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PRIMJENA MATEMATIČKIH MODELA U PLANIRANJU LOGISTIČKIH OPERACIJA

APPLYING MATHEMATICAL MODELS IN PLANNING LOGISTICS OPERATIONS

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Pregledni rad

Sažetak: *Matematički modeli linearnog programiranja mogu se primjenjivati u planiranju logističkih operacija, kako bi se time pomoglo rješavanje logističkih problema. Taj je pristup demonstriran na dva osnovna logistička problema, problemu alokacije resursa i problemu distribucijske mreže. Optimalno rješenje koje zadovoljava postavljena ograničenja dobiveno je primjenom programskog alata na matematičkim modelima problema. Daljnjim poboljšavanjem prikazanih modela omogućila bi se njihova primjena u praksi.*

Ključne riječi: *planiranje, logističke operacije, linearno programiranje, matematički modeli*

Review article

Abstract: *Linear programming mathematical models can be applied in planning logistics operations, in order to facilitate solving logistics problems. This approach is demonstrated on two basic logistics problems – the resource allocation problem and the distribution network problem. The optimal solution that meets the given constraints is generated using a software tool applied to the mathematical models of the problems. Further improvement of the outlined models would enable their practical implementation.*

Key words: *planning, logistics operations, linear programming, mathematical models*

1. INTRODUCTION

Supply chain is a system which generates value by providing for customer's needs. It consists of interactions between parties involved, such as customers, suppliers of raw material, manufacturers, distributers (wholesalers), retailers, logistics operators, carriers, etc. Those interactions generate flows of products (material), information and funds between and within different stages of supply chain [1].

Planning logistics operations, whether considered from the aspect of a particular product or a company, always involves a trade-off between efficiency and effectiveness in compliance with the corporate competitive strategy, subject to a series of constraints, while finding an optimal solution [2]. Solving such type of problems can be facilitated by applying linear programming mathematical models, as discussed in the following paragraphs.

2. PLANNING OF LOGISTICS OPERATIONS

Planning of logistics operations is a part of supply chain management, mainly focused on the following issues:

- Maintaining the right balance among production, inventory and distribution;
- Matching supply and demand by optimal resource allocation;
- Exploiting resources in compliance with the given trade-off between effectiveness and efficiency.

With respect to these issues, applying the linear programming mathematical models in resource allocation planning and distribution network planning will be outlined in further paragraphs.

2.1. Resource allocation planning

Particular resources of the logistics systems could be or could become insufficient to meet requirements, whether the different processes share the same resource or some processes require several different resources, such as:

- quantity of products to be distributed via DC exceeds its capacity,
- quantity of cargo to be transported exceeds max. payload of the vehicle,
- available workforce is insufficient to get the job done within the given time,
- particular equipment act as a bottle-neck in the production line,
- limited amount of certain ingredient or raw material, etc.

Optimal solution in such cases is to allocate available resources in a way that the whole system generates the best total outcome, while the following rule should be taken into account: ...*optimum of the whole is not necessarily a sum of particular optimums of its parts*... [3].

Due to insufficient resources it is not possible to meet the demand of all processes, so those contributing less to the result of the whole are sacrificed for the benefit of those contributing more, but only to the extent that the operating of the whole is not put at question [4]. In this respect, the following issues should be considered:

- How to define relations between processes and resources?
- How to identify bottle-necks?
- How to evaluate outcome of a process?

2.2. Distribution network planning

Performance of a distribution network should be evaluated against customers' needs to be met (competitive strategy, efficiency) and costs of meeting those needs (effectiveness) [5]. Optimal solution in this case means to connect source nodes (production plants, distribution centers, etc.) and destination nodes (warehouses, points of sale (POS), etc.) by transport routes, so that the demand is met at minimum distribution costs.

In planning of a distribution network [6], the following issues should be considered:

- Where to locate source nodes and which capacity to allocate?
- Which sources should supply which destinations?
- Which quantity should be shipped?
- Which transport routes to select?

Mathematical models used to facilitate distribution network planning are based on assumption that supply chain can be depicted as a set of nodes and arcs connecting them into a functional network [1]. Nodes represent resources, while arcs represent physical or logical links between them, enabling flows of material, funds and information throughout the supply chain.

3. APPLYING LINEAR PROGRAMING IN PLANNING OF LOGISTICS OPERATIONS

In planning their logistics operations to satisfy customers demand and maximize profit, companies are faced with a series of constraints, such as capacities of production plants, raw material supply, distribution network design, etc. Linear programming is a powerful tool for obtaining optimal solution which would maximize utility while meeting the constraints given [5].

Applying linear programming mathematical models in planning of logistic operations is outlined by the following example which involves both, resource allocation and distribution network design.

3.1. Defining the resource allocation problem

A company assembles final product out of standardized parts (components), supplied by various vendors. Assembly operations are not demanding in terms of equipment nor infrastructure, but determined mainly by available workforce. The demand of the market is not constant, but can be predicted as listed in Table 1.

Table 1: Demand forecast

Month	Demand (D_t)
1	3.600
2	6.200
3	6.900
4	6.700
5	4.800
6	3.700

The company can handle these fluctuations by building-up inventory during low demand months, increasing production during high demand months or backlogging orders to be delivered late to the customers, but no backlog order must remain at the end of the planning horizon. In order to utilize its resources effectively and efficiently, the company has to set up optimal operational plan (six months period is long enough for this example).

3.1.1. Mathematical model of the problem

Mathematical model that encompasses all the elements of the resource allocation problem (i.e. operational planning problem) is defined by the following mathematical structure:

Input Data

- Selling price .... 40,00 €/unit,
- Initial inventory .... 1.200 units,
- Minimal inventory .... 800 units,
- Initial workforce .... 100 workers.

Objective Function

Objective function is to minimize total cost of the planning period. Since all the demand must be satisfied by the end of the planning horizon with fixed selling price, minimizing cost is equal to maximizing profit. It encompasses the following elements:

- Costs of regular work .... 480,00 €/month,
- Costs of overtime work............................ 4,50 €/hour,
- Costs of reassigning workers .... 300,00 €/worker,
- Costs of inventory .... 2,00 €/unit,
- Stockout/backlog costs .... 4,00 €/unit,
- Material costs .... 10,00 €/unit.

Objective Function is given by the following equation:

$$minF = \sum_{t=1}^{6} 480R_t + \sum_{t=1}^{6} 4{,}5S_t + \sum_{t=1}^{6} 300(pR_t + sR_t) + \\ + \sum_{t=1}^{6} 2Z_t + \sum_{t=1}^{6} 4N_t + \sum_{t=1}^{6} 10\, P_t \quad (1)$$

where

R_t = number of workers in month t, $t = 1,\ldots,6$

pR_t = increase of the workforce in month t, $t = 1,\ldots,6$ (additional workers assigned in the beginning of month t)

sR_t = decrease of the workforce in a month t, $t = 1,\dots,6$ (workers reassigned to other jobs in the beginning of month t)
P_t = production in month t, $t = 1,\dots,6$ (number of units produced in month t)
Z_t = inventory at the end of month t, $t = 1,\dots,6$
N_t = stockout/backlogged orders at the end of month t, $t = 1,\dots,6$
S_t = overtime hours in month t, $t = 1,\dots,6$

Constraints

Quantitative area of allowable values of the decision variables is defined by the following constraints and relations:

1) **The number of workers** R_t in month t is equal to the number of workers R_{t-1} in month t-1, minus sR_t (decrease of the number of workers in the beginning of month t), plus pR_t (increase of the number of workers in month t) respectively:
$$R_t = R_{t-1} + pR_t - sR_t \text{ for } t = 1,\dots,6 \qquad (2)$$
$$R_0 = 100$$
2) **Production** P_t in month t is limited by available work force (number of working hours – regular and overtime) in month t. With respect to the norm, each worker can produce two units daily in regular hours (40 units monthly), and one unit for every four hours overtime.
$$P_t \le 40R_t + \frac{1}{4}S_t \text{ for } t = 1,\dots,6 \qquad (3)$$
3) **Total demand** in month t equals to the sum of current demand D_t in month t and backlogged orders N_{t-1} from the previous month t-1. This demand can be satisfied either by current production P_t and previous month inventory Z_{t-1} or can be partly backlogged N_t.
$$Z_{t-1} + P_t = P_t + N_{t-1} + Z_t - N_t \text{ for } t = 1,\dots,6 \quad (4)$$
$$Z_0 = 1200, N_0 = 0, Z_t \ge 800, N_6 = 0$$
4) **Overtime hours** per worker are limited to maximum 10 hours monthly.
$$S_t \le 10R_t \text{ for } t = 1,\dots,6 \qquad (5)$$

3.1.2. Optimal solution of the problem

The optimal solution of the problem (i.e. operational plan) is the one that yields minimal operating costs over the planning horizon. It is generated out of the mathematical model of the problem, by *MS Excel* spreadsheet optimizer *Solver*, as shown in Table 2 and in Table 3. The equations (1) to (5) of the mathematical model correspond to the *Excel* formulas entered in the respective cells of the spreadsheet.

Table 2: Operational plan for six months period

Month	DECISION VARIABLES							Demand
	pR	sR	R	S	Z	N	P	
	Workforce increase	Workforce decrease	Number of workers	Overtime hours	Inventory	Stockout Backlog	Production	
0	0	0	100	0	1.200	0		
1	37	0	137	0	3.065	0	5.465	3.600
2	0	0	137	0	2.329	0	5.465	6.200
3	0	0	137	0	894	0	5.465	6.900
4	0	0	137	1.366	800	800	5.806	6.700
5	0	17	120	0	800	800	4.800	4.800
6	0	8	112	0	800	0	4.500	3.700

Table 3: Costs, Incomes and Profit over a six-month period

	Costs	Income	Profit
1	137.335	144.000	6.665
2	124.880	248.000	123.120
3	122.009	276.000	153.991
4	134.585	267.200	132.615
5	115.385	192.000	76.615
6	102.850	148.800	45.950
	737.044	**1.276.000**	**538.956**

3.2. Defining the distribution network problem

The demand (cf. Table 1) is generated by points of sale (POS) in nine cities, the company supplies from its production plant located in Bratislava. The demand is satisfied from the distribution centers (DC) the company operates in five of those nine cities. The product is not yet ready to be delivered to the point of sale, until finishing operations are done at DC (sorting, packing into commercial packing materials, labeling, etc.).

The company must decide which of five DCs to use for distribution and which quantity to supply from each DC to the cities, in order to meet the demand at minimum distribution costs.

The month with the highest demand (6.900 units in 3rd month) is taken as a reference, however any other month within the planning horizon could be referred to by the same model.

Relevant input data are given in Table 4. Entries are based on the tariffs of logistic operators, labor and infrastructure costs, as well as the market analysis.

Table 4: Input data

	Unit transport costs (€/unit)											
From / To	Wien	Bratislava	Budapest	Ljubljana	Munich	Nuernberg	Prague	Stutgart	Zagreb	ft (€)	d (€/unit)	K (units)
Wien	9	22	35	32	35	38	33	41	32	15.000	20	3500
Bratislava	22	9	24	35	35	72	33	72	35	10.000	8	3500
Munich	35	35	41	33	9	19	32	19	51	15.000	32	3500
Prague	32	33	32	57	32	51	9	51	57	11.000	30	3500
Zagreb	32	35	32	19	51	76	57	76	9	12.000	32	3500
Demand (units)	900	600	700	500	1200	700	800	900	600			

The problem solubility condition

The problem can be solved if the total distribution capacity of the network (sum of individual DC capacities) is greater than or equal to the total market demand (sum of the demands of individual cities with their gravity zones) [6], as defined by the following mathematical expressions:

$$\text{Let } K=\sum_{i=1}^{n} k_i \text{ and } P=\sum_{j=1}^{m} p_j \tag{6}$$

$$\text{Problem can be solved if } K \geq P \tag{7}$$

where
n = total number of used DCs, $n_{max} = 5$,
k_i = capacity of DC on location i,
K = total distribution capacity,
m = total number of cities to be supplied, $m = 9$,
p_j = demand of POS in city j,
P = total demand of the market.

3.2.1. Mathematical model of the problem

The mathematical model that encompasses relevant elements of the problem (issues not relevant to this problem are disregarded) is defined by the following mathematical structure:

Objective Function

Objective function is to minimize distribution costs, while meeting the demand of the market. It consists of fixed costs of DC operations, if DC is used on the particular location, variable costs of transporting products from the production plant to DC on that location and variable costs of delivering products from DCs to POS in the cities:

$$minF=\sum_{i=1}^{n} l_i(ft_i+r_i d_i)+\sum_{i=1}^{n}\sum_{j=1}^{m} t_{ij} q_{ij} \tag{8}$$

Subject to the constraints:

1) Demand of the market must be satisfied, i.e. POS in each city must be supplied the quantity of units (products) equal to demand in that city:

$$p_j=\sum_{i=1}^{n} q_{ij} \text{ for every } j=1,\dots,m \tag{9}$$

2) Total quantity supplied from each DC is limited by its capacity:

$$k_i l_i \geq \sum_{j=1}^{m} q_{ij} \text{ for every } i=1,\dots,n \tag{10}$$

3) There are five possible locations of DC to be used for distribution. Each location of DC can be used for distribution or not, as defined by location variable (binary variable, 0 means not used, 1 means used):

$$l_i \in \{0,1\} \text{ for every } i=1,\dots,n \tag{11}$$

With simplification:

Inbound flow of DC is equal to outbound flow, i.e. there are no local inventories in DCs:

$$r_i l_i=\sum_{j=1}^{m} q_{ij} \text{ for every } i=1,\dots,n \tag{12}$$

where
l_i = location variable, binary: 1 = DC on location i is used, 0 = not used,
ft_i = fixed cost of DC on location i (€/month),
r_i = realized turnover of DC on location i, (units),
d_i = unit transport cost from the production plant to DC on location i (€/unit),
n = total number of DCs used, n_{max},= 5,
m = total number of cities to be supplied, $m = 9$,
t_{ij} = unit transport cost from DC on location i to city j (€/unit),
q_{ij} = number of units delivered from DC on location i to city j (units).
p_j = demand in city j (units/month),
k_i = capacity of DC on location i (units/month).

3.2.2. Optimal solution of the problem

The optimal solution (i.e. distribution plan) is generated out of the mathematical model of the problem (equations 8, 9, 10, 11 and 12), by *MS Excel* spreadsheet optimizer *Solver*, as shown in Table 5.

Decision variable q_{ij} may take any non-negative integer value, as there is no logical point to deliver negative number of units or less than a whole unit.

It is allowed to round-up fractional values of q_{ij} that may occur, because the values are large enough so that the tolerance corresponds to accuracy of the input data (rounding-up cannot cause major logical error). Defining decision variable l_i as binary (only takes values of 1 or 0), fractional values are excluded.

Table 5: Optimal solution of the problem

	Distribution plan (units)									DC	DC	>
DC Location	Wien	Bratislava	Budapest	Ljubljana	Munich	Nuernberg	Prague	Stuttgart	Zagreb	0 = yes 1 = no	**total flow**	**K**
Wien	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0	0
Bratislava	**900**	**600**	**700**	**500**	**0**	**0**	**800**	**0**	**0**	**1**	3.500	0
Munich	**0**	**0**	**0**	**0**	**1200**	**700**	**0**	**900**	**0**	**1**	2.800	700
Prague	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0	0
Zagreb	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**600**	**1**	600	2.900
Unsatisfied demand (units)	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**			

The demand of the market is satisfied through three DCs, located in Bratislava, Munich and Zagreb, according to the following distribution plan:

1. DC in Bratislava is used to supply local markets of Wien, Bratislava, Budapest, Ljubljana, and Prague;
2. DC in Munich is used to supply local markets of Munich, Nuernberg and Stuttgart;
3. DC in Zagreb is used to supply local market of Zagreb.

All the constraints (equations 9, 10 and 11) are satisfied at total distribution cost of 306.300 € for the given month. Graphical presentation of the optimal solution of the problem is shown in Figure 1.

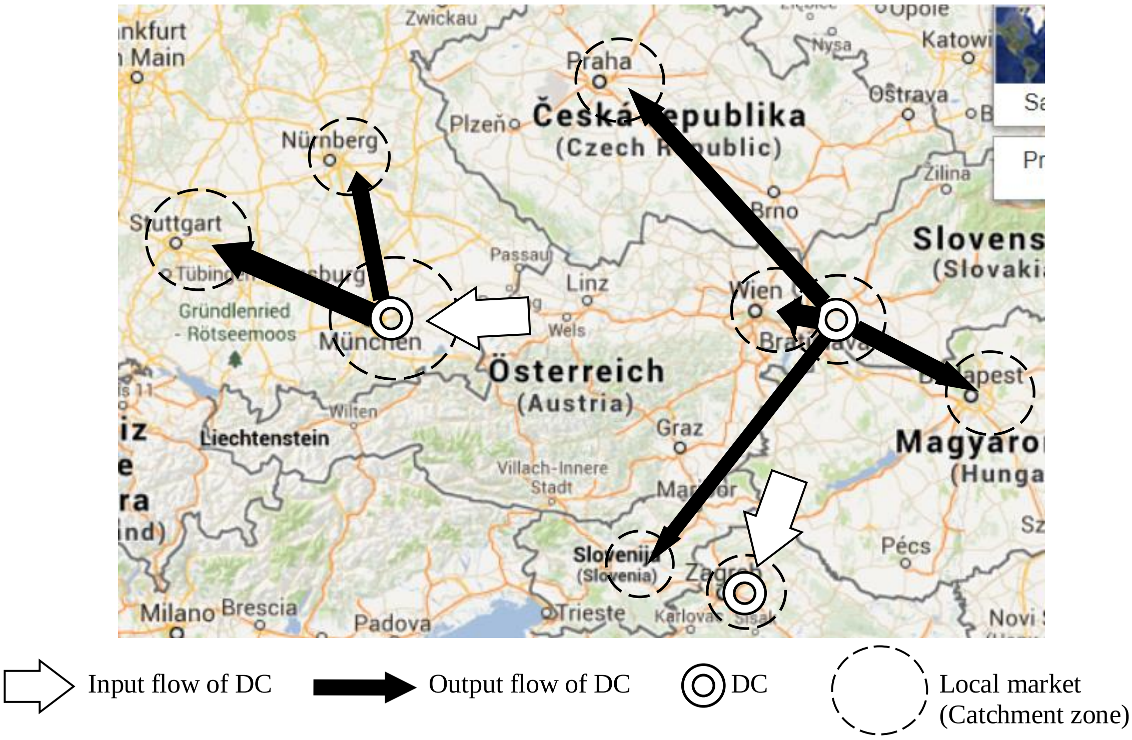

Figure 1: Graphical presentation of the optimal solution

4. CONCLUSION

Planning of logistics operations, as a part of supply chain management, deals with issues that can be mathematically expressed. Based on this assumption, linear programming is applied to obtain the optimal solution of the two basic logistics problems, which would maximize utility (benefit/cost ratio), while meeting the constraints given. For this purpose, two mathematical models are developed, each one representing the respective logistic problem.

Similarity with the real system is limited by accuracy of the input data and the simplifications made in the models. However, it is sufficient to demonstrate the basic principles of the solving method, which is outlined in this paper.

The mathematical models developed and presented in this paper could be improved by adding more elements representing relevant issues of the real system and by obtaining more accurate input data, in order to achieve higher level of similarity. It would enable implementation of this approach in solving practical logistics problems.

As the improvement of the models is only a technical issue, the main point here is to show the possibilities of how to facilitate planning of logistics operations by applying linear programming mathematical models.

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