Observations on some sequences supplied by inequalities

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Abstract. *The convergence of some sequences supplied by inequalities is used in order to prove the convergence of Ishikawa and Mann iterations. Our purpose in this note is to give some observations on these sequences.*

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1. Introduction

Four Lemmas are needed in [1], [2], [3], [4], [5], [6], [8], [9], [11], [14] for the convergence of (Mann) Ishikawa iteration. In this note we will give new proofs for two of them. Also, we will show that two Lemmas are dependent. We will give new applications.

We need Lemma 1 from [14]. We will give a new proof for it.

Lemma 1. *[14] If $(a_n)_n$, $(b_n)_n$ are two real nonnegative sequences satisfying*

$$\begin{aligned} a_{n+1} &\leq a_n + b_n, \forall n \geq 1, \qquad (1)\\ \sum_{n=1}^{\infty} b_n &< \infty, \end{aligned}$$

then $(a_n)_n$ is convergent.

Proof. Let us denote by

$$S_n = \sum_{k=1}^{n} b_k. \qquad (2)$$

We know $\sum_{n=1}^{\infty} b_n < \infty$. For a fixed $\varepsilon > 0$, there exists n_0 such that for all $n, p \geq 1$ with $p - 1 \geq n$, we have

$$S_{n_0+p-1} - S_{n_0+n} < \varepsilon. \qquad (3)$$

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From (1) we know

$$\begin{aligned} a_{n_0+n+1} &\leq a_{n_0+n} + b_{n_0+n}, \\ a_{n_0+n+2} &\leq a_{n_0+n+1} + b_{n_0+n+1}, \\ &\ldots \\ a_{n_0+p} &\leq a_{n_0+p-1} + b_{n_0+p-1}. \end{aligned}$$

Summing, one obtains

$$\begin{aligned} a_{n_0+p} &\leq a_{n_0+n} + \sum_{k=n}^{p-1} b_{n_0+k} \text{ i.e.} \\ a_{n_0+p} - a_{n_0+n} &\leq S_{n_0+p-1} - S_{n_0+n} < \varepsilon. \end{aligned}$$

Hence $(a_n)_n$ is fundamental. Thus the limit of $(a_n)_n$ exists. □

In the proof from [14] $\overline{\lim}$ and $\underline{\lim}$ are used.

If $(a_n)_n$ from (1) has a subsequence which converges to zero, then $(a_n)_n$ will converge to zero. Let us remark that if $(a_n)_n$ is decreasing, then $(a_n)_n$ is not necessarry convergent to zero. Take $a_n = 1 + \frac{1}{n}, b_n = \frac{1}{n^2}$. The assumptions of *Lemma 1* are verified, but $a_n \to 1$.

The following lemma can be found in [16] as Lemma 4. Also, it can be found in [8] as Lemma 1.2, with another proof.

Lemma 2. *[16], [8] Let $(\Psi_n)_n$ be a nonnegative real sequence satisfying*

$$\Psi_{n+1} \leq (1-\lambda_n)\Psi_n + \sigma_n, \tag{4}$$

where $\lambda_n \in (0,1), \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \Psi_n = 0$.

A very useful result is the following Lemma:

Lemma 3. *[6] Let $(\beta_n)_n$ be recursively generated by*

$$\beta_{n+1} = (1-\delta_n)\beta_n + \sigma_n^2, \tag{5}$$

with $n \geq 1, \beta_1 \geq 0, (\delta_n)_n \subset (0,1)$, and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, $\sum_{n=1}^{\infty} \delta_n = \infty$. Then $\beta_n \geq 0$ for $n \geq 1$, and $\beta_n \to 0$ as $n \to \infty$.

Lemma 1 is used for the convergence of (Mann) Ishikawa iteration in [3], [14]. *Lemma 2* is used in [2], [8], [9], [11]. *Lemma 3* is used in [1], [4], [6].

The following result is Lemma 1 from [5], with another proof.

Proposition 1. *[5] Let $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ be three nonnegative sequences which satisfy*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \tag{6}$$

where $\sum_{n=1}^{\infty} a_n b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Then there exists the limit of $(a_n)_n$.

Proof. We have

$$\begin{aligned} a_{n+1} &\leq a_n + b_n a_n + c_n \\ a_{n+2} &\leq a_{n+1} + b_{n+1} a_{n+1} + c_{n+1} \\ &\ldots \\ a_{n+p+1} &\leq a_{n+p} + b_{n+p} a_{n+p} + c_{n+p} \end{aligned}$$

Summing, one obtains

$$\begin{aligned} a_{n+p+1} &\leq a_n + \sum_{k=n}^{n+p} b_k a_k + \sum_{k=n}^{n+p} c_k, i.e. \\ a_{n+p+1} - a_n &\leq \sum_{k=n}^{n+p} b_k a_k + \sum_{k=n}^{n+p} c_k. \end{aligned}$$

We know $\sum_{n=1}^{\infty} a_n b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Let $\varepsilon > 0$ be a fixed number. There exists n_0' such that $\forall n \geq n_0'$, we have $\sum_{k=n}^{n+p} b_k a_k < \varepsilon/2$. For the same $\varepsilon > 0$ there exists a n_0" such that $\forall n \geq n_0$", we have $\sum_{k=n}^{n+p} c_k < \varepsilon/2$. We take $n_0 := \max\{n_0'$, n_0"$\}$. For all $n \geq n_0$ we have

$$a_{n+p+1} - a_n \leq \sum_{k=n}^{n+p} b_k a_k + \sum_{k=n}^{n+p} c_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{7}$$

Thus $(a_n)_n$ is fundamental in $[0, \infty)$. Hence there exists the limit of $(a_n)_n$. □

Proposition 4 is used in [5], for the convergence of Ishikawa iteration with errors introduced in [15].

2. Lemma 3 implies Lemma 2

We need the following Lemma:

Lemma 4. *Let* $(\delta_n)_n, (\sigma_n^2)_n \subset (0, 1)$.*The following relation is true:*

$$\sum_{n=1}^{\infty} \delta_n = \infty, \sum_{n=1}^{\infty} \sigma_n^2 < \infty \Rightarrow \sigma_n^2 = o(\delta_n). \tag{8}$$

Proof. Our assumptions lead us to

$$\sigma_n^2 < \delta_n, \forall n \geq 1. \tag{9}$$

Else$\sum_{n=1}^{\infty} \sigma_n^2 = \infty$, is in contradiction with $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.

We have two cases:

I) The case in which $\lim_{n\to\infty} \sigma_n^2 = 0$ and $\lim_{n\to\infty} \delta_n = 0$. We fix $n \geq 1$. Then there exists $\varepsilon_n > 0$ such that $\sigma_n^2 = \varepsilon_n \delta_n$. We have $\varepsilon_n < \delta_n$, else $\varepsilon_n \geq \delta_n$, hence $\sigma_n^2 < \delta_n \leq \varepsilon_n$. Thus $\sigma_n^2 < \delta_n \varepsilon_n$, which is in contradiction with $\sigma_n^2 = \varepsilon_n \delta_n$. If $n \in N$, then there exists a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n < \delta_n$. Thus

$$0 \leq \lim_{n\to\infty} \varepsilon_n \leq \lim_{n\to\infty} \delta_n = 0. \tag{10}$$

Hence $\sigma_n^2 = o(\delta_n)$.

II) The case in which $\lim_{n\to\infty} \sigma_n^2 = 0$ and $\lim_{n\to\infty} \delta_n \neq 0$. We know $\sigma_n^2 = \varepsilon_n \delta_n$, $\forall n \geq 1$. Then $\lim_{n\to\infty} \varepsilon_n = 0$. Hence $\sigma_n^2 = o(\delta_n)$. □

Remark 1. *The two converses of* Lemma 2 *are not true.*
The first converse is

$$\sigma_n^2 = o(\delta_n), \sum_{n=1}^{\infty} \delta_n = \infty \Rightarrow \sum_{n=1}^{\infty} \sigma_n^2 < \infty. \tag{11}$$

A counterexample is given by: $\delta_n := 1/\sqrt{n}, \sigma_n^2 = 1/n,\ \varepsilon_n = 1/\sqrt{n},\ \forall n \geq 1.$
The second converse is

$$\sigma_n^2 = o(\delta_n), \sum_{n=1}^{\infty} \sigma_n^2 < \infty \Rightarrow \sum_{n=1}^{\infty} \delta_n = \infty. \tag{12}$$

A counterexample is given by: $\delta_n := 1/n^2, \sigma_n^2 = 1/n^3,\ \varepsilon_n = 1/n,\ \forall n \geq 1.$ *We will prove that* Lemma 3 *implies* Lemma 2.

Proposition 2. *Let* $(\beta_n)_n$ *be a nonnegative sequence which satisfies*

$$\beta_{n+1} = (1 - \delta_n)\beta_n + \sigma_n^2, \tag{13}$$

where $(\delta_n)_n, (\sigma_n^2)_n \subset (0, 1),$ *and*

$$\sum_{n=1}^{\infty} \sigma_n^2 \quad < \quad \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \delta_n \quad = \quad \infty. \tag{15}$$

Then the sequence $(\Psi_n)_n$ *which satisfies*

$$\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n^2, \tag{16}$$

converges to zero and

$$\sigma_n^2 = o(\delta_n). \tag{17}$$

Proof. From *Lemma 5*, we have $(4), (5) \Rightarrow (7)$. Let us consider the sequence given by

$$\begin{aligned} \Psi_1 &= \beta_1, \\ \Psi_{n+1} &\leq (1 - \delta_n)\Psi_n + \sigma_n^2. \end{aligned}$$

We have $\Psi_1 \leq \beta_1$. Supposing $\Psi_n \leq \beta_n$, we prove $\Psi_{n+1} \leq \beta_{n+1}$. Thus we have

$$\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n^2 \leq \Psi_{n+1} \leq (1 - \delta_n)\beta_n + \sigma_n^2 = \beta_{n+1}. \tag{18}$$

From *Lemma 3* we know that $\lim_{n\to\infty} \beta_n = 0$. Thus $\lim_{n\to\infty} \Psi_n = 0$. □

3. Applications

Proposition 3. *Let* $(a_n)_n$ *be a nonnegative sequence which satisfies*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, \tag{19}$$

where $(\alpha_n)_n \subset (0,1), \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Proof. We denote $\Psi_n := a_n,\ \lambda_n := \alpha_n,\ \sigma_n := \alpha_n\beta_n$. From $\frac{\sigma_n}{\lambda_n} = \frac{\alpha_n\beta_n}{\alpha_n} = \beta_n$ and from $\lim_{n\to\infty} \beta_n = 0$, we have $\sigma_n = o(\lambda_n)$. *Lemma 2* implies $\lim_{n\to\infty} \Psi_n = 0$, hence $\lim_{n\to\infty} a_n = 0$. □

If $\beta_n = \varepsilon$, $\forall\ n \geq 1$, then we recognize Lemma 1 from [12]. The conclusion is $0 \leq \limsup_{n\to\infty}\ a_n \leq \varepsilon$.

Using *Lemma 2*, we are able to improve Proposition 2.5 from [13]:

Proposition 4. *Let* $(a_n)_n$ *be a nonnegative sequence which satisfies*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n, \tag{20}$$

where $\alpha_n \in (0,1),\ \forall n \geq 1,\ \sum_{n=1}^{\infty} \alpha_n = \infty$, $(c_n)_n$ *is a nonnegative sequence and* $\sum_{n=1}^{\infty} \alpha_n c_n = l$. *Then*

$$\lim_{n\to\infty} a_n = 0. \tag{21}$$

Proof. Let $(\beta_n)_n$ be the sequence given by

$$\begin{aligned} \beta_1 &: = a_1, \\ \beta_{n+1} &: = (1-\alpha_n)\beta_n + \alpha_n c_n, \forall n \geq 1. \end{aligned}$$

We observe that $a_1 \leq \beta_1$. We suppose $a_n \leq \beta_n$ and we prove that $a_{n+1} \leq \beta_{n+1}$. We have

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n \leq a_{n+1} \leq (1-\alpha_n)\beta_n + \alpha_n c_n = \beta_{n+1}. \tag{22}$$

Hence $a_n \leq \beta_n \forall n \geq 1$. From *Lemma 2* with $\delta_n := \alpha_n$ and $\sigma_n^2 := \alpha_n c_n$, we have $\lim_{n\to\infty} \beta_n = 0$. Thus $\lim_{n\to\infty} a_n = 0$. □

In proposition 2.5 from[13] the conclusion is $0 \leq \lim_{n\to\infty} \sup a_n \leq l$.

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