Hyperspherical curves in n-dimensional k-isotropic space I_n^k

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Abstract. *In this paper we give the characterization of hyperspherical curves in n-dimensional k-isotropic space I_n^k.*

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1. Introduction

The n-dimensional k-isotropic space I_n^k is introduced in [5] where it is defined as a pair (A, V) where A is a real n-dimensional affine space and V its corresponding vector space decomposed in a direct sum of subspaces

$$V = U_1 \oplus U_2, \quad \dim U_1 = n - k, \dim U_2 = k.$$

The space U_1 is endowed with a scalar product $\cdot : U_1 \times U_1 \to \mathbb{R}$ which is extended on the whole V by

$$\mathbf{x} \cdot \mathbf{y} = \pi_1(\mathbf{x}) \cdot \pi_1(\mathbf{y}),$$

where $\pi_1 : V \to U_1$ denotes the canonical projection. In such a way a semi-definite scalar product on V is defined.

In this paper we describe the osculating hyperspheres of an admissible curve in the space I_n^k. The theory of curves in I_n^k is developed in [2]. Furthermore, we study the conditions under which an admissible curve is hyperspherical.

2. Osculating hypersphere in I_n^k

As it is shown in [5], a hypersphere in I_n^k is defined in affine coordinates by the equation

$$\sum_{i=1}^{n-k} x_i^2 + 2\sum_{i=1}^{n} \alpha_i x_i + \alpha_0 = 0, \tag{1}$$

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where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

We distinguish the following types of hyperspheres in I_n^k ([5], Theorem 7.1). If $(\alpha_{n-k+1}, \dots, \alpha_n) \neq (0, \dots, 0)$, then by an isotropic motion we obtain the normal form of a parabolic hypersphere of type l, $l \in \{0, \dots, k-1\}$,

$$\sum_{i=1}^{n-k} x_i^2 + \alpha_{n-l} x_{n-l} = 0,$$

Its radius is defined by $-\frac{\alpha_{n-l}}{2}$, $\alpha_{n-l} \neq 0$.
If $\alpha_{n-k+1} = \dots = \alpha_n = 0$, then by an isotropic motion we get a cylindrical hypersphere

$$\sum_{i=1}^{n-k} x_i^2 = r^2.$$

Its radius is defined by r.

Definition 1. *Let $c : I \to I_n^k$ be a regular C^r-curve and $f(x) = 0$ a regular C^r-hypersurface, $r \geq 1$. A point $P_0(t_0)$ is a point of contact of r^{th} order of the curve c and the hypersurface f if the function*

$$F(t) = f(c(t))$$

satisfies $F(t_0) = \ldots = F^{(r)}(t_0) = 0,\ F^{(r+1)}(t_0) \neq 0$.

Definition 2. *Let c be an admissible C^r-curve in I_n^k, $r \geq n$, P_0 a point of c. A hypersphere which has contact of n^{th} order with the curve c in P_0 is called the osculating hypersphere.*

We can write equation (1) of a hypersphere in the following form. Let $P_0 = (p_1, \dots, p_n)$ be a point of hypersphere (1). Then

$$\sum_{i=1}^{n-k} p_i^2 + 2\sum_{i=1}^{n} \alpha_i p_i + \alpha_0 = 0. \tag{2}$$

Subtracting (2) from (1) we get

$$\sum_{i=1}^{n-k} (x_i^2 - p_i^2) + 2\sum_{i=1}^{n} \alpha_i (x_i - p_i) = 0,$$

which can be written as

$$\sum_{i=1}^{n-k} (x_i - p_i)^2 + 2\sum_{i=1}^{n-k} (\alpha_i + p_i)(x_i - p_i) + 2\sum_{i=n-k+1}^{n} \alpha_i (x_i - p_i) = 0.$$

By introducing vectors

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{p} = (p_1, \dots, p_n),$$

$$\mathbf{u} = 2(\alpha_1 + p_1, \dots, \alpha_{n-k} + p_{n-k}, \alpha_{n-k+1}, \dots, \alpha_n)$$

the previous expression can be written in the form

$$(\mathbf{x}-\mathbf{p})^2+\mathbf{u}\cdot_e(\mathbf{x}-\mathbf{p})=0, \tag{3}$$

where square denotes the isotropic square of the given vector and

$$\mathbf{u}\cdot_e(\mathbf{x}-\mathbf{p})=\sum_{i=1}^{n}u_i(x_i-p_i)$$

is introduced as a symbolic Euclidean inner product to make the calculations easier.

Let us consider the function $F:I\to\mathbb{R}$, $F(s)=(c(s)-\mathbf{p})^2+\mathbf{u}\cdot_e(c(s)-\mathbf{p})$, where $c(s):I\to I_n^k$ is an admissible curve parametrized by the arc length, which represents no loss of generality. Let $s_0\in I$ be such that $c(s_0)=\mathbf{p}$. Then $F(s_0)=0$. Let us determine the conditions under which the curve c and hypersphere (3) have the contact of n^{th} order.

By differentiating F we get

$$F'(s)=2(c(s)-\mathbf{p})\cdot\mathbf{t}_1(s)+\mathbf{u}\cdot_e\mathbf{t}_1(s).$$

Hence in $s=s_0$ we have

$$\mathbf{u}\cdot_e\mathbf{t}_1(s_0)=0. \tag{4}$$

By differentiating F again we get

$$F''(s)=2+2(c(s)-\mathbf{p})\cdot c''(s)+\mathbf{u}\cdot_e c''(s),$$

and therefore

$$\mathbf{u}\cdot_e\mathbf{t}_2(s_0)=-2\frac{1}{\kappa_1}(s_0). \tag{5}$$

Let us define the functions $\rho_l,h_l:I\to\mathbb{R}$ by

$$\rho_l(s)=\frac{1}{\kappa_l(s)},\ \ l=1,\dots n-1,$$

$$\mathbf{u}\cdot_e\mathbf{t}_l(s)=-2h_{l-1}(s),\ \ l=1,\dots,n,$$

and let us prove by induction

$$h_l(s_0)=\rho_l(h'_{l-1}+\kappa_{l-1}h_{l-2})(s_0),\ \ l=3,\dots,n-k, \tag{6}$$

$$h_{n-k+l}(s_0)=\rho_{n-k+l}h'_{n-k+l-1}(s_0),\ \ l=1,\dots,k-1, \tag{7}$$

$$h_0(s_0)=0,\ \ h_1(s_0)=\rho_1(s_0),\ \ h_2(s_0)=\rho_1'\rho_2(s_0),$$

The statement has already been proved for $l=0,1$. Now suppose that it is true for $l\le j-1$ and let us prove it for $l=j$. We can write

$$F^{(j)}(s)=2(c(s)-p)\cdot c^{(j)}(s)+\mathbf{u}\cdot_e c^{(j)}(s)+G_j(s). \tag{8}$$

By differentiating the previous expression we get

$$F^{(j+1)}(s)=2\mathbf{t}_1\cdot c^{(j)}(s)+2(c(s)-p)\cdot c^{(j+1)}(s)+\mathbf{u}\cdot_e c^{(j+1)}(s)+G_j'(s).$$

From the construction of the Frenet frame of an admissible curve ([2]) it follows that there exist functions $a_i : I \to \mathbb{R}$ such that

$$c^{(j)}(s) = \sum_{i=1}^{j} a_i(s)\mathbf{t}_i(s), \tag{9}$$

and therefore

$$F^{(j+1)}(s) = 2a_1(s) + 2(c(s) - p) \cdot c^{(j+1)}(s) + \mathbf{u} \cdot_e c^{(j+1)}(s) + G'_j(s).$$

By comparing the previous equation and equation (8) for $j + 1$

$$F^{(j+1)}(s) = 2(c(s) - p) \cdot c^{(j+1)}(s) + \mathbf{u} \cdot_e c^{(j+1)}(s) + G_{j+1}(s) \tag{10}$$

we conclude the following

$$2a_1(s) + G'_j(s) = G_{j+1}(s).$$

Now, let $j \leq n - k$. By sustituting (9) in (10), the condition $F^{(j+1)}(s_0) = 0$ implies

$$\begin{aligned} -2h_j(s_0) \quad = \quad & -\frac{1}{a_j\kappa_j}\Big[-2\sum_{i=2}^{j-1}(a'_i h_{i-1} - a_i\kappa_{i-1}h_{i-2} + a_i\kappa_i h_i) \\ & + a'_j(-2h_{j-1}) - a_j\kappa_{j-1}(-2h_{j-2}) - 2a_1 + G_{j+1}\Big](s_0). \end{aligned} \tag{11}$$

On the other hand, by substituting (9) and the definition of the functions h_i in the expression (8) we get

$$F^{(j)}(s) = 2(c(s) - p) \cdot c^{(j)}(s) + \sum_{i=1}^{j} a_i(s)(-2h_{i-1})(s) + G_j(s),$$

and therefore

$$F^{(j+1)}(s) = 2a_1(s) + 2(c(s) - p) \cdot c^{(j+1)}(s) + \sum_{i=1}^{j} \Big[a'_i(-2h_{i-1}) + a_i(-2h_{i-1})'\Big] + G'_j(s),$$

or in $s = s_0$

$$-2h'_{j-1}(s_0) = -\frac{1}{a_j}\Big[2a_1(s_0) - 2\sum_{i=1}^{j-1}(a'_i h_{i-1} + a_i h'_{i-1})(s_0) - 2a'_j h_{j-1}(s_0) + G'_j(s_0).\Big] \tag{12}$$

Comparing equations (11) and (12) we get recursion (6)

$$h_j(s_0) = \rho_j(h'_{j-1} + \kappa_{j-1}h_{j-2})(s_0).$$

In the same way we can prove recursion (7).

From these conditions we can determine vector $\mathbf{u} = (u_1, \dots, u_n)$. Since

$$\mathbf{u} \cdot_e \mathbf{t}_i = \mathbf{u} \cdot \mathbf{t}_i + \sum_{j=n-k+1}^{n} u_j \mathbf{t}_{ij} = -2h_{i-1}, i = 1, \dots, n,$$

where $\mathbf{t}_{ij}$ denotes the j^{th} coordinate of the vector $\mathbf{t}_i$ according to the base of the space U_2, we have

$$\mathbf{u} \cdot_e \mathbf{t}_n = u_n = -2h_{n-1},$$

and therefore the last component of the vector $\mathbf{u}$ is determined. The components $u_{n-1}, \dots, u_{n-k+1}$ can be determined from the system $TU = H$, where T is the upper triangular matrix with units on the diagonal such that in its j^{th} row there vector $\mathbf{t}_{n-k+j}$, $j = 1, \dots, k$, U is the matrix-column $[u_{n-k+1}, \dots, u_n]^\tau$, and H the matrix-column $-2[h_{n-k+1}, \dots, h_n]^\tau$. If components $u_{n-k+1}, \dots, u_n$, are already determined, the components $u_1, \dots, u_{n-k}$ can be determined from the system

$$\mathbf{u} \cdot \mathbf{t}_i = -\sum_{j=n-k+1}^{n} u_j \mathbf{t}_{ij} - 2h_{i-1}, \;\; i = 1, \dots, n-k.$$

Since the determinant of this system is $det(\pi_1(\mathbf{t}_1), \dots, \pi_1(\mathbf{t}_{n-k})) = 1$, the system has a unique solution.

It can be shown that the equation of the osculating hypersphere obtained in this way coincides with the equations of the osculating hyperspheres in I_3^1 and in I_3^2 ([1], [4]).

If the osculating hypersphere is a hypersphere of type l, $l \in \{1, \dots, k-1\}$, then its equation is

$$(\mathbf{x} - \mathbf{p})^2 + \sum_{i=1}^{n} u_i(x_i - p_i) = 0,$$

where $u_{n-l} \neq 0$, $u_n = \dots = u_{n-l+1} = 0$. We can notice that this holds if and only if $h_{n-1} = \cdots = h_{n-l} = 0$. The radius of such a hypersphere is given by $R = -\frac{u_{n-l}}{2} = h_{n-l-1}$.

3. Hyperspherical curves

Definition 3. *An admissible C^r-curve c, $r \geq n+1$, is called hyperspherical of type l, $l \in \{0, \dots k-1\}$, if it lies on a hypersphere of type l.*

The equation of an osculating hypersphere of type l can be written in the following form

$$(\mathbf{x} - c(s))^2 + \mathbf{u} \cdot (\mathbf{x} - c(s)) + \sum_{i=n-k+1}^{n-l} u_i(x_i - c_i(s)) = 0, \tag{13}$$

where $u_{n-l} \neq 0$. If we define

$$\mathbf{w}(s) = 2\pi_1(c(s)) - \mathbf{u}$$

$$v(s) = c(s)^2 - c(s) \cdot_e \mathbf{u},$$

equation (13) can be written as

$$\mathbf{x}^2 - \mathbf{x} \cdot_e \mathbf{w}(s) + v(s) = 0.$$

If a curve is hyperspherical of type l, then it lies on its osculating hypersphere of type l which does not depend on the point $c(s)$ of a curve. Then the expressions $\mathbf{w}(s)$ and $v(s)$ do not depend on s, i.e., $\mathbf{w}'(s) = 0$, $v'(s) = 0$, $s \in I$. By differentiating we get

$$\mathbf{w}'(s) = 2\pi_1(\mathbf{t}_1(s)) - \mathbf{u}',$$

$$v'(s) = 2c(s) \cdot \mathbf{t}_1(s) - \mathbf{t}_1 \cdot_e \mathbf{u} - c(s) \cdot_e \mathbf{u}'.$$

Since

$$\mathbf{t}_1 \cdot_e \mathbf{u}(s) = 0, \tag{14}$$

we have

$$v'(s) = c(s) \cdot_e \mathbf{w}'(s).$$

Therefore, it is sufficient to show $\mathbf{w}'(s) = 0$. First we can notice that the components of $\mathbf{w}'$ with respect to the basis of U_2 satisfy

$$u'_{n-k+1} = \cdots = u'_{n-l} = 0.$$

However, we can show that the condition $u'_{n-l} = 0$ implies that the derivatives $u'_{n-k+1}, \ldots, u'_{n-l-1}$ are also 0. For an osculating hypersphere of type l we have

$$u_n = \ldots = u_{n-l+1} = 0, \;\; u_{n-l-1} = -2h_{n-l-2} - \mathbf{t}_{n-l-1,n-l} u_{n-l}.$$

By using Frenet's equations and the definition of the functions h_{n-l-1} we get $u'_{n-l-1} = 0$. Analogously we show that the other derivatives are 0.

Furthermore, let us show that the vector

$$\pi_1(2\mathbf{t}_1 - \mathbf{u}')$$

is a zero-vector under the given conditions. Differentiating (14) we get

$$\mathbf{u}' \cdot_e \mathbf{t}_1 = \mathbf{u}' \cdot \mathbf{t}_1 = 2.$$

Therefore, the isotropic scalar product of $\pi_1(2\mathbf{t}_1 - \mathbf{u}')$ and $\mathbf{t}_1$ is equal to 0. Furthermore, from

$$\mathbf{u} \cdot_e \mathbf{t}_2 = -2h_1(s)$$

there follows

$$\mathbf{u}' \cdot_e \mathbf{t}_2 = \mathbf{u}' \cdot \mathbf{t}_2 = 0,$$

so the scalar product of $\pi_1(2\mathbf{t}_1 - \mathbf{u}')$ and $\mathbf{t}_2$ is equal to 0. In the same way we can show that the vector $\pi_1(2\mathbf{t}_1 - \mathbf{u}')$ is orthogonal to each of the vectors $\pi_1(\mathbf{t}_1), \ldots, \pi_1(\mathbf{t}_{n-k})$ which form the basis of U_1. Hence, it is a zero-vector.

Therefore, we have proved the following theorem.

Theorem 1. *An admissible C^r-curve c, $r \geq n+1$, is hyperspherical of type l, $l \in \{0, \dots, k-1\}$, if and only if $h_{n-1} = \cdots = h_{n-l} = 0, R := h_{n-l-1} = const. \neq 0$.*

At the end, let us notice that the proof of the previous theorem also holds for the curves on cylindrical hyperspheres. As before, we can write the cylindrical hypersphere in the form

$$(\mathbf{x} - \mathbf{p})^2 + \mathbf{u} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

by using only the isotropic scalar product. When written in this form, the radius of the cylindrical hypersphere is given by

$$r^2 = \left(\frac{\mathbf{u}}{2}\right)^2,$$

where $\mathbf{u} = \sum_{i=1}^{n-k} -2h_{i-1}\pi_1(\mathbf{t}_i)$. The following theorem is true.

Theorem 2. *An admissible C^r-curve c, $r \geq n+1$, lies on a cylindrical hypersphere if and only if $h_{n-1} = \cdots = h_{n-k} = 0$.*
The radius of that hypersphere is given by $r^2 = \sum_{i=1}^{n-k} h_{i-1}^2$.
We can obtain the same result by noticing that a curve c lies on a cylindrical hypersphere if and only if the projection of c onto the space U_1 lies on the hyperphere of the space U_1 and by applying Euclidean results ([3]).

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