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ENTIRELY CIRCULAR CURVES OF $4^{\mathbf{th}}$–ORDER PRODUCED BY QUADRATIC INVERSION IN THE HYPERBOLIC PLANE

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Abstract. This paper gives an overview of all the entirely circular curves of the the 4^{th}-order which can be constructed by the quadratic H-inversion defined on Cayley-Klein model of hyperbolic plane. It is shown that the generating conic can be only H-parabola and H-circle.

A curve of the hyperbolic plane is said to be entirely circular if it possesses an isotropic asymptote at each intersection point with the absolute [3]. These points can be simple, double, triple or multiple points of the curve. Entirely circular curves of 2^{nd}-order are well known H-circles: hypercycle, cycle and horocycle. So, the entirely circular curve of n^{th}-order in hyperbolic plane is called an H-circle of n^{th}-order.

We shall work on the entirely circular curves of 4^{th}-order. They have eight common points with the absolute conic where at least four are double joint points. Accordinaly, this curve has four real tangential points on the absolute at the simple points of the curve (a single intersection with the absolute is excluded). Some of them can be multiple joint points on the absolute and at the same time simple, double, triple or multiple points of the curve. Theoretically, the following combinations of these eight points on the absolute are possible: (2+2+2+2), (2+3+3), (2+2+4), (4+4), (2+6), (3+5), (8). To find a way for the construction of curves of 4^{th}-order with such multiple joint points on the absolute by synthetical methods can be an extensive work.

This paper deals with entirely circular curves of 4^{th}-order which can be constructed by quadratic inversion defined on Cayley-Klein model of the hyperbolic plane. The construction of all circles of 3^{rd}-order was explored and published in [3].

As it is known, an involutive mapping of the hyperbolic plane (or more generally, of the projective plane) where any point and its image are conjugate with respect to a fixed conic, and simultaneously lie on the lines of a fixed pencil, is called generalized quadratic inversion in the hyperbolic plane, or shorter *generalized H-inversion*. The vertex of the pencil can be either an interior or an exterior point with respect to the absolute or can lie on it. The fixed conic c is called the fundamental conic, and the vertex P of the pencil (P) is called the pole of the H-inversion $\mathscr{I}(P,c)$. The H-inversion is

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restricted to a special hyperbolic case if the fundamental conic is identical with the absolute a. The generating conic denoted by k is a curve of 2^{nd}-order, whose H-inversion image will be constructed. The conic k which produces a curve of 4^{th}-order, denoted by k^4, is not allowed to contain any fundamental point of the inversion, since in that case its image would degenerate.

a) H-inversion $\mathscr{I}(P,a)$

Let $\mathscr{I}(P,a)$ be a restricted H-inversion with pole P on the absolute. For the generating conic k we choose an H-circle which does not contain the point P. The inversion image is the H-circle k^4 with two tangential points on the absolute, which coincides with k. In the case of a horocycle, the image curve hyperosculates the absolute at a quadruple joint point, being a simple one of the curve (Fig. 1). The remaining four absolute points of k^4 are common with the pole P. Since P is a quadruple joint point on the absolute and a double point of the curve (Figure 1 presents a tangential node), it has an isotrope asymptote as a tangent to both branches. The reality of one branch depends upon the existence of real intersection points of the polar line p and the conic k. Depending on the number of multiple joint points the cycle k^4 will be of type (2+2+4) or (4+4) in the case when k is a horocycle. The horocycle in Figure 1 is constructed by a perspective collineation from the absolute with the center point G and the axis g.

It is easy to conclude that k may not be a H-parabola, or any other H-conic, while its single intersection points on the absolute will be single points of its image curve. In this case k^4 will not be an H-circle.

Let $\mathscr{I}(P,a)$ be a H-inversion with pole P being an exterior point of the absolute conic. Three different fundamental real points of this inversion are denoted by P, P_1 and P_2, where P_1 and P_2 lie on the absolute. The inversion image of one conic k is a 4^{th}-order curve with double points in P, P_1 and P_2. The type of a double point depends upon the reality of the intersection points of k with the polar line of the corresponding fundamental point. The double point can be a node, an isolated double point or a cusp, when these intersection points are a pair of different real points, or a pair of conjugate imaginary points, or are coinciding in a joint point, respectively. Generally, the curves k^4 and a are intersecting in those two points, P_1 and P_2, without common tangents, so the image curve is not a H-circle regardless of the type of the generating conic k.

b) H-inversion $\mathscr{I}(P,c)$

Let $\mathscr{I}(P,c)$ be the generalized H-inversion. The construction of the fundamental conic c is shown in [3]. The absolute a is mapped by the H-inversion $\mathscr{I}(P,c)$ onto itself. Each inversion ray intersects the absolute conic at a pair of points which are mapped one onto another. This enables the construction of as many points of the fundamental conic c as necessary. The four intersection points of a and c are denoted by O_1, O_2, P_1 and P_2. The ray of the inversion PP_1 is the tangent of the conic c at the point P_1 (the same holds for the ray PP_2) (Fig. 2).

Now we have to find the conditions for determining the type and the position of the generating conic k to all the fixed elements of the H-inversion in order to produce a H-circle of 4^{th}-order. We denote with P_3 and P_4 the intersection points of the absolute

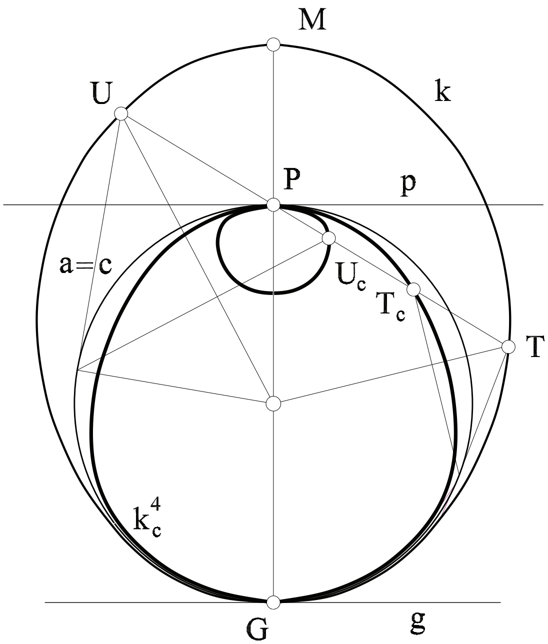

Figure 1.

with the rays PP_1 and PP_2. They are crucial for the H-inversion $\mathscr{I}(P,c)$, while the point P_3 is mapped onto P_1, and P_4 onto P_2.

We shall prove that the generating conic k has to be a 2^{nd}-order H-circle or a H-parabola. Each of them must contain both points, P_3 and P_4. Depending on k two types of k^4 are obtained.

Type 1)

Let k be an $\mathbf{H-parabola}$ through the absolute points P_3 and P_4 and with the absolute tangential point G. Each of two polar lines p_1 and p_2 intersects this parabola in one more point. The type of the double points P_1 and P_2 depends on the reality of these intersection points.

The H-circle k^4 has three double points which are nodes and coincide with the fundamental points of the inversion (P, P_1 and P_2) (Figure 2). Since P_3 is mapped onto P_1, the curve k^4 has with the absolute a triple joint point in P_1. This means that P_1 is a node, a tangential point of the curve with selfintersection. The tangent of one

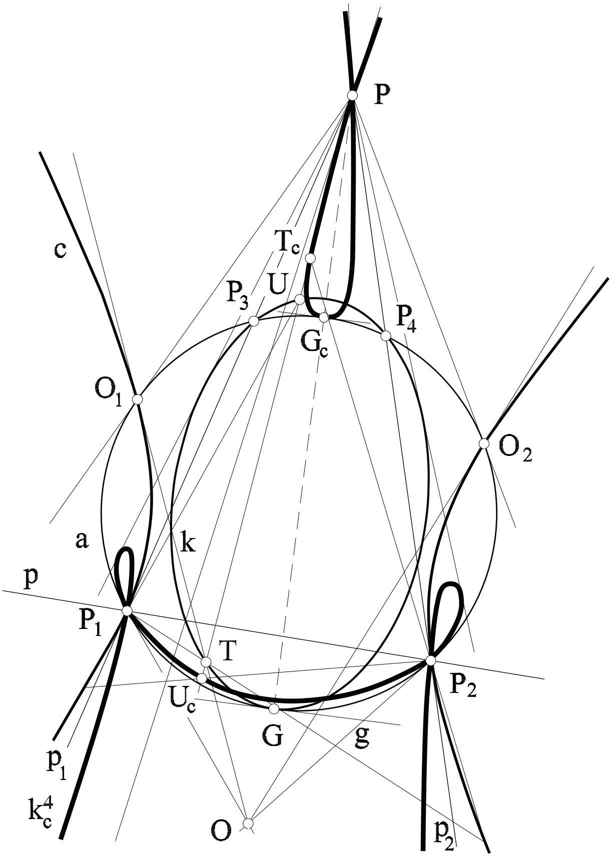

Figure 2.

branch of k^4 at a node is its isotropic asymptote. The same holds for the fundamental point P_2 (Fig. 2). Beside these two triple joint points, the H-cycle has one more double joint point on the absolute as the image of the tangential point G of the H-parabola. Therefore, k^4 is of type (2+3+3).

Based on the above discussion we can conclude that k^4 has a cusp in the absolute point P_1 with an isotropic asymptote as a tangent, if p_1 is a tangent line of k at the point P_3. Therefore k^4 would be also an H-circle.

The most interesting case arises when the absolute tangential point G coincides with one of the fixed points O_1 or O_2 of the H-inversion (Fig. 3). Furthermore, let k intersect the absolute in P_3 and touch the fundamental line p_2 at P_4. The result of the mapping is the entirely circular curve of 4^{th}-order with one double joint and two triple joint points on the absolute. Therefore, it is of type (2+3+3). From the position of k follows that P_1 is a node, which is a tangent point of the curve with selfintersection, and P_2 is a cusp with a tangent which is an isotropic asymptote. In order to prove the later statement let us consider the pencil of straight lines through P_4. It will be mapped into a pencil of conics passing through three fundamental points. The remaining two points of each conic are the intersection points of the given line and the fundamental conic c. Only the line P_1P_4 will be mapped into a conic that splits into two lines: p_1 and t which is determined by P_2 and by the intersection point of P_1P_4 with c. Therefore, the line t will be the tangent in the cusp. Because the vertex of the pencil P_4 has been mapped into P_2, the obtained tangent will be an isotropic asymptote.

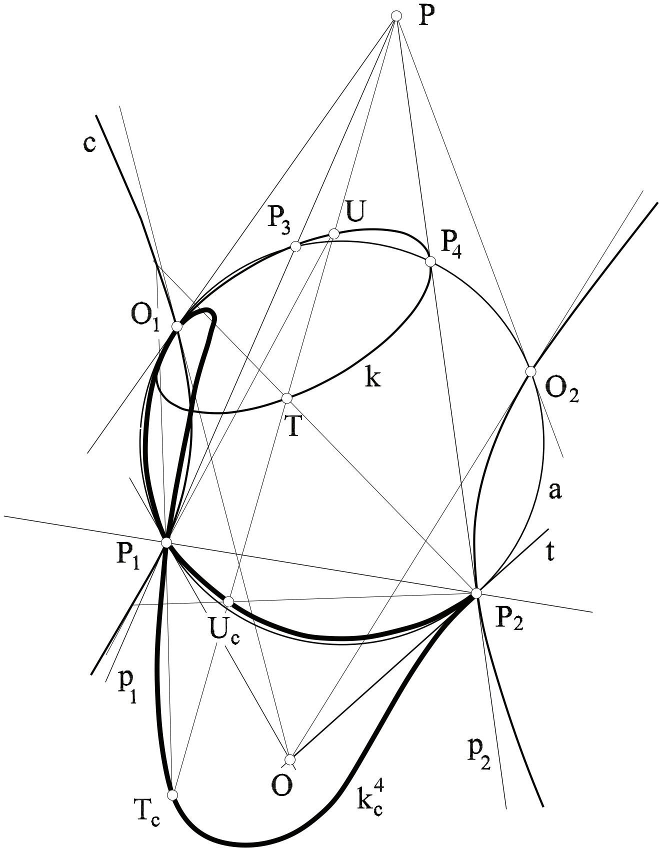

Figure 3.

Type 2)

Let k be an **H − circle**, e.g. a hypercycle (Figure 4). With exception of the tangential points P_3 and P_4 the hypercycle has no other points on the absolute. It can be constructed by a perspective collineation from the absolute with the axis $g \equiv P_3P_4$ and center G. With respect to the absolute a curve k^4 has only two different points at the fundamental points (P_1 and P_2) of the H-inversion, which are quadruple joint into each one. It is a pair of double points of the curve k^4. Consequently, the type of this H-circle is (4+4) (Fig. 4).

Let us prove this for the point P_1. While the polar line p_1 intersects k in two different real points, P_1 is its double joint point on the absolute, even a node. The hypercycle k touches the absolute at the double joint point P_3 which is mapped into P_1. So P_1 will be a quadruple joint point at the absolute - a node, which is a three times osculating point with selfintersection. The same holds for the point P_2. Regarding the previous consideration we can conclude that the pole P in Figure 4 is an isolated double point of the H-circle k^4.

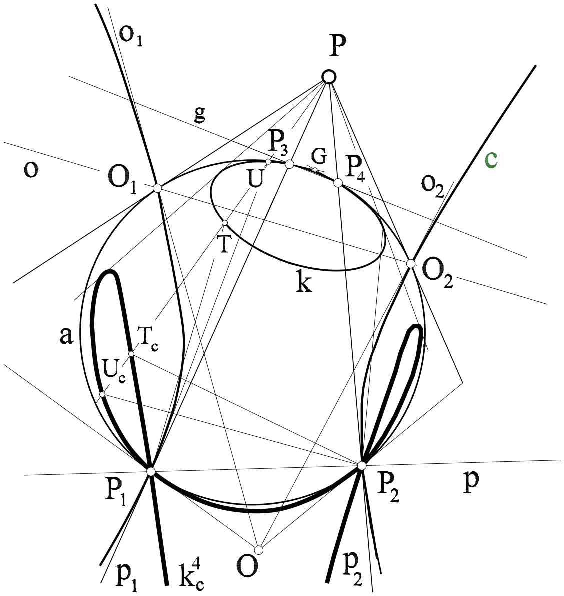

Figure 4.

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