-q,x_{n+1}-x_n\rangle}{\alpha_n(2(1-r)-\alpha_n)} \\
&= 2\lim_{n\to\infty}\frac{\langle Tx_n-q,x_{n+1}-x_n\rangle}{(2(1-r)-\alpha_n)}=0;
\end{aligned}$$

the last equality is true. From *Lemma 4*, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. The sequence $(\|Tx_n-q\|)_n$ is bounded, being in the bounded set B. Hence we have $\lim_{n\to\infty}\langle Tx_n-q,x_{n+1}-x_n\rangle=0$. The assumptions from *Lemma 2* are fulfilled. Hence $\rho_n\to 0$ as $n\to\infty$. Thus $x_n\to q$ as $n\to\infty$. □

A prototype for $(\alpha_n)_n$ is $(1/\sqrt{n})_{n\geq 1}$.

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